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Triangle packings and 1-factors in oriented graphs

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ABSTRACT

An oriented graph is a directed graph which can be obtained from a simple undirected graph by orienting its edges. In this paper we show that any oriented graph G on n vertices with minimum indegree and outdegree at least $(1/2 - o(1))n$ contains a packing of cyclic triangles covering all but at most 3 vertices. This almost answers a question of Cuckler and Yuster and is best possible, since for $n \equiv 3 \bmod 18$ there is a tournament with no perfect triangle packing and with all indegrees and outdegrees $(n-1)/2$ or $(n-1)/2 \pm 1$. Under the same hypotheses, we also show that one can embed any prescribed almost 1-factor, i.e. for any sequence $n_1, \dots, n_t$ with $\sum_{i=1}^{t} n_i \leqslant n - O(1)$ we can find a vertex-disjoint collection of directed cycles with lengths $n_1, \dots, n_t$. In addition, under quite general conditions on the n_i we can remove the $O(1)$ additive error and find a prescribed 1-factor.

1. Introduction

A classical result of Extremal Combinatorics, Dirac's theorem [10], states that a graph G on $n \geqslant 3$ vertices with minimum degree at least $n/2$ contains a Hamiltonian cycle, i.e. a cycle that passes through every vertex of G. This motivates the general question of determining what minimum degree condition one needs to find a certain structure in a graph. An important result of this type is the Hajnal–Szemerédi theorem [15], which states that if G is a graph on n vertices with minimum degree at least $(1 - 1/r)n$ and r divides n then G has a perfect K_r-packing, i.e., a collection of vertex-disjoint copies of the complete graph K_r on r vertices which covers all the vertices of G. (The case $r = 3$ was obtained earlier by Corrádi and Hajnal [8].) This was generalised to packings of arbitrary graphs by

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Komlós, Sarközy and Szemerédi [25]. Confirming a conjecture of Alon and Yuster [5], they proved that for every graph H there is a constant C such that if G is a graph on n vertices with minimum degree at least $(1-1/\chi(H))n+C$ and $|V(H)|$ divides n then G has a perfect H-packing. (Here $\chi(H)$ denotes the chromatic number of H.) Finally, Kühn and Osthus [27] determined the minimum degree needed to find an H-packing up to an additive constant: it is $(1-1/\chi^*(H))n+O(1)$, where $\chi^*(H)$ is a rational number in the range $(\chi(H)-1, \chi(H)]$ that can be calculated when H is given. Another packing result that is closely related to the topic of this paper was obtained by Aigner and Brandt [1] and in a slightly weaker form by Alon and Fischer [2]. Verifying a conjecture of Sauer and Spencer [35], they proved that a graph G on n vertices with minimum degree at least $(2n-1)/3$ contains any graph H on n vertices with maximum degree at most 2.

It is very natural to ask whether these results have analogues for directed graphs. Here instead of degree one may consider the *minimum semidegree* $\delta^0(G)=\min(\delta^+(G),\delta^-(G))$, where $\delta^+(G)$ is the minimum outdegree and $\delta^-(G)$ is the minimum indegree of a digraph G. (We refer the reader to [6] for a comprehensive introduction to the theory of directed graphs.) A directed version of Dirac's theorem was obtained by Ghouila-Houri [11], who showed that any digraph G on n vertices with minimum semidegree at least $n/2$ contains a Hamilton cycle. (When referring to paths and cycles in directed graphs we always mean that these are directed, without mentioning this explicitly.) This result is very closely related to Dirac's theorem, as (some) extremal digraphs can be obtained from extremal graphs for Dirac's theorem by replacing each edge by a pair of arcs, one in each direction; the proof of the upper bound is more complicated, but not unduly so. However, the situation becomes more complicated if one considers *oriented graphs*. An oriented graph is a directed graph that can be obtained from a (simple) undirected graph by orienting its edges. The question concerning the analogue of Dirac's theorem in this case was raised by Thomassen [37], who asked what minimum semidegree forces a Hamilton cycle in an oriented graph. Over the years since the question was posed, a series of improving bounds were obtained in [38,39,13,14], until an asymptotic solution of $(3/8+o(1))n$ was given by Kelly, Kühn and Osthus [20]. Soon after that an exact answer $\lceil\frac{3n-4}{8}\rceil$ for n sufficiently large was proved by Keevash, Kühn and Osthus [19]. This result was recently extended by Kelly, Kühn and Osthus [21], who showed that the same minimum semidegree condition guarantees that G is pancyclic (contains directed cycles with all lengths ℓ, $3\leqslant\ell\leqslant n$).

In this paper we mainly study cycle packings in oriented graphs, although in the concluding remarks we discuss directed graphs as well. Our starting point is the following question posed independently by Cuckler [9] and Yuster [40]. A tournament is an orientation of a complete graph. It is regular if every vertex has equal indegree and outdegree.

Question. Does a regular tournament on n vertices with $n\equiv 3 \bmod 6$ have a perfect packing of cyclic triangles?

We obtain the following general result, which in the case of tournaments 'almost' answers this question.

Theorem 1.1.

(i) *There is some real $c>0$ so that for sufficiently large n, any oriented graph G on n vertices with minimum indegree and outdegree at least $(1/2-c)n$ contains a packing of cyclic triangles covering all but at most 3 vertices.*

(ii) *If $n\equiv 3 \bmod 18$ then there is a tournament T which does not have a perfect packing of cyclic triangles, in which every vertex has indegree and outdegree $(n-1)/2$ or $(n-1)/2\pm 1$.*

Our second result is an attempt to prove a directed analogue of the results of Aigner–Brandt and Alon–Fischer. It shows that the same semidegree condition as above allows one to cover all but a constant number of vertices by cycles of prescribed lengths.

Theorem 1.2. *There exist constants $c, C > 0$ such that for n sufficiently large, if G is an oriented graph on n vertices with minimum indegree and outdegree at least $(1/2 - c)n$ and $n_1, \dots, n_t$ are numbers with $\sum_{i=1}^{t} n_i \leqslant n - C$ then G contains vertex-disjoint cycles of length $n_1, \dots, n_t$.*

Moreover, in some cases our technique allows us to strengthen the previous theorem and to obtain a prescribed 1-*factor*, i.e., a perfect packing by cycles with given lengths. To illustrate this we prove the following result, in which we also assume that G is a tournament to make the proof more convenient to present, although one can remove this assumption.

Theorem 1.3. *For any number M there is $c > 0$ and numbers T and n_0 so that if G is a tournament on $n > n_0$ vertices with minimum indegree and outdegree at least $(1/2 - c)n$ and $n_1, \dots, n_t$ are numbers satisfying $\sum_{i=1}^{t} n_i = n$ then G contains a 1-factor with cycle lengths $n_1, \dots, n_t$ if the following holds: for some $3 \leqslant k \leqslant M$ at least $T \log n$ of the n_i are equal to k and at least T of the n_i lie between $k + 1$ and M.*

The rest of this paper is organised as follows. In the next section we prove Theorem 1.1 using probabilistic arguments (Rödl nibble) together with the idea of 'absorbing structures' introduced by Rödl, Rucinski and Szemerédi. In Section 3 we prove Theorem 3.1, which is the first ingredient in the proof of Theorem 1.2, and a result of independent interest: an asymptotically best possible condition for finding a 1-factor in which all prescribed cycle lengths are long. To deal with short cycles we need the machinery of Szemerédi's Regularity Lemma and the Blow-up Lemma of Komlós, Sárközy and Szemerédi, which we describe in Section 4. Then in Section 5 we prove Theorem 5.1, the second ingredient in the proof of Theorem 1.2, giving an almost perfect packing by k-cycles for any fixed k. Section 6 contains the proofs of Theorems 1.2 and 1.3 and the final section contains some concluding remarks.

Notation. Given two vertices x and y of a directed graph G, we write xy for the edge directed from x to y. We write $N_G^+(x)$ for the outneighbourhood of a vertex x and $d_G^+(x) := |N_G^+(x)|$ for its outdegree. Similarly, we write $N_G^-(x)$ for the inneighbourhood of x and $d_G^-(x) := |N_G^-(x)|$ for its indegree. We write $N_G(x) := N_G^+(x) \cup N_G^-(x)$ for the neighbourhood of x and $d_G(x) := |N_G(x)|$ for its degree. We use $N^+(x)$, etc. whenever this is unambiguous. As is customary in Extremal Graph Theory, our approach to the problems researched will be asymptotic in nature. We thus assume that the order n of a graph G tends to infinity and therefore is sufficiently large whenever necessary. We also assume that the constant c, which controls the deviation of the degrees of G from $n/2$ is sufficiently small. When we speak of 'paths' and 'cycles' in directed graphs it is always to be understood that these are directed paths and cycles. We use the notation $0 < \alpha \ll \beta$ to mean that there is an increasing function $f(x)$ so that the following argument is valid for $0 < \alpha < f(\beta)$. We write $a \pm b$ to denote an unspecified real number in the interval $[a - b, a + b]$.

2. Covering by cyclic triangles

In this section we prove Theorem 1.1. Our arguments combine probabilistic reasoning (Rödl nibble) together with idea of 'absorbing structures' introduced by Rödl, Rucinski and Szemerédi. We divide the exposition into five subsections, that successively treat two simple lemmas, large deviation inequalities, the nibble, our absorbing structure, and the proof of Theorem 1.1.

2.1. Two simple lemmas

Our first lemma shows that, under the hypotheses of Theorem 1.1, there are approximately the same number of cyclic triangles through every vertex.

