European Journal of Combinatorics 30 (2009) 1630–1645

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Two remarks on the Burr–Erdős conjecture

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ARTICLE INFO

Article history:
Available online 5 April 2009

ABSTRACT

The Ramsey number $r(H)$ of a graph H is the minimum positive integer N such that every two-coloring of the edges of the complete graph K_N on N vertices contains a monochromatic copy of H. A graph H is d-degenerate if every subgraph of H has minimum degree at most d. Burr and Erdős in 1975 conjectured that for each positive integer d there is a constant c_d such that $r(H) \leq c_d n$ for every d-degenerate graph H on n vertices. We show that for such graphs $r(H) \leq 2^{c_d\sqrt{\log n}}n$, improving on an earlier bound of Kostochka and Sudakov. We also study Ramsey numbers of random graphs, showing that for d fixed, almost surely the random graph $G(n, d/n)$ has Ramsey number linear in n. For random bipartite graphs, our proof gives nearly tight bounds.

1. Introduction

For a graph H, the *Ramsey number* $r(H)$ is the least positive integer N such that every two-coloring of the edges of complete graph K_N on N vertices contains a monochromatic copy of H. Ramsey's theorem states that $r(H)$ exists for every graph H. A classical result of Erdős and Szekeres, which is a quantitative version of Ramsey's theorem, implies that $r(K_n) \leq 2^{2n}$ for every positive integer n. Erdős showed using probabilistic arguments that $r(K_n) > 2^{n/2}$ for $n > 2$. Over the past sixty years, there has been several improvements on these bounds (see, e.g., [9]). However, despite efforts by various researchers, the constant factors in the above exponents remain the same.

Determining or estimating Ramsey numbers is one of the central problems in combinatorics, see the book *Ramsey theory* [15] for details. Besides the complete graph, the next most classical topic in this area concerns the Ramsey numbers of sparse graphs, i.e., graphs with certain upper bound constraints on the degrees of the vertices. The study of these Ramsey numbers was initiated by Burr and Erdős in 1975, and this topic has since played a central role in graph Ramsey theory.

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0195-6698/$ – see front matter
doi:10.1016/j.ejc.2009.03.004

A graph is *d-degenerate* if every subgraph has a vertex of degree at most d. In 1975, Burr and Erdős [6] conjectured that, for each positive integer d, there is a constant $c(d)$ such that every d-degenerate graph H with n vertices satisfies $r(H) \le c(d)n$. An important special case of this conjecture for bounded degree graphs was proved by Chvátal, Rödl, Szemerédi, and Trotter [8].

Another notion of sparseness was introduced by Chen and Schelp [7]. A graph is *p-arrangeable* if there is an ordering $v_1, \dots, v_n$ of the vertices such that for any vertex v_i, its neighbors to the right of v_i have together at most p neighbors to the left of v_i (including v_i). This is an intermediate notion of sparseness not as strict as bounded degree though not as general as bounded degeneracy. Extending the result of [8], Chen and Schelp proved that there is a constant $c(p)$ such that every p-arrangeable graph H on n vertices has Ramsey number at most $c(p)n$. This gives linear Ramsey numbers for planar graphs and more generally for graphs that can be drawn on a bounded genus surface. This result was later extended by Rödl and Thomas [22], who showed that graphs with no K_p-subdivision are p^8-arrangeable. Recently, Nešetřil and Ossona de Mendez [21] defined the concept of *expansion class* (which is related to the notion of arrangeability) and showed that graphs with bounded expansion have linear Ramsey numbers.

Here we introduce a notion of sparseness that is closely related to arrangeability. The main reason for introducing this notion is that it turns out to be more useful for bounding Ramsey numbers. A graph H is *(d, Δ)-degenerate* if there exists an ordering $v_1, \dots, v_n$ of its vertices such that for each v_i,

1. there are at most d vertices v_j adjacent to v_i with $j < i$, and
2. there are at most Δ subsets $S \subset \{v_1, \dots, v_i\}$ such that $S = N(v_j) \cap \{v_1, \dots, v_i\}$ for some neighbor v_j of v_i with $j > i$, where the *neighborhood* $N(v_j)$ is the set of vertices that are adjacent to v_j.

From the definition, every (d, Δ)-degenerate graph is d-degenerate, and every graph with maximum degree Δ is (Δ, Δ)-degenerate. More interesting but also very simple to show is that every (d, Δ)-degenerate graph is $(\Delta(d-1)+1)$-arrangeable, and every p-arrangeable graph is $(p, 2^{p-1})$-degenerate (see Lemmas 4.1 and 4.2).

While the conjecture of Burr and Erdős is still open, there has been considerable progress on this problem recently. Kostochka and Rödl [18] were the first to prove a polynomial upper bound on the Ramsey numbers of d-degenerate graphs. They showed that $r(H) \le c_d n^2$ for every d-degenerate graph H with n vertices. A nearly linear bound of the form $r(H) \le 2^{c_d (\log n)^{2d/(2d+1)}} n$ was obtained by Kostochka and Sudakov [19]. In [11], the authors proved that $r(H) \le 2^{c_d \sqrt{\log n}} n$ for every bipartite d-degenerate graph H with n vertices. Here we show how to use the techniques developed in [11] to generalize this result to all d-degenerate graphs.

Theorem 1.1. *For each positive integer d there is a constant c_d such that every (d, Δ)-degenerate graph H with order n satisfies $r(H) \le 2^{c_d \sqrt{\log \Delta}}\, n$. In particular, $r(H) \le 2^{c_d \sqrt{\log n}}\, n$ for every d-degenerate graph H on n vertices.*

This result follows from Theorem 2.1, which gives a more general bound on the Ramsey number which also incorporates the chromatic number of the graph H.

We next discuss Ramsey numbers and arrangeability of sparse random graphs. The *random graph $G(n, p)$* is the probability space of labeled graphs on n vertices, where every edge appears independently with probability $p = p(n)$. We say that the random graph possesses a graph property $\mathcal{P}$ *almost surely*, or a.s. for brevity, if the probability that $G(n, p)$ has property $\mathcal{P}$ tends to 1 as n tends to infinity. It is well known and easy to show that if $p = d/n$ with $d > 0$ fixed, then, a.s. $G(n, d/n)$ will have maximum degree $\Theta(\log n / \log\log n)$. Moreover, as shown by Ajtai, Komlós, and Szemerédi [2], for $d > 1$ fixed, a.s. the random graph $G(n, d/n)$ contains a subdivision of K_p with p almost as large as its maximum degree. Therefore one cannot use known results to give a linear bound on the Ramsey number of $G(n, d/n)$. Here we obtain such a bound by proving that the random graph $G(n, d/n)$ a.s. has bounded arrangeability.

Theorem 1.2. *There are constants $c_1 > c_2 > 0$ such that for $d \ge 1$ fixed, a.s. the random graph $G(n, d/n)$ is $c_1 d^2$-arrangeable but not $c_2 d^2$-arrangeable.*

Theorem 1.2 is closely related to a question of Chen and Schelp, who asked to estimate the proportion of d-degenerate graphs which have bounded arrangeability. The following is an immediate corollary of this theorem and the result of Chen and Schelp.

Corollary 1.3. *For each $d \geq 1$, there is a constant c_d such that a.s. the Ramsey number of $G(n, d/n)$ is at most $c_d n$.*

One would naturally like to obtain good bounds on the Ramsey number of $G(n, d/n)$. To accomplish this, we prove a stronger version of Theorem 1.2, showing that for $d \geq 10$ a.s. $G(n, d/n)$ is $(16d, 16d)$-degenerate. We also modify a proof of Graham, Rödl, Ruciński, to prove that there is a constant c such that for every (d, Δ)-degenerate H with n vertices and chromatic number q, $r(H) \leq \left(d^d \Delta\right)^{c \log q} n$. In the other direction, it follows from a result of Graham, Rödl, and Ruciński [13] that a.s. $G(n, d/n)$ has Ramsey number at least $2^{cd} n$ for some absolute positive constant c. From these results we get the following quantitative version of Corollary 1.3.

Theorem 1.4. *There are positive constants c_1, c_2 such that for $d \geq 2$ and n a.s.*

$$2^{c_1 d} n \leq r(G(n, d/n)) \leq 2^{c_2 d \log^2 d} n.$$

In the case of random bipartite graphs, we can obtain nearly tight bounds. In another paper, Graham, Rödl, and Ruciński [14] adapt their proof of a lower bound for Ramsey numbers to work also for random bipartite graphs. The random bipartite graph $G(n, n, p)$ is the probability space of labeled bipartite graphs with n vertices in each class, where each of the n^2 edges appears independently with probability p.