Lemma 2.1. *Suppose $c > 0$ and G is an oriented graph on n vertices with minimum indegree and outdegree at least $(1/2 - c)n$. Then every vertex x of G belongs to at least $(1/8 - 2c)n^2$ and at most $(1/8 + 2c)n^2$ cyclic triangles.*

Proof. To prove this lemma, we need to estimate $e(N^+(x), N^-(x))$, the number of edges in G going from the outneighbourhood of x to the inneighbourhood of x.

$$\begin{aligned} e\big(N^+(x), N^-(x)\big) &\geqslant \sum_{y \in N^+(x)} \big(d^+(y) - \big|V(G) \setminus \big(N^+(x) \cup N^-(x)\big)\big|\big) - e\big(N^+(x)\big) \\ &\geqslant \big|N^+(x)\big|\big((1/2-c)n - n + \big|N^+(x)\big| + \big|N^-(x)\big|\big) - \big|N^+(x)\big|^2/2 \\ &= \big|N^+(x)\big|\big((1/2-c)n - n + \big|N^+(x)\big|/2 + \big|N^-(x)\big|\big) \\ &\geqslant (1/2-c)n\left((1/2-c)n - n + \frac{3}{2}(1/2-c)n\right) \\ &= (1/2-c)n(1/4-5c/2)n \geqslant (1/8-2c)n^2. \end{aligned}$$

By symmetry we can also estimate $e(N^-(x), N^+(x)) \geqslant (1/8-2c)n^2$. Therefore

$$\begin{aligned} e\big(N^+(x), N^-(x)\big) &\leqslant \big|N^+(x)\big|\big|N^-(x)\big| - e\big(N^-(x), N^+(x)\big) \\ &\leqslant n^2/4 - (1/8-2c)n^2 = (1/8+2c)n^2. \qquad \square \end{aligned}$$

Our next lemma will allow us to find cyclic triangles on ‘most’ pairs of vertices. Suppose that G is an oriented graph on n vertices. We say that an edge e of G is a-good if there are at least an cyclic triangles containing e; otherwise we say it is a-bad. Also, given a vertex x we say that a vertex y is a-good for x if an edge between x and y is a-good; otherwise we say it is a-bad for x.

Lemma 2.2. *Suppose $c > 0$ and G is an oriented graph on n vertices with minimum indegree and outdegree at least $(1/2-c)n$. For any $a > 0$ and vertex x, there are at most $(2a+4c)n$ a-bad vertices for x in each of $N^+(x)$ and $N^-(x)$, so the total number of a-bad vertices for x is at most $(4a+10c)n$.*

Proof. Let S be the set of a-bad vertices for x that belong to $N^+(x)$. Then, by definition, any $y \in S$ has at most an outneighbours in $N^-(x)$. By averaging there is some $y \in S$ with at most $|S|/2$ outneighbours in S, and for this y we have

$$\begin{aligned} (1/2-c)n \leqslant \big|N^+(y)\big| &= \big|N^+(y) \cap N^-(x)\big| + \big|N^+(y) \cap N^+(x)\big| + \big|N^+(y) \setminus N(x)\big| \\ &\leqslant an + \big(\big|N^+(x)\big| - |S|/2\big) + \big|V(G) \setminus N(x)\big| = an - |S|/2 + n - \big|N^-(x)\big| \\ &\leqslant an + (1/2+c)n - |S|/2, \end{aligned}$$

so $|S| \leqslant (2a+4c)n$. Similarly there are at most $(2a+4c)n$ a-bad vertices for x that belong to $N^-(x)$. Since $|V(G) \setminus N(x)| \leqslant 2cn$ there are at most $(4a+10c)n$ a-bad vertices for x. $\square$

2.2. Large deviation inequalities

We will use the following three large deviation estimates. The first one is a classical Chernoff-type bound see, e.g., [4, Appendix A].

Theorem 2.3. *Suppose $a > 0$ and $X_1, \dots, X_m$ are independent identically distributed random variables with $\mathbb{P}(X_i = 1) = p$ and $\mathbb{P}(X_i = 0) = 1 - p$. Then*

1. $\mathbb{P}(\sum_{i=1}^m X_i < pm - a) < e^{-a^2/2pm}$, *and*
2. $\mathbb{P}(\sum_{i=1}^m X_i \geqslant pm + a) < e^{-a^2/2pm + a^3/2(pm)^2}$.

Another useful inequality is due to Azuma (see, e.g., [4,16]).

Theorem 2.4. *Let $t_1, \dots, t_n$ be a family of independent indicator random variables. Suppose that real-valued function $X = X(t_1, \dots, t_n)$ is c-Lipschitz, i.e., changing the value of any t_i can change the value of X by at most c. Then*

$$\mathbb{P}\big(|X - \mathbb{E}X| > a\big) \leqslant 2e^{-a^2/2nc^2}.$$

The third inequality we need was proved by Kim and Vu [22] (see also Chapter 7.8 in [4]). Suppose $f(x_1, \dots, x_m) = \sum_{e \in H} \prod_{i \in e} x_i$ is a homogeneous polynomial of degree k defined by a k-uniform hypergraph H on $[m]$. Let $X_1, \dots, X_m$ be independent identically distributed random variables with $\mathbb{P}(X_i = 1) = p$, $\mathbb{P}(X_i = 0) = 1 - p$ and let $Y = f(X_1, \dots, X_m)$. For $J \subset [m]$ define $\partial_J f$ to be the partial derivative of f with respect to the variables $\{x_i\}_{i \in A}$ and the *non-zero influence* $\mathbb{E}'Y = \max_{|A| \geqslant 1} \mathbb{E}\partial_A f$.

Theorem 2.5. *For $t > 1$,*

$$\mathbb{P}\big(|Y - \mathbb{E}Y| > (2k)!t^k\sqrt{\mathbb{E}Y\mathbb{E}'Y}\big) \leqslant 16e^{-t+(k-1)\log m}.$$

2.3. *The nibble*

The 'nibble' is a term referring to a semi-random construction method, used by Rödl [32] in proving the existence of asymptotically good designs. Several researchers realised that this method applies in a more general setting, dealing with matchings in uniform hypergraphs. The following theorem is due to Pippenger, following Frankl and Rödl, with further refinements by Pippenger and Spencer [30]. We refer the reader to the presentation given in [12] and in [4, pp. 54–58]. For a pair of vertices x, y of a hypergraph H, the common degree $d(x, y)$ is the number of edges of H containing both x and y.

Theorem 2.6. *For any $\varepsilon > 0$ and number r there is $\delta > 0$ and a number d so that the following holds for $n > D > d$. Any r-uniform hypergraph H on n vertices such that any vertex x has degree $d(x) = (1 \pm \delta)D$ and any pair of vertices x, y has common degree $d(x, y) < \delta D$ contains a matching covering at least $(1 - \varepsilon)n$ vertices.*

The result which appears in [4] deals with covering of vertices of hypergraph rather than matchings. It states that a k-uniform hypergraph H on n vertices with all degrees $(1 + o(1))D$ and all codegrees $o(D)$ has a collection of $(1 + o(1))n/k$ edges which covers all its vertices. It is easy to check that deletion of all pairs of intersecting edges from this collection gives a matching covering $(1 - o(1))n$ vertices of H.

We also need a further property of the matching which comes out of the proof of Theorem 2.6 by means of a semi-random 'nibble'. The matching is constructed in a series of 'bites', in which we choose each remaining available edge independently with some probability $p = \Theta(n^{-2})$ (which shrinks by a constant factor with each step) and delete any pair of edges that intersect. It is shown that with probability at least 0.9 (say) each bite preserves certain regularity properties in the hypergraph that allow the nibble to proceed. The parameters of the proof are such that we may assume that the first bite constructs a matching of size βn with $\varepsilon \ll \beta \ll 1$.

2.4. *The absorbing structure*

The nibble can be used to cover all but $o(n)$ vertices, and to make further progress we will need a mechanism that will allow us to gradually 'absorb' the remaining vertices into our triangle packing. Our approach was inspired by ideas used in [33,34] to obtain results on matchings and Hamiltonian hypergraphs.

Suppose G is an oriented graph and $Q = \{v_1, v_2, v_3, v_4\}$ is a quadruple of vertices in G. We say that the disjoint sets $a_1a_2a_3$, $b_1b_2b_3$, $c_1c_2c_3$ are an absorbing triple of triangles for Q if each of the following triples is a cyclic triangle in G: $a_1a_2a_3$, $b_1b_2b_3$, $c_1c_2c_3$, $v_1a_1b_1$, $v_2c_1a_2$, $v_3b_2c_2$, $v_4a_3b_3$. The motivation for this definition is that if we have a set of disjoint cyclic triangles $C_1, \dots, C_t$ that includes

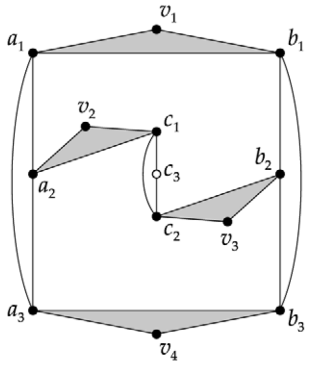

Fig. 1.

$a_1a_2a_3$, $b_1b_2b_3$, $c_1c_2c_3$ and is disjoint from Q, then we can enlarge our collection by replacing $a_1a_2a_3$, $b_1b_2b_3$, $c_1c_2c_3$ by $v_1a_1b_1$, $v_2c_1a_2$, $v_3b_2c_2$, $v_4a_3b_3$. Thus Q is absorbed and the vertex c_3 is lost, for a net gain of one triangle (see Fig. 1).

The following lemma shows that there are many absorbing triples for every quadruple Q.

Lemma 2.7. *There is some $c > 0$ and number n_0 such that if G is an oriented graph on $n > n_0$ vertices with minimum indegree and outdegree at least $(1/2 - c)n$ then for any quadruple of vertices Q there are at least $(n/100)^9$ absorbing triples for Q in G.*

Proof. We use the above notation and greedily construct the absorbing triples by repeated application of Lemma 2.2.