Theorem 1.5. *There are positive constants c_1, c_2 such that for each $d \geq 1$ and n a.s.*

$$2^{c_1 d} n \leq r(G(n, n, d/n)) \leq 2^{c_2 d} n.$$

The rest of this paper is organized as follows. In the next section we present a proof of Theorem 2.1 which implies Theorem 1.1 on the Ramsey number for (d, Δ)-degenerate graphs. In Section 3, we prove another bound on Ramsey numbers for (d, Δ)-degenerate graphs which is sometimes better than Theorem 2.1. In Section 4, we prove results on the random graphs $G(n, d/n)$ and $G(n, n, d/n)$. The last section of this paper contains some concluding remarks. Throughout the paper, we systematically omit floor and ceiling signs whenever they are not crucial for the sake of clarity of presentation. We also do not make any serious attempt to optimize absolute constants in our statements and proofs. All logarithms in this paper are base 2.

2. Proof of Theorem 1.1

The main result of this section is the following general bound on Ramsey numbers.

Theorem 2.1. *There is a constant c such that for $0 < \delta \leq 1$, every (d, Δ)-degenerate graph H with chromatic number q and order n satisfies*

$$r(H) < 2^{cq3^q d/\delta} \Delta^{c\delta} n.$$

Note that a greedy coloring shows that every d-degenerate graph has chromatic number at most $d + 1$. Theorem 1.1 follows from the above theorem by letting $\delta = 1/\sqrt{\log \Delta}$.

The first result that we need is a lemma from [11] whose proof uses a probabilistic argument known as *dependent random choice*. Early versions of this technique were developed in the papers [12,17,24]. Later, variants were discovered and applied to various Ramsey-type problems (see, e.g., [19,3,25,11], and their references). We include the proof here for the sake of completeness. Given a vertex subset T of a graph G, the *common neighborhood* $N(T)$ *of* T is the set of all vertices of G that are adjacent to T, i.e., to *every* vertex in T.

Lemma 2.2. *If $\epsilon > 0$ and $G = (V_1, V_2; E)$ is a bipartite graph with $|V_1| = |V_2| = N$ and at least ϵN^2 edges, then for all positive integers t, x, there is a subset $A \subset V_2$ with $|A| \geq \epsilon^{2t} N/2$ such that for all but at most $2\epsilon^{-2t} \left(\frac{x}{N}\right)^{2t} \binom{N}{t}$ t-sets S in A, we have $|N(S)| \geq x$.*

Proof. Let T be a subset of $2t$ random vertices of V_1, chosen uniformly with repetitions. Set $A = N(T)$, and let X denote the cardinality of $A \subset V_2$. By linearity of expectation and by convexity of $f(z) = z^{2t}$,

$$\mathbb{E}[X] = \sum_{v \in V_2} \left(\frac{|N(v)|}{N}\right)^{2t} = N^{-2t} \sum_{v \in V_2} |N(v)|^{2t} \geq N^{1-2t} \left(\frac{\sum_{v \in V_2} |N(v)|}{N}\right)^{2t} \geq \epsilon^{2t} N.$$

Let Y denote the random variable counting the number of t-sets in A with fewer than x common neighbors. For a given t-set S, the probability that S is a subset of A is $\left(\frac{|N(S)|}{N}\right)^{2t}$. Therefore, we have

$$\mathbb{E}[Y] \leq \binom{N}{t} \left(\frac{x-1}{N}\right)^{2t}.$$

Using linearity of expectation, we have

$$\mathbb{E}\left[X - \frac{\mathbb{E}[X]}{2\mathbb{E}[Y]} Y - \mathbb{E}[X]/2\right] = 0.$$

Therefore, there is a choice of T for which this expression is nonnegative. Then

$$|A| = X \geq \frac{1}{2}\mathbb{E}[X] \geq \frac{1}{2}\epsilon^{2t} N$$

and

$$Y \leq 2X\mathbb{E}[Y]/\mathbb{E}[X] \leq 2N\mathbb{E}[Y]/\mathbb{E}[X] < 2\epsilon^{-2t} \left(\frac{x}{N}\right)^{2t} \binom{N}{t},$$

completing the proof. □

We use this lemma to deduce the following:

Lemma 2.3. *For every 2-edge-coloring of K_N and integers $y \geq t \geq q \geq 2$ there is a color and nested subsets of vertices $A_1 \subset \cdots \subset A_q$ with $|A_1| \geq 2^{-4tq} N$ such that the following holds. For each $i < q$, all but at most $2^{4t^2 q} y^{2t} N^{-t}$ subsets of A_i of size t have at least y common neighbors in A_{i+1} in this color.*

Proof. For $j \in \{0, 1\}$, let G_j denote the graph of color j. Let $B_1 = V(K_N)$. We will pick subsets $B_1 \supset B_2 \supset \cdots \supset B_{2q-2}$ such that for each $i \in [2q-3]$, we have $|B_{i+1}| \geq |B_i|/2^{2t+2}$ and there is a color $c(i) \in \{0, 1\}$ such that there are less than $\left(8y^2/|B_i|\right)^t$ t-sets $S \subset B_{i+1}$ which have less than y common neighbors in B_i in graph $G_{c(i)}$.

Having already picked B_i, we now show how to pick $c(i)$ and B_{i+1}. Arbitrarily partition B_i into two subsets $B_{i,1}$ and $B_{i,2}$ of equal size. Let $c(i)$ denote the densest of the two colors between $B_{i,1}$ and $B_{i,2}$. By Lemma 2.2 with $\epsilon = 1/2$, there is a subset $B_{i+1} \subset B_{i,2} \subset B_i$ with $|B_{i+1}| \geq 2^{-2t-1}|B_{i,2}| = 2^{-2t-2}|B_i|$ such that for all but at most

$$2 \cdot 2^{2t} \left(\frac{y}{|B_{i,2}|}\right)^{2t} \binom{|B_{i,2}|}{t} \leq 2^{3t} y^{2t} |B_i|^{-t} = \left(8y^2/|B_i|\right)^t$$

t-sets $S \subset B_{i+1}$, S has at least y common neighbors in B_i in graph $G_{c(i)}$.

We have completed the part of the proof where we constructed the nested subsets $B_1 \supset \cdots \supset B_{2q-2}$ and the colors $c(1), \ldots, c(2q-3)$. Notice that $|B_{2q-2}| \geq 2^{-(2t+2)(2q-3)} N \geq 2^{-4tq+6} N$. So for all but at most

$$\left(8y^2/|B_i|\right)^t \leq \left(2^{4tq-3} y^2/N\right)^t \leq 2^{4t^2 q} y^{2t} N^{-t}$$

t-sets $S \subset B_{i+1}$, S has at least y common neighbors in B_i by color $c(i)$. Since the sets $B_1 \supset \cdots \supset B_{2q-2}$ are nested, this also implies that for all but at most $2^{4t^2q}y^{2t}N^{-t}$ t-sets $S \subset B_{i+1}$, S has at least y common neighbors in B_j in graph $G_{c(i)}$ for each $j \le i$.

By the pigeonhole principle, one of the two colors is represented at least $q-1$ times in the sequence $c(1), \ldots, c(2q-3)$. We suppose without loss of generality that 0 is this popular color. Let $A_q = B_1$, i_j denote the jth smallest positive integer such that $c(i_j) = 0$, and $A_{q-j} = B_{i_j+1}$ for $1 \le j \le q-1$. By the above discussion, it follows that $A_1 \subset \cdots \subset A_q$, $|A_1| \ge |B_{2q-2}| \ge 2^{-4t^2q}N$, and, for each positive integer $i < q$, all but at most $2^{4t^2q}y^{2t}N^{-t}$ subsets of A_i of size t are adjacent to at least y vertices in A_{i+1} in graph G_0. This completes the proof. □

The previous lemma shows that in every 2-edge-coloring of the complete graph K_N there is a monochromatic subgraph G and large nested vertex subsets $A_1 \subset \cdots \subset A_q$ such that almost every t-set in A_i has a large common neighborhood in A_{i+1} in graph G. The next lemma is the most technical part of the proof. It says that if a graph G has such vertex subsets $A_1 \subset \cdots \subset A_q$, then for $1 \le i \le q$ there are large subsets $V_i \subset A_i$ such that almost every d-set in $\bigcup_{\ell \ne i} V_\ell$ has a large common neighborhood in V_i. These vertex subsets $V_1, \ldots, V_q$ will be used to show that G contains all (d, Δ)-degenerate graphs on n vertices with chromatic number at most q.