1. Pick a_1 to be 1/8-good for v_1 and disjoint from Q. There are at least $n - (4/8 + 10c)n - 4 > (1/2 - 11c)n$ possible choices.
2. Pick a_2 to be 1/16-good for a_1 and v_2 (and disjoint from $Q \cup \{a_1\}$: we will not keep repeating this condition). There are at least $n - 2(4/16 + 10c)n - 5 > (1/2 - 21c)n$ possible choices.
3. Pick a_3 to be 1/128-good for v_4 and so that $a_1a_2a_3$ is a cyclic triangle. Since a_2 is 1/16-good for a_1 there are at least $n/16 - (4/128 + 10c)n - 6 > (1/32 - 11c)n$ possible choices.
4. Pick b_3 so that $v_4a_3b_3$ is a cyclic triangle. Since a_3 is 1/128-good for v_4 there are at least $n/128 - 7$ possible choices.
5. Pick b_1 to be 1/64-good for b_3 and so that $v_1a_1b_1$ is a cyclic triangle. Since a_1 is 1/8-good for v_1 there are at least $n/8 - (4/64 + 10c)n - 8 > (1/16 - 11c)n$ possible choices.
6. Pick b_2 to be 1/512-good for v_3 and so that $b_1b_2b_3$ is a cyclic triangle. Since b_1 is 1/64-good for b_3 there are at least $n/64 - (4/512 + 10c)n - 9 > (1/128 - 11c)n$ possible choices.
7. Pick c_2 so that $v_3b_2c_2$ is a cyclic triangle. Since b_2 is 1/512-good for v_3 there are at least $n/512 - 10$ possible choices.
8. Pick c_1 to be 1/128-good for c_2 and so that $v_2c_1a_2$ is a cyclic triangle. Since a_2 is 1/16-good for v_2 there are at least $n/16 - (4/128 + 10c)n - 11 > (1/32 - 11c)n$ possible choices.
9. Pick c_3 so that $c_1c_2c_3$ is a cyclic triangle. Since c_1 is 1/128-good for c_2 there are at least $n/128 - 12$ possible choices.

Note that given three cyclic triangles there are $3! = 6$ ways to chose which one of them is going to be $a_1a_2a_3$, $b_1b_2b_3$, $c_1c_2c_3$. Then for every cyclic triangle there are $3! = 6$ different labeling of its vertices. This implies that each configuration of three cyclic triangles was counted at most 6^4 times. Hence the number of absorbing triples is at least

$$6^{-4}\cdot\left(\frac{1}{2}-11c\right)n\cdot\left(\frac{1}{2}-21c\right)n\cdot\left(\frac{1}{32}-11c\right)n\cdot\left(\frac{n}{128}-7\right)\cdot\left(\frac{1}{16}-11c\right)n\cdot\left(\frac{1}{128}-11c\right)n$$
$$\cdot\left(\frac{n}{512}-10\right)\cdot\left(\frac{1}{32}-11c\right)n\cdot\left(\frac{n}{128}-12\right)=\left(6^{-4}2^{-46}-O(c)\right)n^9>(n/100)^9. \qquad \square$$

Next we use the previous lemma to show that a random selection of vertex-disjoint cyclic triangles will have many absorbing triples for every quadruple of vertices of G.

Lemma 2.8. *Suppose $0<1/n_0\ll c\ll c_2\ll 1$ and G is an oriented graph on $n>n_0$ vertices with minimum indegree and outdegree at least $(1/2-c)n$. Suppose we form a collection of vertex-disjoint cyclic triangles C by choosing each cyclic triangle in G independently with probability $p=c_2/n^2$ and deleting any pair of triangles that intersect. Then with probability at least 0.9 we have $|C|=m=(1\pm c_2^{1/2})c_2n/24$ and for any quadruple of vertices Q there are at least $10^{-16}m^3$ absorbing triples for Q in C.*

Proof. Let C' be a collection of cyclic triangles formed by choosing each cyclic triangle of G randomly and independently with probability $p=c_2/n^2$. By Lemma 2.1, every vertex of G is contained in $(1/8\pm 2c)n^2$ cyclic triangles and therefore the number of cyclic triangles in G is $T=(1/8\pm 2c)n^2\cdot n/3=(1\pm 16c)n^3/24$. Applying Chernoff bounds (mentioned in Section 2.2) we obtain that

$$|C'|=(1\pm c)pT=(1\pm 20c)pn^3/24=(1\pm 20c)c_2n/24$$

with high probability. Let Z be the number of pairs of intersecting triangles in C'. Since the total number of such pairs is clearly at most n^5 we have that $\mathbb{E}Z<p^2n^5=c_2^2n$. Hence, $c_2\ll 1$ together with Markov's inequality gives that $Z<c_2^{3/2}n/100$ with probability at least 0.95. Since $c\ll c_2$, by definition of C, we obtain

$$m=|C|\geqslant|C'|-2Z>\left(1\pm c_2^{1/2}\right)c_2n/24.$$

Given a quadruple of vertices Q, let A_Q be the set of absorbing triples for Q. By Lemma 2.7 we have $|A_Q|>(n/100)^9$. Let X_Q be the random variable counting the number of absorbing triples for Q that belong to C'. Then $\mathbb{E}X_Q=p^3|A_Q|>(c_2n)^3/10^{18}$. We can write $X_Q=\sum_{S\in A_Q}\prod_{T\in S}I_T$ where I_T is the indicator random variable for the event that triangle T is chosen for C'. Since X_Q is a homogeneous polynomial of degree three, we can estimate the probability that X_Q is small by Theorem 2.5. Since the number of absorbing triples for Q which contains a given triangle is clearly at most n^6 it is easy to see that $\max_{|J|=1}\mathbb{E}\partial_JX_Q$ is at most $p^2n^6=(c_2n)^2$. Similarly, there are at most n^3 absorbing triples containing a given pair of triangles and therefore $\max_{|J|=2}\mathbb{E}\partial_JX_Q\leqslant pn^3=c_2n$. This implies that the non-zero influence $\mathbb{E}'X_Q=\max_{|J|\geqslant 1}\mathbb{E}\partial_JX_Q$ is bounded by $p^2n^6=(c_2n)^2$. Now we apply Theorem 2.5 with $k=3$ and $t=10^{-5}(c_2n)^{1/6}$. With this choice of t we have $720t^3\sqrt{\mathbb{E}X_Q\,\mathbb{E}'X_Q}<\mathbb{E}X_Q/2$, so we can estimate

$$\mathbb{P}(X_Q<\mathbb{E}X_Q/2)\leqslant 16e^{-t+2\log n^3}\ll n^{-4}.$$

Taking a union bound over all quadruples of vertices of G, we obtain that with high probability $X_Q>p^3|A_Q|/2>m^3/10^{15}$ for every Q. Note also that deletion of any triangle from C' can destroy at most $|C'|^2<2m^2$ absorbing triples for Q. Since we delete at most $2Z<c_2^{3/2}n/50<c_2m/2$ triangles to form C and since $c_2\ll 1$, we still have at least $m^3/10^{15}-c_2m^3>m^3/10^{16}$ absorbing triples for each Q. $\square$

2.5. *Proof of Theorem 1.1*

(i) Choose constants to satisfy the hierarchy $0<1/n\ll c\ll c_1\ll c_2\ll 1$ and suppose G is an oriented graph on n vertices with minimum indegree and outdegree at least $(1/2-c)n$. Consider the hypergraph H on the same vertex set of G whose edges are all cyclic triangles in G. By Lemma 2.1

every vertex x in H has degree $d_H(x) = (1/8 \pm 2c)n^2$. Also, for any pair of vertices x, y we have $d_H(x, y) \leqslant n - 2 \ll n^2/8$. Applying Theorem 2.6, we can cover all but at most $(1 - c_1)n$ vertices with vertex-disjoint cyclic triangles.

Furthermore, as explained in the paragraph after Theorem 2.6, we may assume that the first bite of the nibble was obtained by choosing each cyclic triangle in G with probability c_2/n^2 and deleting any pair of triangles that intersect. Since the bite was valid for the nibble with probability at least 0.9, we can also assume that it is an absorbing collection C as given by Lemma 2.8. Now, as long as there at least 4 uncovered vertices, we can repeatedly choose a quadruple Q from these vertices and increase our triangle packing by using an absorbing triple for Q from C. Note that at each such iteration, we can only use absorbing triples from C no triangle of which has yet been used in previous rounds. Since at each iteration we use three triangles and each triangle can participate in at most $|C|^2 = m^2$ absorbing triples for Q, we destroy at most $3m^2$ absorbing triples for Q at every round. As the number of rounds is at most $c_1 n \ll m = \Omega(c_2 n)$ there are still at least $10^{-16}m^3 - c_1 nm^2 > 10^{-17}m^3$ absorbing triples remaining untouched during the whole procedure. This shows that our process can be continued until only 3 vertices will remain uncovered, which completes the proof of the first part.

(ii) To prove the second part of the theorem, partition a set of n vertices with $n = 18k+3$ into three sets V_0, V_1, V_2 of size $|V_0| = 6k$, $|V_1| = 6k + 1$, $|V_2| = 6k + 2$. Construct a tournament T as follows. Between the classes we orient all pairs from V_i to V_{i+1}, where addition is mod 3. Inside each class we place a tournament that is as regular as possible, i.e., in V_1 all indegrees and outdegrees are $3k$, in V_0 each vertex has indegree and outdegree $3k$ and $3k - 1$ in some order, and in V_2 each vertex has indegree and outdegree $3k$ and $3k + 1$ in some order. Then every vertex in T either has indegree and outdegree $9k + 1$ or indegree and outdegree $9k$ and $9k + 2$ in some order. However, any collection of vertex-disjoint cyclic triangles in T must leave at least 3 vertices uncovered. To see this, notice that a cyclic triangle must either have one point in each part or all three points in one of V_i, and so however many triangles we remove from T the class sizes will always be different mod 3.