Lemma 2.4. *Let $d, q, \Delta \ge 2$ be integers and $0 < \delta \le 1$. Let $t = (3^q - 1)d/\delta + d$, $y = 2^{-5qt}\Delta^{-\delta}N$, and $x = y^4N^{-3}$. Suppose $x \ge 2t$. Let $G = (V, E)$ be a graph with nested vertex subsets $A_1 \subset \cdots \subset A_q$ with $|A_1| \ge 2^{-4tq}N$ such that for each i, all but at most $2^{4t^2q}y^{2t}N^{-t}$ subsets of A_i of size t have at least y common neighbors in A_{i+1}. Then there are vertex subsets $V_i \subset A_i$ for $1 \le i \le q$ such that $|V_i| \ge x$ and the number of d-sets in $\bigcup_{\ell \ne i} V_\ell$ with fewer than x common neighbors in V_i is less than $(2\Delta)^{-d}\binom{x}{d}$.*

Proof. We will first pick some constants. Let $r_0 = t$ and for $1 \le j \le q$, let $t_j = 2 \cdot 3^{q-j}d/\delta$ and $r_j = r_{j-1} - t_j = (3^{q-j} - 1)d/\delta + d$. In particular, we have $r_q = d$ and $t_j \ge 2r_j$.

Let

$$b_{i,i} = 2q\,(2x/y)^{t_i}\binom{N}{r_i} \quad \text{and} \quad b_{i,j} = 2q\left(\frac{r_{j-1}}{y}\right)^{t_j} b_{i,j-1} \quad \text{for } i < j.$$

Let

$$c_0 = 2^{-t^2q}y^t \quad \text{and} \quad c_j = 2q\left(\frac{r_{j-1}}{y}\right)^{t_j} c_{j-1}.$$

By the hypothesis of the lemma, we have nested subsets $A_1 \subset \cdots \subset A_q$ with $|A_1| \ge 2^{-4tq}N$ such that for each i, all but at most $2^{4t^2q}y^{2t}N^{-t} \le 2^{-t^2q}y^t = c_0$ subsets of A_i of size t are adjacent to at least y vertices in A_{i+1}. Let $A_{i,0} = A_i$ for each i. We will prove by induction on j that there are subsets $A_{1,j}, \ldots, A_{q,j}$ for $1 \le j \le q$ that satisfy the following properties.

1. For $1 \le i, j \le q$, $A_{i,j} \subset A_{i,j-1}$.
2. For $0 \le j < \ell < i \le q$, $A_{\ell,j} \subset A_{i,j}$.
3. $|A_{i,j}| \ge y$ for all $i > j$ and $|A_{i,j}| \ge 2x - t_i$ for $i \le j$.
4. For each $i \le j$, the number of r_j-sets in $\bigcup_{\ell \ne i} A_{\ell,j}$ that have less than $2x - t_i$ common neighbors in $A_{i,j}$ is at most $b_{i,j}$.
5. For $j < i < q$, the number of r_j-sets in $A_{i,j}$ with less than y common neighbors in $A_{i+1,j}$ is at most c_j.

It is easy to see that the desired properties hold for $j = 0$. Assume that we have already found the subsets $A_{i,j-1}$ for $1 \le i \le q$. We now show how to pick the subsets $A_{i,j}$. Pick a subset S_j of $A_{j,j-1}$ of size t_j uniformly at random. We will let $A_{i,j} = A_{i,j-1} \cap N(S_j)$ for $i \ne j$ and $A_{j,j} = A_{j,j-1} \setminus S_j$.

Let X_j be the random variable that counts the number of r_j-sets in $\bigcup_{\ell\neq j} A_{\ell,j}$ with at most $2x - t_j$ common neighbors in $A_{j,j}$. Equivalently, X_j is the number of r_j-sets in $\bigcup_{\ell\neq j} A_{\ell,j}$ with at most $2x$ common neighbors in $A_{j,j-1}$. The number of r_j-sets in $\bigcup_{\ell\neq j} A_{\ell,j-1}$ is at most $\binom{N}{r_j}$, and the probability that a given r_j-set $R \subset \bigcup_{\ell\neq j} A_{\ell,j-1}$ with at most $2x$ common neighbors in $A_{j,j-1}$ is also contained in $\bigcup_{\ell\neq j} A_{\ell,j}$ (which corresponds to $S_j \subset N(R)$) is at most $\binom{2x}{t_j} / \binom{|A_{j,j-1}|}{t_j}$. By linearity of expectation, we have

$$\mathbb{E}[X_j] \le \frac{\binom{2x}{t_j}}{\binom{|A_{j,j-1}|}{t_j}} \binom{N}{r_j} \le \left(\frac{2x}{y}\right)^{t_j} \binom{N}{r_j} = b_{j,j}/2q.$$

For $i < j$, let $Y_{i,j}$ be the random variable that counts the number of r_{j-1}-sets containing S_j in $\bigcup_{\ell\neq i} A_{\ell,j-1}$ with less than $2x - t_i$ common neighbors in $A_{i,j-1}$. Since the number of r_{j-1}-sets in $\bigcup_{\ell\neq i} A_{\ell,j-1}$ that have less than $2x - t_i$ common neighbors in $A_{i,j-1}$ is at most $b_{i,j-1}$, then

$$\mathbb{E}[Y_{i,j}] \le \frac{\binom{r_{j-1}}{t_j}}{\binom{|A_{j,j-1}|}{t_j}} b_{i,j-1} \le \left(\frac{r_{j-1}}{y}\right)^{t_j} b_{i,j-1} = b_{i,j}/2q.$$

Note that $A_{\ell,j}$ is disjoint from S_j for each ℓ. So if T is a subset of $\bigcup_{\ell\neq i} A_{\ell,j}$ with cardinality r_j, then T is disjoint from S_j and so $T \cup S_j$ is a subset of $\bigcup_{\ell\neq i} A_{\ell,j-1}$ with cardinality $r_j + t_j = r_{j-1}$ satisfying

$$|N(T \cup S_j) \cap A_{i,j-1}| = |N(T) \cap N(S_j) \cap A_{i,j-1}| = |N(T) \cap A_{i,j}|.$$

Hence, $Y_{i,j}$ is also an upper bound on the number of r_j-sets in $\bigcup_{\ell\neq i} A_{\ell,j}$ with less than $2x - t_i$ common neighbors in $A_{i,j}$.

For $j < i < q$, let $Z_{i,j}$ be the random variable that counts the number of r_{j-1}-sets in $A_{i,j-1}$ that contain S_j and have less than y common neighbors in $A_{i+1,j-1}$. Since c_{j-1} is an upper bound on the number of r_{j-1}-sets in $A_{i,j-1}$ that have less than y common neighbors in $A_{i+1,j-1}$ and $S_j \subset A_{j,j-1} \subset A_{i,j-1}$, then

$$\mathbb{E}[Z_{i,j}] \le \frac{\binom{r_{j-1}}{t_j}}{\binom{|A_{j,j-1}|}{t_j}} c_{j-1} \le \left(\frac{r_{j-1}}{y}\right)^{t_j} c_{j-1} = c_j/2q.$$

Since $A_{i,j}$ is disjoint from S_j, if $T \subset A_{i,j}$ has cardinality r_j, then $T \cup S_j \subset A_{i,j-1}$ has cardinality $r_j + t_j = r_{j-1}$ and satisfies

$$|N(T \cup S_j) \cap A_{i+1,j-1}| = |N(T) \cap N(S_j) \cap A_{i+1,j-1}| = |N(T) \cap A_{i+1,j}|.$$

Hence, $Z_{i,j}$ is an upper bound on the number of subsets of $A_{i,j}$ of size r_j with less than y common neighbors in $A_{i+1,j}$.