3. Long cycles

Our first ingredient in the proof of Theorem 1.2 will be the following theorem, which shows that the minimum semidegree threshold for finding a 1-factor in which all the prescribed cycle lengths are large is asymptotically $3n/8$. The lower bound is given by a construction in [13] (see also [20]) of an oriented graph with minimum semidegree $\sim 3n/8$ and with no 1-factor at all. Hence it only remains to prove the upper bound.

Theorem 3.1. *For any $\delta > 0$ there are numbers M and n_0 so that if G is an oriented graph on $n > n_0$ vertices with minimum indegree and outdegree at least $(3/8 + \delta)n$ and $n_1, \ldots, n_t$ are numbers satisfying $n_i \geqslant M$ for $1 \leqslant i \leqslant t$ and $\sum_{i=1}^{t} n_i = n$ then G contains a 1-factor with cycle lengths $n_1, \ldots, n_t$.*

Our proof combines the partitioning argument similar to that used in [2] together with the result of [21]. In Theorem 8 of [21] it was proved that for all sufficiently large n every oriented graph G on n vertices with minimum semidegree at least $(3n - 4)/8$ contains an ℓ-cycle for all $3 \leqslant \ell \leqslant n$. We also need another large deviation inequality, for the hypergeometric random variable X with parameters (n, m, k), which is defined as follows. Fix $S \subset [n]$ of size $|S| = m$. Pick a random $T \subset [n]$ of size $|T| = k$. Define $X = |T \cap S|$. Then $\mathbb{E}X = km/n$. We have the following 'Chernoff bound' approximation for $0 < a < 3/2$ (see [16, pp. 27–29]):

$$\mathbb{P}\big(|X - \mathbb{E}X| > a\mathbb{E}X\big) < 2e^{-\frac{a^2}{3}\mathbb{E}X}. \tag{1}$$

The following lemma is an immediate consequence of the previous inequality.

Lemma 3.2. *For any $\alpha, \beta > 0$ there is a number n_0 so that the following holds. Suppose G is an oriented graph on $n > n_0$ vertices with minimum indegree and outdegree at least αn and $\beta n < m < (1 - \beta)n$. Then there is a partition of $V(G)$ as $A \cup B$ with $|A| = m$ and $|B| = n - m$ so that $G[A]$ has minimum indegree and outdegree at least $(\alpha - n^{-1/3})m$ and $G[B]$ has minimum indegree and outdegree at least $(\alpha - n^{-1/3})(n - m)$.*

Proof of Theorem 3.1. Without loss of generality suppose that $n_1 \geqslant n_i$ for $1 \leqslant i \leqslant t$. Consider two cases.

Case 1. If $n_1 > (1-\delta/2)n$ then $\sum_{i\geqslant 2} n_i < \delta n/2$ and we can use Theorem 8 in [21] (mentioned above) to choose disjoint cycles of length $n_2, \dots, n_t$ one by one. Indeed, it is possible since during this process the semidegree of the oriented graph which remains is always at least $(3/8+\delta)n - \sum_{i\geqslant 2} n_i > 3n/8$. In particular, the oriented graph on n_1 vertices which we obtain in the end has minimum semidegree larger than $3n/8 \geqslant 3n_1/8$ and therefore has a Hamilton cycle, so we are done. Note that this argument works as long as the minimum semidegree of the graph is at least $(3/8+\delta/2)n$, which will be used in the analysis of the second case.

Case 2. If $n_1 \leqslant (1-\delta/2)n$ we can partition $[t] = I \cup J$ so that $n_I = \sum_{i\in I} n_i$ and $n_J = \sum_{i\in J} n_i$ are both at most $(1-\delta/2)n$ (we may assume $\delta < 1/3$). Then by Lemma 3.2 there is a partition of the vertices of G into sets V_I of size n_I and V_J of size n_J so that $G[V_I]$ has minimum semidegree at least $(3/8+\delta-n^{-1/3})n_I$ and $G[V_J]$ has minimum semidegree at least $(3/8+\delta-n^{-1/3})n_J$.

Now we repeat the above splitting procedure for both V_I and V_J, repeatedly partitioning while Case 2 holds. Each time the number of vertices in the part which was split is reduced by a factor of $(1-\delta/2)$ and no part in our process ever has size smaller than M. Therefore, for any part S in the final partition the induced graph $G[S]$ has minimum indegree and outdegree at least

$$\left(3/8+\delta-M^{-1/3}\sum_{i=0}^{\infty}(1-\delta/2)^{i/3}\right)|S| = \left(3/8+\delta-\frac{1}{1-(1-\delta/2)^{1/3}}M^{-1/3}\right)|S|.$$

For large enough M this is more than $(3/8+\delta/2)|S|$, so we can use the argument of Case 1 to find the required cycles. □

4. Regularity

The second ingredient in the proof of Theorem 1.2 will be a theorem giving an almost perfect packing of k-cycles when k is fixed and the number of vertices n is large. The proof of this theorem will use the machinery of Szemerédi's Regularity Lemma and the Blow-up Lemma of Komlós, Sárközy and Szemerédi, which we will now describe. We will be quite brief, so for more details and motivation we refer the reader to the surveys [26] for the Regularity Lemma and [23] for the Blow-up Lemma.

We start with some definitions. The density of a bipartite graph $G = (A, B)$ with vertex classes A and B is defined to be

$$d_G(A, B) := \frac{e_G(A, B)}{|A||B|}.$$

We often write $d(A, B)$ if this is unambiguous. Given $\varepsilon > 0$, we say that G is *ε-regular* if for all subsets $X \subseteq A$ and $Y \subseteq B$ with $|X| > \varepsilon|A|$ and $|Y| > \varepsilon|B|$ we have that $|d(X, Y) - d(A, B)| < \varepsilon$. Given $d \in [0, 1]$ we say that G is *(ε, d)-super-regular* if it is ε-regular and furthermore $d_G(a) \geqslant (d-\varepsilon)|B|$ for all $a \in A$ and $d_G(b) \geqslant (d-\varepsilon)|A|$ for all $b \in B$. (This is a slight variation of the standard definition of (ε, d)-super-regularity where one requires $d_G(a) \geqslant d|B|$ and $d_G(b) \geqslant d|A|$.)

The Diregularity Lemma is a version of the Regularity Lemma for digraphs due to Alon and Shapira [3] (with a similar proof to the undirected version). We will use the following degree form of the Diregularity Lemma, which can be easily derived from the standard version, in exactly the same manner as the undirected degree form. (See e.g. [29] for a sketch proof.)

Lemma 4.1 *(Degree form of the Diregularity Lemma). For every $\varepsilon \in (0, 1)$ and $M' > 0$ there are numbers M and n_0 such that if G is a digraph on $n \geqslant n_0$ vertices and $d \in [0, 1]$, then there is a partition of the vertices of G into $V_0, V_1, \dots, V_s$ and a spanning subdigraph G' of G such that the following holds*:

- $M' \leqslant s \leqslant M$,

- $|V_0| \leqslant \varepsilon n$,
- $|V_1| = \cdots = |V_s|$,
- $d^+_{G'}(x) > d^+_G(x) - (d+\varepsilon)n$ *for all vertices* $x \in G$,
- $d^-_{G'}(x) > d^-_G(x) - (d+\varepsilon)n$ *for all vertices* $x \in G$,
- *for all* $i = 1, \ldots, s$ *the digraph* $G'[V_i]$ *is empty,*
- *for all* $1 \leqslant i, j \leqslant s$ *with* $i \neq j$ *the bipartite graph whose vertex classes are* V_i *and* V_j *and whose edges are all the edges in* G' *directed from* V_i *to* V_j *is* ε*-regular and has density either* 0 *or density at least* d.

Given clusters $V_1, \ldots, V_s$ and a digraph G', the *reduced digraph* R' *with parameters* (ε, d) is the digraph whose vertex set is $[s]$ and in which ij is an edge if and only if the bipartite graph whose vertex classes are V_i and V_j and whose edges are all the edges in G' directed from V_i to V_j is ε-regular and has density at least d. (So ij is an edge in R' if and only if there is an edge from V_i to V_j in G'.) It is easy to see that the reduced digraph R' obtained from the Regularity Lemma 'inherits' the minimum degree of G, in that $\delta^+(R')/|R'| > \delta^+(G)/|G| - d - 2\varepsilon$ and $\delta^-(R')/|R'| > \delta^-(G)/|G| - d - 2\varepsilon$. However, R' is not necessarily oriented even if the original digraph G is. The next lemma from [20] shows that by discarding edges with appropriate probabilities one can go over to a reduced oriented graph $R \subseteq R'$ which still inherits the minimum degree and density of G.

Lemma 4.2. *For every* $\varepsilon \in (0,1)$ *there exist numbers* $M' = M'(\varepsilon)$ *and* $n_0 = n_0(\varepsilon)$ *such that the following holds. Let* $d \in [0,1]$ *with* $\varepsilon \ll d$*, let* G *be an oriented graph of order* $n \geqslant n_0$ *and let* R' *be the reduced digraph with parameters* (ε, d) *obtained by applying Lemma* 4.1 *to* G *with parameters* ε, d *and* M'*. Then* R' *has a spanning oriented subgraph* R *such that* $\delta^+(R) \geqslant (\delta^+(G)/|G| - (d+3\varepsilon))|R|$ *and* $\delta^-(R) \geqslant (\delta^-(G)/|G| - (d+3\varepsilon))|R|$.

We conclude this section with the Blow-up Lemma of Komlós, Sárközy and Szemerédi [24].

Lemma 4.3. *Given a graph* F *on* $[s]$ *and positive numbers* d, Δ*, there is a positive real* $\eta_0 = \eta_0(d, \Delta, s)$ *such that the following holds for all positive numbers* $\ell_1, \ldots, \ell_s$ *and all* $0 < \eta \leqslant \eta_0$*. Let* F' *be the graph obtained from* F *by replacing each vertex* $i \in F$ *with a set* V_i *of* ℓ_i *new vertices and joining all vertices in* V_i *to all vertices in* V_j *whenever* ij *is an edge of* F*. Let* G' *be a spanning subgraph of* F' *such that for every edge* $ij \in F$ *the bipartite graph consisting of all the edges of* G' *between the sets* V_i, V_j *is* (η, d)*-super-regular. Then* G' *contains a copy of every subgraph* H *of* F' *with* $\Delta(H) \leqslant \Delta$*. Moreover, this copy of* H *in* G' *maps the vertices of* H *to the same sets* V_i *as the copy of* H *in* F'*, i.e. if* $h \in V(H)$ *is mapped to* V_i *by the copy of* H *in* F'*, then it is also mapped to* V_i *by the copy of* H *in* G'.

Note that the 'moreover' part of this statement does not appear in the usual formulation of the Blow-up Lemma but is stated explicitly in its proof.

5. Short cycles

We now come to the second ingredient in the proof of Theorem 1.2, which is the following statement, providing an almost perfect packing by k-cycles, when k is fixed and n is large.

Theorem 5.1. *For any number* $k \geqslant 3$ *there is some real* $c > 0$ *and numbers* C *and* n_0 *so that if* G *is an oriented graph on* $n > n_0$ *vertices with minimum indegree and outdegree at least* $(1/2 - c)n$ *then* G *contains vertex-disjoint* k*-cycles covering all but at most* C *vertices.*

Since the case $k = 3$ was already proved in Theorem 1.1, in the rest of this section we assume that $k \geqslant 4$. First we note that the following result is an immediate consequence of Lemma 4.3 (the Blow-up Lemma).

Corollary 5.2. *Suppose* H *is an oriented graph with parts* V_0, V_1, V_2 *of equal size, so that all edges go from* V_i *to* V_{i+1} *(addition* mod 3*). Suppose also that the underlying graphs between each pair of classes are* (η, d)*-*

super-regular, for some $\eta \ll d < 1$. Then H has a perfect packing by cyclic triangles (with one vertex in each class).

Next we use this corollary to obtain an almost perfect packing by k-cycles under an additional semidegree assumption on the parts.

Theorem 5.3. *Suppose numbers k, m and reals δ, d, η satisfy $0 < 1/m \ll \delta \ll \eta \ll d \ll 1/k \leqslant 1/4$ and H is an oriented graph whose vertices are partitioned into three parts V_0, V_1, V_2 of sizes $0.9m \leqslant |V_i| \leqslant m$ satisfying*

(i) *$H[V_i]$ has minimum (total) degree at least $(1-\delta)|V_i|$ for $i = 0, 1, 2$;*
(ii) *the edges of H between V_i and V_{i+1} are all directed from V_i to V_{i+1};*
(iii) *the underlying graphs between each pair of classes (V_i, V_{i+1}) are (η, d)-super-regular.*

Then H contains a packing of k-cycles covering all but at most $3k$ vertices.

Proof. First it will be useful to see how to find such k-cycles in the oriented graph K which is obtained from H by adding all directed edges from V_i to V_{i+1}. We will choose our k-cycles to have $k-2$ points in one class and 1 point in each of the other two classes. Let n_i be the number of cycles with $k-2$ points in V_i. We need to choose n_i so that $|V(H)|/k - 3 \leqslant n_0 + n_1 + n_2 \leqslant |V(H)|/k$ subject to the conditions $|V_i| \geqslant (k-2)n_i + \sum_{j \neq i} n_j = (k-3)n_i + n_0 + n_1 + n_2$. We may take $n_i = \lfloor (|V_i| - |V(H)|/k)/(k-3) \rfloor$. Indeed, then $n_i = (|V_i| - |V(H)|/k)/(k-3) - x_i$ for some $0 \leqslant x_i < 1$. With $x = x_0 + x_1 + x_2$, we have

$$n_0 + n_1 + n_2 = \big(|V_0| + |V_1| + |V_2| - 3\big|V(H)\big|/k\big)/(k-3) - x_0 - x_1 - x_2 = \big|V(H)\big|/k - x.$$

Since, by definition $0 \leqslant x < 3$, this implies that $|V(H)|/k - 3 \leqslant n_0 + n_1 + n_2 \leqslant |V(H)|/k$, and that

$$|V_i| - (k-3)n_i = \big|V(H)\big|/k + (k-3)x_i = n_0 + n_1 + n_2 + x + (k-3)x_i \geqslant n_0 + n_1 + n_2.$$

Since $k \geqslant 4$, we also have $n_i > (0.9 - 3/k)m/(k-3) > m/4k$. In order to form the cycles in oriented graph K it clearly suffices to find n_i disjoint directed paths of length $k-2$ in V_i. Let P_i be an arbitrary subset of V_i of size precisely $(k-2)n_i$. Ignore the direction of the edges and consider the induced subgraph $H[P_i]$. By assumption (i) of the theorem, the degree of every vertex in this graph is at least $|P_i| - \delta|V_i|$. Since $|P_i| = (k-2)n_i = \Theta(m)$ and $\delta \ll 1/k$, every vertex in $H[P_i]$ has at least $(1 - \frac{1}{k-2})|P_i|$ neighbours. Thus, applying the Hajnal–Szemerédi theorem (mentioned in the introduction) we can find a collection of disjoint cliques of size $k-2$ covering all vertices of P_i. With the directions each of these cliques becomes a tournament (i.e. a complete oriented graph) and it is well known that any tournament has a Hamiltonian path (i.e. a path containing all of its vertices). Given these paths, we may assign each path an arbitrary pair of vertices in V_{i-1} and V_{i+1} to form the required k-cycles in the oriented graph K.

Now we will use the Blow-up Lemma to show that the same strategy works even when the edges from V_i to V_{i+1} no longer form a complete bipartite graph, but do form a super-regular pair. We start by picking randomly disjoint sets $S_{i,j} \subset V_i$ of size $|S_{i,j}| = n_j$ for all $j \neq i \in \{0, 1, 2\}$. For sufficiently large m, by the Chernoff bound for hypergeometric distributions (see inequality (1)), we can assume that $|N^+(v) \cap S_{i+1,i}| > (d-\eta)n_i - m^{2/3}$ and $|N^-(v) \cap S_{i-1,i}| > (d-\eta)n_i - m^{2/3}$ for every $i = 0, 1, 2$ and $v \in V_i$. Next let P_i be an arbitrary subset of $V_i \setminus \bigcup_{j \neq i} S_{i,j}$ of size $(k-2)n_i$. Using the same argument as in the previous paragraph we can find disjoint paths $P_{i,1}, \ldots, P_{i,n_i}$ each of length $k-2$ covering all vertices of P_i. Denote the first and last vertices of $P_{i,j}$ by $x_{i,j}$ and $y_{i,j}$, respectively and let $X_i = \{x_{i,1}, \ldots, x_{i,n_i}\}$, $Y_i = \{y_{i,1}, \ldots, y_{i,n_i}\}$. Note that these sets have size linear in m.

By regularity, there are at most $\eta|V_i|$ vertices $v \in V_i$ with $|N^+(v) \cap X_{i+1}| < (d-2\eta)|X_{i+1}|$ and at most $\eta|V_i|$ vertices $v \in V_i$ with $|N^-(v) \cap Y_{i-1}| < (d-2\eta)|Y_{i-1}|$. Call such vertices *bad*. For $i \neq j \in \{0, 1, 2\}$ define new sets $S'_{i,j}$ and paths $P'_{i,t}$ as follows. Choose sets $B_{i,j} \subseteq S_{i,j}$, $j \neq i$ containing all bad vertices in $S_{i,j}$ with $|B_{i,j}|$ equal to the first number larger than $2\eta|V_i|$ that is divisible by $k-2$.

For every i and $j \neq i$ choose a collection of paths $\{P_{i,\ell}\}, \ell \in C_{i,j}$ containing only good vertices such that the sets of indices $C_{i,j}$, $j \neq i$ are disjoint and have size $|C_{i,j}| = |B_{i,j}|/(k-2)$. Note that this is possible since the number of bad vertices is at most $2\eta m$, the number of paths $P_{i,\ell}$ is $n_i = \Theta(m/k)$ and $\eta \ll 1/k$. For $j \neq i$ remove the paths $\{P_{i,\ell}\}, \ell \in C_{i,j}$ from P_i, adding their vertices to $S_{i,j}$, and replace the vertices lost from P_i with $\bigcup_{j\neq i} B_i^j$. Delete the vertices of $B_{i,j}$ from $S_{i,j}$ and call the new set $S'_{i,j}$. Note that the size of $S'_{i,j}$ is still n_i and it now contains only good vertices. Since there are at least $4\eta|V_i|$ vertices in $\bigcup_{j\neq i} B_{i,j}$ and $\delta \ll \eta$ we can again use the same argument as above to find disjoint paths of length $k-2$ covering all vertices in $\bigcup_{j\neq i} B_{i,j}$. Add these new paths instead of the paths $P_{i,\ell}$, $\ell \in \bigcup_{j\neq i} C_{i,j}$ which were removed and call the new collection of paths $P'_{i,1}, \dots, P'_{i,n_i}$. Also, let $X'_i = \{x'_{i,1}, \dots, x'_{i,n_i}\}$ and $Y'_i = \{y'_{i,1}, \dots, y'_{i,n_i}\}$ be the sets of first and last vertices for the new collection of paths $\{P'_{i,t}\}$.