For $i < j$, let $F_{i,j}$ be the event that every r_{j-1}-set in $A_{j,j-1}$ containing S_j has less than $2x - t_i$ common neighbors in $A_{i,j-1}$. The number of r_{j-1}-sets in $A_{j,j-1}$ with less than $2x - t_i$ common neighbors in $A_{i,j-1}$ is at most $b_{i,j-1}$. The number of subsets of $A_{j,j-1}$ of size r_{j-1} containing a fixed subset of size t_j is $\binom{|A_{j,j-1}|-t_j}{r_{j-1}-t_j}$ and there are $\binom{r_{j-1}}{t_j}$ subsets of size t_j in an r_{j-1}-set. Hence, the number of t_j-sets in $A_{j,j-1}$ for which every r_{j-1}-set in $A_{j,j-1}$ containing it has less than $2x - t_i$ common neighbors in $A_{i,j-1}$ is at most

$$\frac{\binom{r_{j-1}}{t_j}}{\binom{|A_{j,j-1}|-t_j}{r_{j-1}-t_j}} b_{i,j-1}.$$

Since there are a total of $\binom{|A_{j,j-1}|}{t_j}$ possible t_j-sets in $A_{j,j-1}$ that can be picked for S_j and $|A_{j,j-1}| \geq y$, then the probability of event $F_{i,j}$ is at most

$$\begin{aligned}
\frac{\binom{r_{j-1}}{t_j}}{\binom{|A_{j,j-1}|-t_j}{r_{j-1}-t_j}} b_{i,j-1} \binom{|A_{j,j-1}|}{t_j}^{-1} &\leq \frac{r_{j-1}^{t_j}}{t_j!} \cdot \frac{(r_{j-1}-t_j)!}{(y/2)^{r_{j-1}-t_j}} \cdot \frac{t_j!}{(y/2)^{t_j}} \cdot b_{i,j-1} = 2^{r_{j-1}} r_{j-1}^{t_j} r_j! y^{-r_{j-1}} b_{i,j-1} \\
&\leq t^{t_i} y^{-r_{j-1}} b_{i,j-1} = t^{t_i} y^{-r_{j-1}} b_{i,i} \prod_{i \leq \ell < j-1} \frac{b_{i,\ell+1}}{b_{i,\ell}} \\
&= t^{t_i} y^{-r_{j-1}} 2q\,(2x/y)^{t_i} \binom{N}{r_i} \prod_{i \leq \ell < j-1} 2q \left(\frac{r_\ell}{y}\right)^{t_{\ell+1}} \\
&\leq t^{t_i} y^{-r_{j-1}} (2q)^q\,(2x/y)^{t_i}\, N^{r_i} \left(\frac{r_i}{y}\right)^{t_{i+1}+\cdots+t_{j-1}} \\
&= (2t)^{t_i} (2q)^q\,(x/y)^{t_i}\,(N/y)^{r_i}\, r_i^{r_i-r_{j-1}} < t^{3t_i} (2q)^q\,(x/y)^{t_i}\,(N/y)^{r_i} \\
&= t^{3t_i} (2q)^q\,(N/y)^{r_i-3t_i} \leq t^{3t_i} (2q)^q \left(2^{5tq}\right)^{-5t_i/2} < \frac{1}{2q}.
\end{aligned}$$

Here we used $t = r_0 \geq r_1 \cdots \geq r_q$, $r_{j-1} = t_j + r_j$ and $t_j \geq 2r_j$ for $1 \leq j \leq q$, and the inequality $2^{r_{j-1}} r_{j-1}^{t_j} r_j! \leq t^{t_i}$. This inequality can be obtained as follows

$$2^{r_{j-1}} r_{j-1}^{t_j} r_j! \leq 2^{r_{j-1}} r_{j-1}^{t_j} r_j^{r_j} \leq 2^{r_{j-1}} r_{j-1}^{t_j+r_j} = (2r_{j-1})^{r_{j-1}} \leq t_{j-1}^{r_{j-1}} \leq t^{t_i}.$$

If there is an r_{j-1}-set T in $A_{j,j-1}$ containing S_j with at least $2x - t_i$ common neighbors in $A_{i,j-1}$, then S_j has at least $2x - t_i$ common neighbors in $A_{i,j-1}$ and $|A_{i,j}| = |A_{i,j-1} \cap N(S_j)| \geq 2x - t_i$. Therefore, if $|A_{i,j}| < 2x - t_i$, then event $F_{i,j}$ occurs.

Let G_j be the event that every r_{j-1}-set in $A_{j,j-1}$ containing S_j has less than y common neighbors in $A_{j+1,j-1}$. The number of r_{j-1}-sets in $A_{j,j-1}$ that have less than y common neighbors in $A_{j+1,j-1}$ is at most c_{j-1}. The number of subsets of $A_{j,j-1}$ of size r_{j-1} containing a fixed set of size t_j is $\binom{|A_{j,j-1}|-t_j}{r_{j-1}-t_j}$ and there are $\binom{r_{j-1}}{t_j}$ subsets of size t_j in an r_{j-1}-set. Hence, the number of t_j-sets in $A_{j,j-1}$ for which every r_{j-1}-set in $A_{j,j-1}$ containing it has less than y common neighbors in $A_{j+1,j-1}$ is at most

$$\frac{\binom{r_{j-1}}{t_j}}{\binom{|A_{j,j-1}|-t_j}{r_{j-1}-t_j}} c_{j-1}.$$

Since there are a total of $\binom{|A_{j,j-1}|}{t_j}$ possible t_j-sets in $A_{j,j-1}$ that can be picked for S_j and $|A_{j,j-1}| \geq y$, then the probability of event $G_{i,j}$ is at most

$$\begin{aligned}
\frac{\binom{r_{j-1}}{t_j}}{\binom{|A_{j,j-1}|-t_j}{r_{j-1}-t_j}} c_{j-1} \binom{|A_{j,j-1}|}{t_j}^{-1} &\leq \frac{r_{j-1}^{t_j}}{t_j!} \cdot \frac{(r_{j-1}-t_j)!}{(y/2)^{r_{j-1}-t_j}} \cdot \frac{t_j!}{(y/2)^{t_j}} \cdot c_{j-1} = 2^{r_{j-1}} r_{j-1}^{t_j} r_j! y^{-r_{j-1}} c_{j-1} \\
&\leq t^{2t} y^{-r_{j-1}} c_{j-1} = t^{2t} y^{-r_{j-1}} c_0 \prod_{\ell=1}^{j-1} \frac{c_\ell}{c_{\ell-1}} \\
&= t^{2t} y^{-r_{j-1}} 2^{-t^2 q} y^t \prod_{\ell=1}^{j-1} 2q \left(\frac{r_{\ell-1}}{y}\right)^{t_\ell} = t^{2t} 2^{-t^2 q} (2q)^{j-1} \prod_{\ell=1}^{j-1} (r_{\ell-1})^{t_\ell} \\
&< t^{2t} 2^{-t^2 q} (2q)^{j-1} t^t < t^{4t} 2^{-t^2 q} < \frac{1}{2q},
\end{aligned}$$

where we used $t = r_0 \geq r_1 \cdots \geq r_q$ and $r_{j-1} = t_j + r_j$.

If there is an r_{j-1}-set T in $A_{j,j-1}$ containing S_j with at least y common neighbors in $A_{j+1,j-1}$, then S_j has at least y common neighbors in $A_{j+1,j-1}$ and $|A_{j+1,j}| = |A_{j+1,j-1} \cap N(S_j)| \geq y$. Therefore, if $|A_{j+1,j}| < y$, then event G_j occurs.

Note that each of the discrete random variables $X_j, Y_{i,j}, Z_{i,j}$ are nonnegative. Markov's inequality for nonnegative random variables says that if X is a nonnegative discrete random variable and $c \geq 1$, then the probability that $X > c\mathbb{E}[X]$ is less than $\frac{1}{c}$. So each of the following five types of events have probability less than $\frac{1}{2q}$ of occurring:

1. $F_{i,j}$ with $i < j$,
2. G_j,
3. $X_j > 2q\mathbb{E}[X_j]$,
4. $Y_{i,j} > 2q\mathbb{E}[Y_{i,j}]$ with $i < j$,
5. $Z_{i,j} > 2q\mathbb{E}[Z_{i,j}]$ with $j < i$.