Now consider three new 3-partite oriented graphs H_0, H_1, H_2 defined as follows. The parts of H_i are $S'_{i-1,i}$, $S'_{i+1,i}$ and an auxiliary set of size n_i, which we may label as $[n_i]$. For any $t \in [n_i]$ the outneighbourhood of t in H_i is $N^+_H(y'_{i,t}) \cap S'_{i+1,i}$ and the inneighbourhood of t in H_i is $N^-(x'_{i,t}) \cap S'_{i-1,i}$. We also include in H_i all the edges from $S'_{i+1,i}$ to $S'_{i-1,i}$ that were present in H. We claim that the underlying graph of each H_i is $(d, 18k\eta)$-super-regular. Note that all three parts of H_i have size $n_i > m/4k$, and therefore any set containing at least $18k\eta n_i$ vertices from one of the parts may be considered as a subset of H of size at least ηm. Thus the regularity condition follows easily from the corresponding η-regularity condition for H. Next we need to check the degree condition in both directions for each of the three directed bipartite graphs $(Y'_i, S'_{i+1,i})$, $(S'_{i+1,i}, S'_{i-1,i})$, $(S'_{i-1,i}, X'_i)$. Recall that the in and out neighbourhoods of all vertices of H in the set $S_{j,i}$ formerly had size at least $(d-\eta)n_i - m^{2/3}$. Since $n_i > m/4k$ and we only swapped $|B_{j,i}| < 2\eta m + k < 9k\eta n_i$ vertices, every vertex still has at least $(d-10k\eta)n_i$ in and out neighbours in $S'_{j,i}$. Also, since vertices in $S'_{i-1,i}$ are good they had at least $(d-2\eta)|X_i|$ outneighbours in X_i. Again, we only removed at most $|\bigcup_{j\neq i} B_{i,j}| \leqslant 2(2\eta m + k) < 17k\eta n_i = 17k\eta|X_i|$ vertices from X_i to create X'_i. Therefore, every vertex in $S'_{i-1,i}$ has at least $(d-18k\eta)|X'_i|$ outneighbours in X'_i. Similar reasoning also shows that every vertex in $S'_{i+1,i}$ has at least $(d-18k\eta)|Y'_i|$ inneighbours in Y'_i. This establishes super-regularity. Now by Corollary 5.2 we can perfectly cover each H_i by vertex-disjoint cyclic triangles. This translates into the required collection of k-cycles in H. $\square$

Combining Theorem 5.3 with a technique similar to that used in [28,18] we can now prove Theorem 5.1.

Proof of Theorem 5.1. Choose constants that satisfy $1/n_0 \ll c \ll 1/M$, $1/M' \ll \varepsilon \ll d \ll \alpha \ll 1/k$. Apply Lemma 4.1 to obtain a partition of the vertices of G into $V_0, V_1, \dots, V_s$ and let R' be the corresponding reduced digraph with parameters (ε, d) on $[s]$. Let R be the reduced oriented graph obtained by applying Lemma 4.2 to R'. Then the minimum indegree and outdegree in R are at least $(1/2 - c - 3\varepsilon - d)s > (1/2 - 2d)s$, so by Theorem 1.1 we can cover all but at most αs vertices in R by vertex-disjoint cyclic triangles $T_1, \dots, T_t$. (Although Theorem 1.1 allows us to cover all but at most 3 vertices of R, we only need this weaker bound which follows immediately from Theorem 2.6.) Let $(V_{i,0}, V_{i,1}, V_{i,2})$ be the three clusters of the regular partition which correspond to the vertices of the cyclic triangle T_i in R. Since the oriented graph R is a subgraph of the digraph R' we have that the edges of G from $V_{i,j}$ to $V_{i,j+1}$ from an ε-regular bipartite subgraph with density at least d. By regularity, there are at most $2\varepsilon|V_{i,j}|$ vertices $v \in V_{i,j}$ such that $|N^+(v) \cap V_{i,j+1}| < (d-\varepsilon)|V_{i,j+1}|$ or $|N^-(v) \cap V_{i,j-1}| < (d-\varepsilon)|V_{i,j-1}|$. We delete sets of size $\lfloor 2\varepsilon|V_{i,j}|\rfloor$ from $V_{i,j}$ that contain all these vertices. Thus we obtain triples $(U_{i,0}, U_{i,1}, U_{i,2})$ for $1 \leqslant i \leqslant t$ in which each $U_{i,j}$ has the same size $m = \Omega(n/M)$, and the edges between $U_{i,j}$ to $U_{i,j+1}$ are all directed from $U_{i,j}$ to $U_{i,j+1}$. Furthermore, since we deleted at most 2ε-proportion of each cluster, it is easy to see that the underlying graph of edges between $U_{i,j}$ to $U_{i,j+1}$ forms a $(2\varepsilon, d/2)$-super-regular pair. Also, since $c \ll 1/M$, there is a constant $\delta \ll \varepsilon$ such that the minimum degree of all induced subgraphs $G[U_{i,j}]$ is at least $|U_{i,j}| - cn \geqslant (1-\delta)|U_{i,j}|$. Let U_0 denote the vertices that do not belong to any triple. Then U_0 contains the exceptional class V_0, which has size at most εn, the classes V_i corresponding to vertices of R not covered by cyclic triangles, which have total size at most αn, and the vertices deleted to construct sets $U_{i,j}$, whose number is at most $2\varepsilon \sum_i |V_i| = 2\varepsilon n$. Therefore $|U_0| \leqslant \varepsilon n + \alpha n + 2\varepsilon n < 2\alpha n$.

Now we partition every set $U_{i,j}$ as $U'_{i,j} \cup U''_{i,j}$ by putting each vertex randomly and independently into either class with probability 1/2. Then, with high probability we have the following properties:

1. $|U'_{i,j}|, |U''_{i,j}| = m/2 \pm m^{2/3}$ for every i, j;
2. every vertex in $U_{i,j}$ has at least $dm/5$ outneighbours in each of $U'_{i,j+1}$, $U''_{i,j+1}$ and inneighbours in each of $U'_{i,j-1}$, $U''_{i,j-1}$;
3. for any vertex x, there are at least $(n/50)^{k-1}$ k-cycles containing x in which all vertices, except possibly x, are in $\bigcup_{i,j} U'_{i,j}$.

The first two properties are simple applications of Chernoff bounds. For the third property we use Azuma's inequality. Fix x and let X be the random variable which counts the number of k-cycles whose all vertices, except possibly x, are in $\bigcup_{i,j} U'_{i,j}$. Since $|U_0| < 2\alpha n$ there are at most $2\alpha n^{k-1}$ such cycles containing x and some vertex from U_0. Therefore, by Lemma 6.2 (proved in the next section) there are at least $(n/10)^{k-1} - 2\alpha n^{k-1}$ k-cycles in which all vertices, except possibly x, are in $\bigcup_{i,j} U'_{i,j}$. This implies that $\mathbb{E}X > 2^{-k+1}((1/10)^{k-1} - 2\alpha)n^{k-1} > (n/25)^{k-1}$. Also, note that X is a c-Lipchitz random variable with $c = n^{k-2}$. Indeed, there are most n^{k-2} k-cycles containing x and any given vertex v. Hence moving v from $U'_{i,j}$ to $U''_{i,j}$ or vice versa can change the value of X by at most n^{k-2}. Now by Azuma's inequality (Theorem 2.4) we have

$$\mathbb{P}\big(X < (n/50)^{k-1}\big) < \mathbb{P}\big(|X - \mathbb{E}X| > (n/50)^{k-1}\big) < 2e^{-(n/50)^{2(k-1)}/2n \cdot n^{2(k-2)}} < e^{-\Omega(n)} \ll 1/n.$$

Next we greedily cover the vertices in U_0 by disjoint k-cycles, so that for each $x \in U_0$ we use k-cycle in which all other vertices are in $\bigcup_{i,j} U'_{i,j}$. We do this in such a way to minimise the maximum number of vertices used in any one of $U'_{i,j}$. When we come to cover some $x \in U_0$, there are at least $(n/50)^{k-1}$ allowable k-cycles (property 3 above). Of these, at most $kn^{k-2}|U_0| \leqslant 2\alpha k n^{k-1}$ intersect a k-cycle that has already been used to cover a vertex that came before x, and at most $\frac{2}{3}(n/50)^{k-1}$ intersect one of the heaviest (with respect to the number of vertices already used) $50^{1-k}n/m$ classes $U'_{i,j}$ (since each $|U'_{i,j}| < 2m/3$). This means we can choose a k-cycle that is disjoint from those already chosen and does not intersect one of the $50^{1-k}n/m$ heaviest classes, and so the number of vertices used in any class will remain bounded by $\frac{k|U_0|}{50^{1-k}n/m} < k50^k \alpha m$.

To finish the proof it is enough to show that when we restrict to the uncovered vertices from each triple $(U_{i,0}, U_{i,1}, U_{i,2})$ we obtain a triple satisfying the hypotheses of Lemma 5.3. To see this recall that $|U_{i,j}| = m$ and $\alpha \ll 1$, so the number of uncovered vertices in each class is at least $m - k50^k\alpha m > 0.9m$. Super-regularity follows from property 2 of the random partition, regularity of all pairs $(U_{i,j}, U_{i,j+1})$ and the fact that we do not touch any vertex from $(U''_{i,0}, U''_{i,1}, U''_{i,2})$. By Lemma 5.3 we can cover all but at most $3k$ vertices in each triple by disjoint k-cycles, so at most $C = 3kt$ vertices remain uncovered. □

6. Covering by prescribed cycles

We have now assembled the two main ingredients for the proof of Theorem 1.2, which we give in the first subsection of this section. We have also done most of the preparation for the proof of Theorem 1.3: we will present a few more lemmas towards this end in the second subsection, and then prove the theorem in the third subsection.

6.1. Proof of Theorem 1.2

Choose M so that Theorem 3.1 applies with $\delta = 1/10$ and then c', C' so that Theorem 5.1 holds with parameters c', C', for all $k \leqslant M$ and n sufficiently large. Set $c = c'/2$, $C = MC'$. Suppose that n is sufficiently large, G is an oriented graph on n vertices with minimum semidegree at least $(1/2-c)n$, and $n_1,\dots,n_t$ are numbers with $\sum_{i=1}^t n_i \leqslant n-C$. Let N_k be the number of the n_i equal to k, for $k \leqslant M$, and let $N_L = \sum_{n_i > M} n_i$. If there is any k such that $N_k < cn/4M^2$ or if $N_L < cn/4$ we can greedily pack the appropriate cycles using the previously mentioned Theorem 8 from [21], which says that an oriented graph on n vertices with minimum semidegree at least $3n/8$ contains cycles of all lengths between 3 and n. After that we will be left with minimum semidegree at least $(1/2-c)n - \sum_{k=3}^M k(cn/4M^2) - cn/4 \geqslant (1/2-c)n - cn/4 - cn/4 = (1/2-3c'/4)n$. Thus we may reduce to the case when each N_k for $k \leqslant M$ is either 0 or at least $cn/4M^2$ and N_L is either 0 or at least $cn/4$. Next we randomly partition the remaining vertices, so that we allocate $kN_k + C'$ vertices for the purpose of embedding k-cycles for each $k \leqslant M$, and N_L vertices for the purpose of embedding all 'long' cycles of length larger than M. Lemma 3.2 implies that there is a choice of partition so that each part has proportional semidegree at least $1/2 - c'$, and then Theorems 5.1 and 3.1 allow us to embed the k-cycles and the long cycles. This completes the proof.

6.2. Absorbing cycles

When we have cycles of different lengths it is also useful to consider the following kind of absorption. We say that a cycle F absorbs a path P (disjoint from F) if $F \cup P$ spans a (non-induced) cycle of length $|F|+|P|$. We present several lemmas in this subsection that culminate in proving the existence of a structure that is absorbing in this sense.

Lemma 6.1. *Suppose $0 < c < 10^{-4}$ and G is an oriented graph on n vertices with minimum indegree and outdegree at least $(1/2-c)n$. Then G has the following properties.*

(1) *Any $A \subset V(G)$ spans at least $e(A) \geqslant |A|(|A|/2 - cn)$ edges.*
(2) *For any (not necessarily disjoint) subsets S, T of $V(G)$ of size at least $(1/2-c)n$ there are at least $n^2/60$ directed edges from S to T.*
(3) *For any (not necessarily disjoint) subsets S, T of $V(G)$ of size at least $(1/2-c)n$ there are at least $10^{-5}n^3$ cyclic triangles that contain an edge from S to T.*

Proof. By deleting vertices if necessary we may assume that $|S| = |T| = (1/2-c)n$.

(1) Since $|N(x)| \geqslant (1-2c)n$ for every vertex x we obtain

$$e(A) = \sum_{x\in A} \bigl|N(x)\cap A\bigr|/2 \geqslant \sum_{x\in A} \bigl(\bigl|N(x)\bigr| + |A| - n\bigr)/2 \geqslant |A|\bigl((1-2c)n + |A| - n\bigr)/2$$
$$= |A|\bigl(|A|/2 - cn\bigr).$$

(2) Suppose first that $|S\cap T| > n/5$. Then, using the estimate from part (1), we get $e(S,T) \geqslant e(S\cap T) \geqslant (n/5)(n/10 - cn) > n^2/60$. Otherwise $|\overline{S\cup T}| \leqslant n - (|S|+|T|-|S\cap T|) \leqslant n - (2(1/2-c)n - n/5) = (1/5+2c)n$. Therefore we can write

$$e(S,\bar S) = \sum_{x\in S} \bigl|N^+(x)\bigr| - e(S) > (1/2-c)n|S| - |S|^2/2 = \bigl((1/2-c)n\bigr)^2/2$$

and

$$e(S,T) > e(S,\bar S) - |S||\overline{S\cup T}| > \bigl((1/2-c)n\bigr)^2/2 - (1/2-c)n(1/5+2c)n > n^2/60.$$

(3) From part (2) there are at least $n^2/60$ edges from S to T. By Lemma 2.2, from any vertex $v \in S$ we have at most $(2a+4c)n$ outgoing edges which are a-bad. Taking $a = 1/300$ we obtain that at most $|S|n/100 \leqslant n^2/200$ of edges from S to T are 1/300-bad. Every 1/300-good edge is contained in at least $n/300$ cyclic triangles, each of which may be counted at most 3 times, so we get at least $(1/60 - 1/200)(1/900)n^3 > 10^{-5}n^3$ suitable triangles. □

Lemma 6.2. *Suppose $0 < c < 10^{-4}$, $k \geqslant 3$, and n is sufficiently large. If G is an oriented graph on n vertices with minimum indegree and outdegree at least $(1/2 - c)n$ then any vertex x of G belongs to at least $(n/10)^{k-1}$ k-cycles. More generally, if $t \geqslant 1$ and $k \geqslant t + 2$ then any path on t vertices belongs to at least $(n/10)^{k-t}$ k-cycles.*

Proof. To construct a k-cycle through x we start by greedily picking a path of $k - 2$ vertices starting at x. When $k = 3$ this is just the point x. For $k \geqslant 4$, note that by the outdegree condition we have at least $(1/2 - c)n - k$ choices at every step, so this gives at least $\prod_{i=0}^{k-4}((1/2 - c)n - k)$ such paths. Given a path P of length $k - 2$, from x to some final point y, we may complete P to a k-cycle by choosing an edge from $N^+(y)$ to $N^-(x)$ which does not use any vertex of P. Clearly there are at most kn edges incident to the vertices on the path P. Hence, by Lemma 6.1, there are at least $n^2/60 - kn$ edges from $N^+(y)$ to $N^-(x)$ disjoint from P. Altogether we get at least $(n^2/60 - kn)\prod_{i=0}^{k-4}((1/2 - c)n - k) > (n/10)^{k-1}$ cycles. The estimate for the number of k-cycles containing a given path of length t can be obtained similarly. □

Lemma 6.3. *Suppose $0 < c < 10^{-4}$, $k \geqslant 3$, $\ell \geqslant k + 3$ and n is sufficiently large. If G is an oriented graph on n vertices with minimum indegree and outdegree at least $(1/2 - c)n$ and P is any path on k vertices in G then there are at least $(n/100)^{\ell-k}$ cycles C of length $\ell - k$ so that $P \cup C$ spans a (non-induced) cycle of length ℓ.*

Proof. Let S be the outneighbourhood of the last vertex of P and T the inneighbourhood of the first vertex of P. Suppose first that $\ell > k + 3$. By part (2) of Lemma 6.1 there are at least $n^2/60$ edges xy with $x \in T$ and $y \in S$. Also, by Lemma 6.2 each such xy is contained in at least $(n/10)^{\ell-k-2}$ cycles of length $\ell - k$. Altogether this gives at least $(n^2/60)(n/10)^{\ell-k-2}$ cycles of length $\ell - k$. Since at most $kn^{\ell-k-1}$ of these cycles intersect the path P, there are at least $(n^2/60)(n/10)^{\ell-k-2} - kn^{\ell-k-1} > (n/10)^{\ell-k}$ cycles of length $\ell - k$ containing an edge from T to S and disjoint from P. Clearly, each such cycle together with P spans a cycle of length ℓ.

Now suppose that $\ell = k+3$. By assertion (3) of Lemma 6.1 there are at least $10^{-5}n^3$ cyclic triangles that contain an edge from T to S. At most kn^2 of these triangles use a point from P, so at least $10^{-5}n^3 - kn^2 > 10^{-6}n^3 = (n/100)^{\ell-k}$ are disjoint from P. These triangles together with P span cycles of length ℓ. □

Now by Lemmas 6.2 and 6.3, the same argument that we used in Lemma 2.8, using simply Chernoff bounds rather than the Kim–Vu inequality, leads to the following lemma.

Lemma 6.4. *For any $k \geqslant 3$ and $M \geqslant k + 1$ there is some $c > 0$ and number n_0 such that if G is an oriented graph on $n > n_0$ vertices with minimum indegree and outdegree at least $(1/2 - c)n$ then the following holds. Suppose we form a collection of vertex-disjoint k-cycles $\mathcal{C}$ by choosing each k-cycle independently with some probability p and deleting any pair of k-cycles that intersect. Write $a_k n^k$ for the number of k-cycles in G (where $a_k > k^{-1}10^{-k+1}$ by Lemma 6.2). If $\frac{\log n}{n^k} \ll p \ll 1/n^{k-1}$ then with high probability we have $|\mathcal{C}| = m \sim a_k pn^k$ and for any path P on $\ell - k$ vertices with $k + 1 \leqslant \ell \leqslant M$ there are at least $200^{-k}m$ absorbing k-cycles for P in $\mathcal{C}$.*

6.3. Proof of Theorem 1.3

Choose constants with the hierarchy $M \ll c^{-1} \ll C \ll T \ll n_0$. By assumption at least T of the n_i lie between $k + 1$ and M. We may relabel so that $k + 1 \leqslant n_i \leqslant M$ for $1 \leqslant i \leqslant T$. Next by Lemma 6.4 we choose a collection $\mathcal{C}$ of $|\mathcal{C}| = T\log n + T$ vertex-disjoint k-cycles such that for any path P in G on $\ell - k$ vertices with $k + 1 \leqslant \ell \leqslant M$ there are at least $200^{-k}m$ absorbing k-cycles for P in $\mathcal{C}$. Let G' be the restriction of G to the vertices not covered by cycles in $\mathcal{C}$. This is a tournament on $n' = n - kT(\log n + 1)$ vertices with minimum semidegree at least $(1/2 - c)n - kT(\log n + 1) > (1/2 - 2c)n'$. Let $n'_1, \dots, n'_{t'}$ be the sequence obtained from $n_1, \dots, n_t$ by removing $n_1, \dots, n_T$ and

$T\log n$ occurrences of k. Note that $\sum_j n'_j = n' - \sum_{i=1}^{T}(n_i - k) < n' - C$. Therefore, we can apply Theorem 1.2 and find a packing of cycles in G' of length $n'_1, \ldots, n'_{t'}$ covering all but a set U of $\sum_{i=1}^{T}(n_i - k)$ vertices. Note that G' restricted to U is a tournament, and so contains a Hamilton path. We can partition this path into T paths with $n_1 - k, \ldots, n_T - k$ vertices. Finally, we can apply the absorbing property of $\mathcal{C}$ to repeatedly combine a path on $n_i - k$ leftover vertices with a k-cycle in $\mathcal{C}$ to form an n_i-cycle, for $1 \leqslant i \leqslant T$. This completes the proof.