Since there are a total of $j - 1 + 1 + 1 + j - 1 + q - j = q + j \leq 2q$ events of the above five types, there is a positive probability that none of these events occur. Hence, there is a choice of S_j for which none of the events $F_{i,j}$ occur, the event G_j does not occur,

$$X_j \leq 2q\mathbb{E}[X_j] \leq b_{j,j}, \qquad Y_{i,j} \leq 2q\mathbb{E}[Y_{i,j}] \leq b_{j,j} \quad \text{for } i < j, \quad \text{and} \quad Z_{i,j} \leq 2q\mathbb{E}[Z_{i,j}] \leq c_j \quad \text{for } j < i.$$

Recall that $A_{i,j} = A_{i,j-1} \cap N(S_j)$ if $i \neq j$ and $A_{j,j} = A_{j,j-1} \setminus S_j$. Hence $A_{i,j} \subset A_{i,j-1}$ for $1 \leq i, j \leq q$, which is the first of the five desired properties. By the induction hypothesis, $A_{\ell,j-1} \subset A_{i,j-1}$ for $j - 1 < \ell < i$, and so for $j < \ell < i$, $A_{\ell,j} = A_{\ell,j-1} \cap N(S_j) \subset A_{i,j-1} \cap N(S_j) = A_{i,j}$, which is the second of the desired properties follows.

Note that $|A_{i,j}| \geq 2x - t_i$ for $i < j$ since $F_{i,j}$ does not occur, and $|A_{j,j}| = |A_{j,j-1}| - t_j \geq y - t_j \geq 2x - t_j$. For $i > j$, since $A_{j+1,j} \subset \cdots \subset A_{q,j}$ and event G_j does not occur, then $|A_{i,j}| \geq |A_{j+1,j}| \geq y$. This demonstrates the third of the five desired properties. The upper bounds on X_j and the $Y_{i,j}$ show the fourth desired property and the upper bound on the $Z_{i,j}$ shows the fifth desired property. Hence, by induction on j, the $A_{i,j}$ have the desired properties.

We let $V_i = A_{i,q}$ for $1 \leq i \leq q$. For each i, we have $|V_i| \geq 2x - t_i \geq x$ and all but less than $b_{i,q}$ d-sets in $\bigcup_{\ell \neq i} V_\ell$ have at least x common neighbors in V_i. To complete the proof, it suffices to show that $b_{i,q} < (2\Delta)^{-d}\binom{x}{d}$. Using $t = r_0 \geq r_1 \ldots \geq r_q = d$, $r_{i-1} = t_i + r_i$ and $t_i \geq 2r_i$ for $1 \leq i \leq q$, we have

$$\begin{aligned}
b_{i,q} &= b_{i,i} \prod_{i \leq \ell \leq q-1} \frac{b_{i,\ell+1}}{b_{i,\ell}} = 2q\,(2x/y)^{t_i} \binom{N}{r_i} \prod_{i \leq \ell \leq q-1} 2q \left(\frac{r_\ell}{y}\right)^{t_{\ell+1}} \\
&\leq (2q)^q\,(2x/y)^{t_i}\,N^{r_i} \left(\frac{r_i}{y}\right)^{t_{i+1}+\cdots+t_q} \\
&= (2q)^q\,(2x)^{t_i}\,y^{r_q - r_{i-1}} N^{r_i} r_i^{t_{i+1}+\cdots+t_q} = (2q)^q 2^{t_i} r_i^{r_i - d} x^{t_i} y^{d - r_{i-1}} N^{r_i} \leq 2^{qtt_i} x^{t_i} y^{d - r_{i-1}} N^{r_i} \\
&= 2^{qtt_i} x^d \left(y^4 N^{-3}\right)^{t_i - d} y^{d - r_{i-1}} N^{r_i} = 2^{qtt_i} x^d (y/N)^{3t_i - 3d - r_i} = 2^{qtt_i} x^d (2^{-5qt}\Delta^{-\delta})^{3t_i - 3d - r_i} \\
&\leq 2^{qtt_i} x^d 2^{-5qtt_i} \Delta^{-2d} = 2^{-4qtt_i} \Delta^{-2d} x^d < (2\Delta)^{-d} \binom{x}{d}. \quad \square
\end{aligned}$$

We now present the proof of Theorem 2.1. Given a 2-coloring of the edges of K_N with $N \geq 2^{25q3^q d/\delta} \Delta^{4\delta} n$ we must show that it contains a monochromatic copy of every (d, Δ)-degenerate graph H with n vertices and chromatic number q. Using Lemmas 2.3 and 2.4 (with $y = 2^{-5tq}\Delta^{-\delta}N$ and $t \leq 3^q d/\delta$), we can find vertex subsets $V_1, \ldots, V_q$, each of cardinality at least

$$x = (y/N)^4 N = (2^{-5qt}\Delta^{-\delta})^4 N \geq 2^{5qt} n > 4n,$$

and a monochromatic subgraph G of the 2-edge-coloring such that for each i the number of d-sets $S \subset \bigcup_{\ell \neq i} V_\ell$ with fewer than x common neighbors in V_i is less than $(2\Delta)^{-d}\binom{x}{d}$. Then the following embedding lemma shows that G contains a copy of H and completes the proof.

Lemma 2.5. *Let H be a (d, Δ)-degenerate graph with n vertices and chromatic number q. Let G be a graph with vertex subsets $V_1, \ldots, V_q$ such that for each i, $|V_i| \geq x \geq 4n$ and the number of d-sets $S \subset \bigcup_{\ell \neq i} V_\ell$ with fewer than x common neighbors in V_i is less than $(2\Delta)^{-d}\binom{x}{d}$. Then G contains a copy of H.*

Proof. Call a d-set $S \subset \bigcup_{\ell \neq i} V_\ell$ *good with respect to i* if $|N(S) \cap V_i| \geq x$, otherwise it is *bad with respect to i*. Also, a subset $S \subset \bigcup_{\ell \neq i} V_\ell$ with $|S| < d$ is *good with respect to i* if it is contained in less than $(2\Delta)^{|S|-d}\binom{x}{d-|S|}$ d-sets in $\bigcup_{\ell \neq i} V_\ell$ which are bad with respect to i. A vertex $v \in V_k$ with $k \neq i$ is *bad with respect to i and a subset* $S \subset \bigcup_{\ell \neq i} V_\ell$ with $|S| < d$ if S is good with respect to i but $S \cup \{v\}$ is not. Note that, for any subset $S \subset \bigcup_{\ell \neq i} V_\ell$ with $|S| < d$ that is good with respect to i, there are at most $\frac{x}{2\Delta}$ vertices that are bad with respect to S and i. Indeed, if not, then there would be more than

$$\frac{x/(2\Delta)}{d-|S|}(2\Delta)^{|S|+1-d}\binom{x}{d-|S|-1} \geq (2\Delta)^{|S|-d}\binom{x}{d-|S|}$$

subsets of $\bigcup_{\ell \neq i} V_\ell$ of size d containing S that are bad with respect to i, which would contradict S being good with respect to i.

Since H is (d, Δ)-degenerate, it has an ordering of its vertices $\{v_1, \ldots, v_n\}$ such that each vertex v_k has at most d neighbors v_ℓ with $\ell < k$ and there are at most Δ subsets $S \subset \{v_1, \ldots, v_k\}$ such that $S = N(v_j) \cap \{v_1, \ldots, v_k\}$ for some neighbor v_j of v_k with $j > k$. Since H has chromatic number q, there is a partition $U_1 \cup \cdots \cup U_q$ of the vertex set of H such that each U_i is an independent set. For $1 \leq j \leq n$, let $r(j)$ denote the index r of the independent set U_r containing vertex v_j. Let $N^-(v_k)$ be all the neighbors v_ℓ of v_k with $\ell < k$. Let $L_h = \{v_1, \ldots, v_h\}$. An *embedding* of a graph H in a graph G is an injective mapping $f : V(H) \to V(G)$ such that $(f(v_j), f(v_k))$ is an edge of G if (v_j, v_k) is an edge of H. In other words, an embedding f demonstrates that H is a subgraph of G. We will use induction on h to find an embedding f of H in G such that for $1 \leq i \leq q, f(U_i) \subset V_i$ and for every vertex v_k and every $h \in [n]$, the set $f(N^-(v_k) \cap L_h)$ is good with respect to $r(k)$.