7. Concluding remarks

In [9], Cuckler raises the question of counting perfect packings of k-cycles in regular tournaments (when k does not divide n a 'perfect' packing is defined to have size $\lfloor n/k \rfloor$). He conjectures that for odd k the number of such packings is $n!^{(k-1)/k}(2 + o(1))^{-n}$, which is asymptotically the number of perfect k-cycle packings which one expects to have in a random tournament. Somewhat surprisingly, he shows that this is no longer true if k is even. In the same paper Cuckler also gives this estimate for counting triangle packings of size $n/3 - o(n/\log n)$ in regular tournaments. Our proof of Theorem 1.1 can be used to show that any oriented graph G of order n with minimum semidegree $(1/2 + o(1))n$ has $n!^{2/3}(2 + o(1))^{-n}$ triangle packings covering all but at most 3 vertices (which are 'perfect' when n is not divisible by 3). An upper bound on the number of packings follows simply from the well-known fact that the number of cyclic triangles in any oriented graph of order n is at most $(1 + o(1))n^3/24$. For completeness we give the short proof of this fact here. Consider an oriented graph G on n vertices with C cyclic triangles and T transitive triangles, so that $C + T = \binom{n}{3}$. Double-counting directed paths with two edges gives

$$3C + T = \sum_{v \in V(G)} d^+(v)d^-(v) \leqslant \sum_{v \in V(G)} \big(d^+(v) + d^-(v)\big)^2/4 \leqslant n \cdot n^2/4 = n^3/4.$$

Therefore $C \leqslant \frac{1}{2}\big(n^3/4 - \binom{n}{3}\big) = (1 + o(1))n^3/24$. Now when we construct a packing by choosing one triangle every time, it follows that for the ith triangle we have at most $(1 + o(1))(n - 3(i - 1))^3/24$ choices. Dividing by the number of different orderings of the same packing, which is at least $(n/3 - 1)!$, we see that there are at most

$$\frac{(1 + o(1))^n}{(n/3 - 1)!} \prod_{i=1}^{n/3} \frac{(n - 3(i - 1))^3}{24} = \frac{(1 + o(1))^n}{(n/3 - 1)!} 2^{-n} 3^{-n/3} n! = n!^{2/3}\big(2 + o(1)\big)^{-n}$$

different packing of cyclic triangles covering all but at most 3 vertices.

Next we present a sketch proof for the lower bound. Suppose that the nibble consists of m 'bites' of size $b_1, \ldots, b_m$ with each $b_i = o(n)$ and $\sum b_i = n/3 - o(n)$. From the analysis of the nibble in [4] it is easy to see that at iteration i we have an oriented graph on $n - \sum_{j<i} 3b_j$ vertices which has $(1 + o(1))\frac{(n - \sum_{j<i} 3b_j)^3}{24}$ cyclic triangles; then we pick b_i such triangles uniformly at random. Therefore, the number of ordered choices of $n - o(n)$ vertex-disjoint cyclic triangles in G is at least $(1 + o(1))^n \prod_{i=1}^{m}((n - \sum_{j<i} 3b_j)^3/24)^{b_i}$. Since for $b = o(a)$ we have

$$\big(a^3/24\big)^b = \big(1 + o(1)\big)^b 24^{-b} a(a - 1) \cdots (a - 3b + 1),$$

we conclude that the number of ordered choices of $n - o(n)$ vertex-disjoint cyclic triangles in G is at least

$$\big(1 + o(1)\big)^n 24^{-\sum_i b_i} n! = \big(1 + o(1)\big)^n 24^{-n/3} n!.$$

We can use the absorption argument to convert each of these families of $n - o(n)$ vertex-disjoint cyclic triangles into triangle packings that cover all but at most 3 vertices. Note that since the absorption process involves only $o(n)$ vertices, each such packing is obtained in at most $\binom{n/3}{o(n)} = e^{o(n)}$ ways. Note also that for each packing its triangles can have at most $(n/3)!$ different orders. Dividing by this number we obtain that the number of triangle packings covering all but at most 3 vertices in G is at least $(1+o(1))^n e^{-o(n)} 24^{-n/3} n!/(n/3)! = (1/2+o(1))^n n!^{2/3}$, as required.

As discussed in the introduction, it is also natural to consider questions which we study in this paper for digraphs, rather than oriented graphs. It appears that such digraph problems tend to be quite closely connected to known extremal results about graphs, and can therefore be answered using these results. Here we mention two illustrative examples:

1. A digraph G on n vertices with minimum total degree (this is a minimum of $d^+(x)+d^-(x)$ over all $x \in G$) at least $(4/3+c)n$ has a perfect packing of transitive triangles, when $n > n_0(c)$ is divisible by 3 and sufficiently large. To see this consider a random ordering $<$ of the vertices of G and let G' be the graph whose edges are all those edges ij of G with $i < j$. It is easy to check that with high probability every vertex of G' has degree at least $(2/3+c/4)n$. Now the Corrádi–Hajnal theorem (mentioned in the introduction) gives a perfect triangle packing in G', which by definition corresponds to a packing of digraph G by transitive triangles.
2. A digraph G on n vertices with minimum semidegree (which is defined in the introduction) at least $(2/3+c)n$ has a perfect packing of cyclic triangles, when $n > n_0(c)$ is divisible by 3 and sufficiently large. To see this, randomly partition the vertices of G into three disjoint sets $V_0 \cup V_1 \cup V_2$ with $|V_i| = n/3$, $i = 0, 1, 2$. Let G' be the 3-partite graph whose edges are the edges of G which go from V_i to V_{i+1} (addition mod 3). Again, it is easy to check using large deviation inequalities that every vertex of G' in V_i has at least $(2/3+c/2)|V_j|$ neighbours in each V_j, $j \neq i$. A result of Johansson [17] implies that G' has a perfect triangle packing, which by definition gives a cyclic triangle packing in G.

Moreover, one can easily show that both these results are asymptotically best possible by taking corresponding construction for graphs and replacing each edge by two directed edges with opposite directions.

Note that in Theorem 1.1 we find a triangle packing which covers all but three vertices. What happens if we slightly relax this requirement? In particular, what minimum semidegree condition in an oriented graph of order n will give a cyclic triangle packing that covers all but at most $o(n)$ vertices? We have no good conjecture for this problem. The following construction shows that the minimum semidegree should be at least $4n/9 - o(n)$. Suppose $\varepsilon > 0$ and divide a set of n vertices into three sets V_0, V_1, V_2 with sizes $|V_0| = (1/3 - \varepsilon/2)n$, $|V_1| = n/3$, $|V_2| = (1/3+\varepsilon/2)n$. Define an oriented graph G as follows. Between the classes we take all possible edges and orient them from V_i to V_{i+1} (addition mod 3). Inside each class we place an oriented graph that has no cyclic triangle with minimum indegree and outdegree as large as possible. For example, a circulant construction gives such an oriented graph on m vertices with every indegree and outdegree equal to $\lceil m/3 \rceil - 1$, and this cannot be improved if the Caccetta–Häggkvist conjecture [7] is true. Then G has minimum indegree and outdegree at least $(4/9 - \varepsilon)n$. Since any cyclic triangle must use one vertex from each class, any collection of vertex-disjoint cyclic triangles leaves at least εn vertices uncovered.

Another interesting variation is to consider what minimum semidegree condition is needed to find certain structures in a tournament. Here one would expect a smaller value than that needed for oriented graphs. For example, a tournament with minimum semidegree at least $n/4$ contains a Hamilton cycle. This is because such a tournament must be strongly connected, i.e. for any ordered pair of vertices (x, y) there is a directed path from x to y, and such a tournament is Hamiltonian by Camion's theorem [9] (see also [6, p. 16]). On the other hand, we recall that for oriented graphs one needs minimum semidegree at least $3n/8$ to get the same conclusion. It would be interesting to find what minimum semidegree in a tournament of order n will give a cyclic triangle packing that covers all but at most $o(n)$ vertices. Here it is easy to show a lower bound of $n/3$ (note it is again smaller than that obtained above for oriented graphs).

Finally, we remark that it is natural to investigate what assumptions other than minimum semidegree suffice to find certain 1-factors in tournaments. Reid [31] discusses the following problem attributed to Bollobás: given a positive integer m, what is the least integer $g(m)$ such that all but a finite number of $g(m)$-connected tournaments contain a 1-factor with m cycles? (A digraph is k-connected if deleting fewer than k vertices always results in a strongly connected digraph.) Reid [31] shows that

$g(2) = 2$ and $m \leqslant g(m) \leqslant 3m - 4$, but determining $g(m)$ exactly is open for $m \geqslant 3$. Song [36] considers the related problem of determining the least integer $f(m)$ such that all but a finite number of $f(m)$-connected tournaments contain a *prescribed* 1-factor with m cycles, i.e. for any $n_1, \ldots, n_m \geqslant 3$ with $n_1 + \cdots + n_m = n$ (the number of vertices) there is a packing with cycles of lengths $n_1, \ldots, n_m$. Clearly, by definition, $g(m) \leqslant f(m)$. Song [36] shows that $f(2) = 2$ and conjectures that in general $f(m) = g(m)$.

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