By our definition, the empty set is good with respect to each i, $1 \leq i \leq q$. We will embed the vertices of H by increasing order of their indices. Suppose we are embedding v_h. Then, by the induction hypothesis, for each vertex v_k, the set $f(N^-(v_k) \cap L_{h-1})$ is good with respect to $r(k)$. Since the set $f(N^-(v_h) \cap L_{h-1}) = f(N^-(v_h))$ is good with respect to $r(h)$, it has at least x common neighbors in $V_{r(h)}$. Also, there are at most Δ subsets $S \subset L_h$ for which there is a neighbor v_j of v_h with $j > h$ such that $L_h \cap N(v_j) = S$. So there are at most Δ sets $f(N^-(v_j) \cap L_{h-1})$ where v_j is a neighbor of v_h and $j > h$. Note that $r(j) \neq r(h)$ for v_j a neighbor of v_h with $j > h$. By the induction hypothesis each such set $f(N^-(v_j) \cap L_{h-1})$ is good with respect to $r(j)$, so there are at most $\Delta\frac{x}{2\Delta} = x/2$ vertices in $V_{r(h)}$ which are bad with respect to at least one of the pairs $f(N^-(v_j) \cap L_{h-1})$ and $r(j)$. This implies that there are at least $x - x/2 - (h-1) > x/4$ vertices in $V_{r(h)}$ in the common neighborhood of $f(N^-(v_h))$ which are not occupied yet and are good with respect to all of the above pairs $f(N^-(v_j) \cap L_{h-1})$ and $r(j)$. Any of these vertices can be chosen as $f(v_h)$. When the induction is complete, $f(v_h)$ is adjacent to $f(N^-(v_h))$ for every vertex v_h of H. Hence, the mapping f provides an embedding of H as a subgraph of G. □

3. Another bound for Ramsey numbers

The following theorem is a generalization of a bound by Graham, Rödl, and Ruciński [13] on Ramsey numbers for graphs of bounded maximum degree. The proof is a minor variation of their proof.

Theorem 3.1. *The Ramsey number of every (d, Δ)-degenerate H with n vertices and chromatic number q satisfies $r(H) \leq \left(2^{7d+8}d^{3d+2}\Delta\right)^{\log q} n$.*

We will need the following lemma. The *edge density* between a pair of vertex subsets W_1, W_2 of a graph G is the fraction of pairs $(w_1, w_2) \in W_1 \times W_2$ that are edges of G.

Lemma 3.2. *Let $\epsilon > 0$ and H be a (d, Δ)-degenerate graph with n vertices and chromatic number q. Let G be a graph on $N \geq 4\epsilon^{-d}qn$ vertices. If every pair of disjoint subsets $W_1, W_2 \subset V(G)$ each with cardinality at least $\frac{1}{2}\epsilon^d\Delta^{-1}q^{-2}N$ has edge density at least ϵ between them, then G contains H as a subgraph.*

Proof. Let $v_1, \ldots, v_n$ be a (d, Δ)-degenerate ordering for H. Let $L_j = \{v_1, \ldots, v_j\}$. Let $V(H) = U_1 \cup \cdots \cup U_q$ be a partition of the vertex set of H into q color classes which are independent sets. For $i > j$, let $N(i,j) = N(v_i) \cap L_j$ denote the set of neighbors v_h of v_i with $h \le j$ and $d_{i,j} = |N(i,j)|$. Arbitrarily partition $V(G) = V_1 \cup \cdots \cup V_q$ into q subsets of size N/q. We will find an embedding $f : V(H) \to V(G)$ of H such that if $v_i \in U_k$, then $f(v_i) \in V_k$. For $1 \le i \le n$ and $v_i \in U_k$, let $T_{i,0} = V_k$. We will embed the vertices $v_1, \ldots, v_n$ of H one by one in increasing order. We will prove by induction on j that at the end of step j, we will have vertices $f(v_1), \ldots, f(v_j)$ and sets $T_{i,j}$ for $i > j$ such that

$$|T_{i,j}| \ge \epsilon^{d_{i,j}} |T_{i,0}| = \epsilon^{d_{i,j}} N/q \ge \epsilon^d N/q$$

and the following holds. For $h, \ell \le j$, $(f(v_h), f(v_\ell))$ is an edge of G if (v_h, v_ℓ) is an edge of H, and for $i > j$ and $v_i \in U_k$, $T_{i,j}$ is the subset of V_k consisting of those vertices adjacent to $f(v_p)$ for every vertex $v_p \in N(i,j)$. In particular, we have that if $i, i' > j$ are such that $v_i, v_{i'}$ lie in the same independent set U_k and $N(i,j) = N(i',j)$, then $T_{i,j} = T_{i',j}$. Note that any vertex in $T_{i,j} \setminus \{f(v_1), \ldots, f(v_j)\}$ can be used to embed v_i.

The base case $j = 0$ for the induction clearly holds. Our induction hypothesis is that we have vertices $f(v_1), \ldots, f(v_{j-1})$ and sets $T_{i,j-1}$ for $i > j-1$ with $|T_{i,j-1}| \ge \epsilon^{d_{i,j-1}} N/q$ such that if (v_h, v_ℓ) is an edge of H and $h, \ell < j$, then $(f(v_h), f(v_\ell))$ is an edge of G, and if (v_h, v_ℓ) is an edge of H with $h < j \le \ell$, then $f(v_h)$ is adjacent to every element of $T_{\ell,j-1}$. It is sufficient to find a vertex $w \in T_{j,j-1} \setminus \{f(v_1), \ldots, f(v_{j-1})\}$ such that for each v_i adjacent to v_j with $i > j$, the number of elements of $T_{i,j-1}$ adjacent to w is at least $\epsilon|T_{i,j-1}|$. Indeed, if we find such a vertex w, we let $f(v_j) = w$ and for $i > j$, we let $T_{i,j} = N(w) \cap T_{i,j-1}$ if v_i is adjacent to v_j in H and otherwise $T_{i,j} = T_{i,j-1}$, which completes step j. For v_i adjacent to v_j with $i > j$, let $X_{i,j}$ denote the set of vertices in $T_{j,j-1}$ with less than $\epsilon|T_{i,j-1}|$ neighbors in $T_{i,j-1}$. If there is a $X_{i,j}$ with cardinality at least $\frac{1}{2q\Delta}|T_{j,j-1}|$, then letting $W_1 = X_{i,j}$ and $W_2 = T_{i,j-1}$, the edge density between W_1 and W_2 is less than ϵ and W_1, W_2 each has cardinality at least $\frac{1}{2}\epsilon^d \Delta^{-1} q^{-2} N$, contradicting the assumption of the lemma. So each of the sets $X_{i,j}$ has cardinality less than $\frac{1}{2q\Delta}|T_{j,j-1}|$.

Since H is (d, Δ)-degenerate, there are at most Δ vertex subsets $S \subset L_j$ with $v_j \in S$ for which there is a vertex v_i with $i > j$ and $N(i,j) = S$. As we already mentioned, if $i, i' > j-1$ are such that $v_i, v_{i'}$ lie in the same independent set U_k and $N(i, j-1) = N(i', j-1)$, then $T_{i,j-1} = T_{i',j-1}$. If furthermore $v_i, v_{i'}$ are neighbors of v_j, then $X_{i,j} = X_{i',j}$. Since there are q sets U_k and at most Δ sets $S \subset L_j$ with $v_j \in S$ for which there is a vertex v_i with $i > j$ and $N(i,j) = S$, then there are at most $q\Delta$ distinct sets of the form $X_{i,j}$. Hence, at least

$$|T_{j,j-1}| - q\Delta \frac{1}{2q\Delta}|T_{j,j-1}| - (j-1) > \frac{1}{2}|T_{j,j-1}| - n \ge \frac{1}{2}\epsilon^d q^{-1} N - n \ge n$$

vertices $w \in T_{j,j-1}$ can be chosen for $f(v_j)$, which by induction on j completes the proof. □

We now mention some useful terminology from [10] that we need before proving Theorem 3.1. For a graph $G = (V, E)$ and disjoint subsets $W_1, \ldots, W_t \subset V$, the *density* $d_G(W_1, \ldots, W_t)$ between the $t \ge 2$ vertex subsets $W_1, \ldots, W_t$ is defined by

$$d_G(W_1, \ldots, W_t) = \frac{\sum_{i<j} e(W_i, W_j)}{\sum_{i<j} |W_i||W_j|}.$$

If $|W_1| = \cdots = |W_t|$, then

$$d_G(W_1, \ldots, W_t) = \binom{t}{2}^{-1} \sum_{i<j} d_G(W_i, W_j).$$

Definition 3.3. For $\alpha, \rho, \epsilon \in [0, 1]$ and a positive integer t, a graph $G = (V, E)$ is $(\alpha, \rho, \epsilon, t)$-**sparse** if for all subsets $U \subset V$ with $|U| \ge \alpha|V|$, there are disjoint subsets $W_1, \ldots, W_t \subset U$ with $|W_1| = \cdots = |W_t| = \lceil \rho|U| \rceil$ and $d_G(W_1, \ldots, W_{t_i}) \le \epsilon$.

By averaging, if $\alpha' \geq \alpha$, $\rho' \leq \rho$, $\epsilon' \geq \epsilon$, $t' \leq t$, and G is $(\alpha, \rho, \epsilon, t)$-sparse, then G is also $(\alpha', \rho', \epsilon', t')$-sparse. To prove Theorem 3.1, we use the following simple lemma from [10].

Lemma 3.4 ([10]). *If G is $(\alpha, \rho, \epsilon/4, 2)$-sparse, then G is also $\left((\frac{2}{\rho})^{h-1}\alpha, 2^{1-h}\rho^h, \epsilon, 2^h\right)$-sparse for each positive integer h.*

For this paragraph, let $\epsilon = \frac{1}{32q^2d}$, $x = 4\epsilon^{-d}\Delta q^2$, and $y = (2x)^{\log q}$. Note that Lemma 3.2 demonstrates that if a graph G on $N \geq xn$ vertices does not contain a (d, Δ)-degenerate graph H with order n and chromatic number q, then G is $(xn/N, x^{-1}, \epsilon, 2)$-sparse. By Lemma 3.4 with $\alpha = xn/N$, $h = \log q$, $\rho = 1/x$, this implies that if a graph G on $N \geq yn$ vertices does not contain a (d, Δ)-degenerate graph H with order n and chromatic number q, then G is $\left(yn/N, y^{-1}, 4\epsilon, q\right)$-sparse. Hence, as long as $N \geq yn$, then there are vertex subsets $W_1, \ldots, W_q$ of G with the same size which is at least N/y such that $d_G(W_1, \ldots, W_q)$ is at most 4ϵ. Consider a red-blue edge-coloring of K_N with

$$N = \left(2^{7d+8}d^{3d+2}\Delta\right)^{\log q} n \geq 8\left(8q^2(32dq^2)^d\Delta\right)^{\log q} n = 8\left(8\epsilon^{-d}\Delta q^2\right)^{\log q} n = 8yn,$$

where we use the fact that the chromatic number q of a d-degenerate graph satisfies $q \leq d + 1 \leq 2d$. If the red graph does not contain H, then there are disjoint subsets $W_1, \ldots, W_q$ of $V(K_N)$ each with the same cardinality which is at least $y^{-1}N \geq 8n$ such that $d_{G(R)}(W_1, \ldots, W_q)$ is at most 4ϵ, where $G(R)$ denotes the graph of color red. Hence, the total number of red edges whose vertices are in different W_is is at most $4\epsilon q^2|W_1|^2$. For each W_i, delete those $|W_i|/2$ vertices of W_i which have the largest number of neighbors in $\bigcup_{j\neq i} W_j$ in the red graph, and let Y_i be the remaining vertices of W_i. Notice that $|Y_i| = |W_i|/2 \geq 4n$ and every vertex of Y_i is in at most $8\epsilon q^2|W_i| = \frac{|Y_i|}{2d}$ red edges with vertices in $\bigcup_{j\neq i} W_j$ since otherwise the number of edges between $W_i \setminus Y_i$ and $\bigcup_{j\neq i} W_j$ is more than $|W_i \setminus Y_i| 8\epsilon q^2|W_i| = 4\epsilon q^2|W_1|^2$, contradicting the fact that the number $\sum_{i<j} e(W_i, W_j)$ of red edges with vertices in different subsets is at most $4\epsilon q^2|W_1|^2$. Therefore, applying the following lemma to the blue graph, there is a monochromatic blue copy of H, completing the proof of Theorem 3.1. □

Lemma 3.5. *Suppose H is a d-degenerate graph with n vertices and chromatic number q. If G is a graph with disjoint vertex subsets $Y_1, \ldots, Y_q$ with $|Y_1| = \cdots = |Y_q| \geq 4n$ such that each vertex in Y_i is adjacent to all but at most $\frac{|Y_i|}{2d}$ vertices of $\bigcup_{j\neq i} Y_j$, then H is a subgraph of G.*

Proof. Let $v_1, \ldots, v_n$ be an ordering of the vertices of H such that for each vertex v_i, there are at most d neighbors v_j of v_i with $j < i$. Let $V(H) = U_1 \cup \cdots \cup U_q$ be a partition of the vertex set of H into independent sets. We will find an embedding $f : V(H) \to V(G)$ of H such that if $v_i \in U_k$, then $f(v_i) \in Y_k$. We will embed the vertices $v_1, \ldots, v_n$ of H one by one in increasing order. Suppose we have already embedded $v_1, \ldots, v_{i-1}$ and we try to embed $v_i \in U_k$. Consider a vertex v_j adjacent to v_i with $j < i$. We have $v_j \notin U_k$ since U_k is an independent set. Therefore, $f(v_j) \notin Y_k$ and is adjacent to all but at most $\frac{|Y_k|}{2d}$ vertices in Y_k. Since there are at most d such vertices v_j adjacent to v_i with $j < i$, then there are at least $|Y_k| - d\frac{|Y_k|}{2d} - (i-1) = \frac{|Y_k|}{2} - (i-1) > n$ vertices in $Y_k \setminus \{f(v_1), \ldots, f(v_{i-1})\}$ which are adjacent to $f(v_j)$ for all neighbors v_j of v_i with $j < i$. Since any of these vertices can be chosen for $f(v_i)$, this completes the proof by induction. □

4. Random graphs

In this section we discuss the arrangeability of sparse random graphs. Our results imply linear upper bounds on Ramsey numbers of these graphs. We start the section with two simple lemmas relating (d, Δ)-degeneracy with p-arrangeability.

Lemma 4.1. *If a graph is (d, Δ)-degenerate, then it is $(\Delta(d-1)+1)$-arrangeable.*

Proof. Let $v_1, \ldots, v_n$ be a (d, Δ)-degenerate ordering of the vertices of a graph G. Then for each i, there are at most Δ subsets S, each of cardinality at most d, such that $S = N(v_j) \cap \{v_1, \ldots, v_i\}$ for some neighbor v_j of v_i with $j > i$. Therefore, for any vertex v_i, its neighbors to the right of v_i have together at most $\Delta(d-1)+1$ neighbors to the left of v_i (including v_i), and so the graph G is $(\Delta(d-1)+1)$-arrangeable. □

Lemma 4.2. *If a graph is p-arrangeable, then it is $(p, 2^{p-1})$-degenerate.*

Proof. Let $v_1, \ldots, v_n$ be a p-arrangeable ordering of the vertices of a graph G. For any vertex v_i, its neighbors to the right of v_i have together at most p neighbors to the left of v_i (including v_i). Let $N(j, i)$ denote the set of neighbors v_k of v_j with $k \le i$. For every neighbor v_j of v_i with $j > i$, the set $N(j, i)$ lies in a set of size p that contains v_i, so there are at most 2^{p-1} such sets $N(j, i)$. Let v_i be the neighbor of v_j which has maximum index $i < j$. Then using the p-arrangeability property for v_i, we get that the number of neighbors of v_j in $\{v_1, \ldots, v_j\}$ is $|N(j, i)| \le p$. Hence, the ordering demonstrates that G is $(p, 2^{p-1})$-arrangeable. □

We prove for $d \ge 10$ that a.s. the random graph $G(n, d/n)$ is $256d^2$-arrangeable. This follows from Lemma 4.1 and Theorem 4.8 below which says that for $d \ge 10$ a.s. $G(n, d/n)$ is $(16d, 16d)$-degenerate. Theorems 4.8 and 3.1 together imply Theorem 1.4, which says that $G(n, d/n)$ almost surely has Ramsey number at most $2^{cd\log^2 d}n$, where c is an absolute constant. Since the random bipartite graph $G(n, n, d/n)$ is a subgraph of $G(2n, d/n)$, Theorem 4.8 implies that almost surely $G(n, n, d/n)$ is $(32d, 32d)$-degenerate. Together with Theorem 2.1, this proves Theorem 1.5, which says that $G(n, n, d/n)$ almost surely has Ramsey number at most $2^{cd}n$, where c is an absolute constant.

The ordering of the vertices of $G(n, d/n)$ used to prove Theorem 4.8 is a careful modification of the ordering by decreasing degrees. Let A be the set of vertices of degree more than $16d$. It is easy to show that a.s. A is quite small. We then enlarge A to a set $F(A)$ that we show has the property that no vertex in the complement of $F(A)$ has more than one neighbor in $F(A)$ and a.s. $|F(A)| \le 4|A|$. Since $|F(A)|$ is small enough, a.s. any set with size $|F(A)|$ (so, in particular, the set $F(A)$ itself) is sparse enough that the subgraph of $G(n, d/n)$ induced by it is $(2, 3)$-degenerate. We first order the set $F(A)$ and then add the remaining vertices of $G(n, d/n)$ arbitrarily. We use this vertex order to demonstrate that a.s. $G(n, d/n)$ is $(16d, 16d)$-degenerate.

Before proving Theorem 4.8, we need several simple lemmas.

Lemma 4.3. *Almost surely there are at most $2^{4-8d}n$ vertices of $G(n, d/n)$ with degree larger than $16d$.*

Proof. Let A be the subset of $s = 2^{4-8d}n$ vertices of largest degree in $G = G(n, d/n)$ and D be the minimum degree of vertices in A. So there are at least $sD/2$ edges that have at least one vertex in A. Consider a random subset A' of A with size $|A|/2$. Every edge which contains a vertex of A has a probability at least 1/2 of having exactly one vertex in A'. This can be easily seen by considering the cases where the edge lies entirely in A and where the edge has exactly one vertex in A. So there is a subset $A' \subset A$ of size $|A|/2$ such that the number m of edges between A' and $V(G) \setminus A'$ satisfies $m \ge sD/4 = |A'|D/2$.

We now give an upper bound on the probability that $D \ge 16d$. Each subset A' of $G(n, d/n)$ of size $s/2$ has probability at most

$$\binom{\frac{s}{2}(n-\frac{s}{2})}{m}(d/n)^m \le \left(\frac{esn}{2m}\right)^m (d/n)^m \le \left(\frac{2sd}{m}\right)^m \le \left(\frac{8d}{D}\right)^m \le 2^{-4ds}$$

of having at least $m \ge (s/2)(16d)/2 = 4sd$ edges between A' and $V(G) \setminus A'$. Therefore the probability that there is a subset A' of size $s/2$ which has at least $4sd$ edges between A' and $V(G) \setminus A'$ is at most

$$\binom{n}{s/2}2^{-4ds} < \left(\frac{2en}{s}\right)^{s/2} 2^{-4ds} \le \left(\frac{2^{3-8d}n}{s}\right)^{s/2} = o(1),$$

completing the proof. □

Lemma 4.4. *If a graph $G = (V, E)$ of order n has less than $\frac{9}{8}n$ edges, then it contains a vertex of degree at most one or contains a vertex of degree two whose both neighbors have degree two.*

Proof. Suppose for contradiction that G has minimum degree at least 2 and there is no vertex of degree 2 whose both neighbors have degree two. Let $V_1 \subset V$ be those vertices with degree 2 and $V_2 = V \setminus V_1$ be those vertices of degree at least 3. Let $x = |V_1|$. Since every vertex in V_1 has degree at least two and every vertex in V_2 has degree at least three, then the number of edges of G is at least $\frac{2x+3(n-x)}{2} = \frac{3n}{2} - \frac{x}{2}$. Since we assumed that every vertex of degree two has at most one neighbor with degree two, then the subgraph of G induced by V_1 has maximum degree at most one. Therefore, V_1 spans at most $x/2$ edges. Since the vertices in V_1 have degree 2, then the number of edges of G with at least one vertex in V_1 is $2x - e(V_1) \geq \frac{3}{2}x$. Hence, the number of edges of G is at least $\max\left(\frac{3n}{2} - \frac{x}{2}, \frac{3}{2}x\right)$. Regardless of the value of x, this number is always at least $\frac{9n}{8}$, a contradiction. □

Lemma 4.5. *If for $r, s \geq 1$, every subgraph of a graph $G = (V, E)$ has a vertex with degree at most one or a vertex with degree at most s all of whose neighbors have degree at most r, then G is $(s, r+1)$-degenerate.*

Proof. Pick out vertices $v_n, v_{n-1}, \ldots, v_1$ one by one such that for each j, in the subgraph of G induced by $V \setminus \{v_n, v_{n-1}, \ldots, v_{j+1}\}$, the vertex v_j has degree at most one or it has degree at most s and all of its neighbors have degree at most r. Let $L_j = \{v_1, \ldots, v_j\}$. First note that this ordering has the property that each vertex v_j has at most s neighbors v_i with $i < j$ since its degree in the subgraph of G induced by L_j is at most s. Let $N_1(v_j)$ be those vertices v_k with $k > j$ that are adjacent to v_j and have a neighbor in L_{j-1}. The cardinality of $N_1(v_j)$ is at most r since otherwise the vertex $v_h \in N_1(v_j)$ with the largest index h has at least two neighbors in L_h and has a neighbor $v_j \in L_h$ which has degree more than r in L_h, contradicting how we chose v_h. The vertices v_k with $k > j$ that are adjacent to v_j and have degree one in the subgraph of G induced by L_k satisfy $N(v_k) \cap L_j = \{v_j\}$. Therefore, for each j, there are at most $r + 1$ sets $S \subset L_j$ for which $S = N(v_k) \cap L_j$ for some vertex v_k adjacent to v_j with $k > j$. Hence, this ordering shows that G is $(s, r + 1)$-degenerate. □

Lemma 4.6. *Almost surely every subgraph G' of $G(n, d/n)$ with $t \leq (5d)^{-9}n$ vertices has average degree less than $9/4$.*

Proof. Let S be a subset of size t with $t \leq (5d)^{-9}n$. The probability that S has at least $m = \frac{9}{8}t$ edges is at most $\binom{\binom{t}{2}}{m}(d/n)^m$. Therefore, by the union bound, the probability that there is a subset of size t with at least $m = \frac{9}{8}t$ edges is at most

$$\binom{n}{t}\binom{\binom{t}{2}}{m}\left(\frac{d}{n}\right)^m \leq \left(\frac{en}{t}\right)^t \left(\frac{et^2}{2m}\right)^m \left(\frac{d}{n}\right)^m = \mathrm{e}^t \left(\frac{4e}{9}\right)^{9t/8} \left(\frac{n}{t}\right)^t \left(\frac{\mathrm{d}t}{n}\right)^{9t/8}$$
$$\leq 5^t \left(\frac{n}{t}\right)^t \left(\frac{\mathrm{d}t}{n}\right)^{9t/8} = 5^t (d^9 t/n)^{t/8}.$$

Summing over all $t \leq (5d)^{-9}n$, one easily checks that the probability that there is an induced subgraph with at most $(5d)^{-9}n$ vertices and average degree at least $9/4$ is $o(1)$, completing the proof. □

For a graph $G = (V, E)$ and vertex subset $S \subset V$, let $F(S)$ denote a vertex subset formed by adding vertices from $V \setminus S$ with at least two neighbors in S one by one until no vertex in $V \setminus F(S)$ has at least two neighbors in $F(S)$. It is not difficult to see that $F(S)$ is uniquely determined by S.

Lemma 4.7. *Almost surely every vertex subset S of $G(n, d/n)$ with cardinality $t \leq (5d)^{-10}n$ has $|F(S)| \leq 4t$.*

Proof. Suppose not, and consider the set T of the first $4t$ vertices of $F(S)$. By definition, the number of edges in the subgraph induced by T is at least $2(|T| - |S|) \geq \frac{3}{2}|T|$, so the average degree of this induced subgraph is at least 3. But Lemma 4.6 implies that a.s. every induced subgraph of $G(n, d/n)$ with at most $4t$ vertices has average degree less than 3, a contradiction. □

Theorem 4.8. *For $d \geq 10$, the graph $G(n, d/n)$ is almost surely $(16d, 16d)$-degenerate.*

Proof. Let A be the $(5d)^{-10}n$ vertices of largest degree in $G(n, d/n)$. Since $(5d)^{-10}n \geq 2^{4-8d}n$ for $d \geq 10$, by Lemma 4.3, a.s. all vertices not in A have degree at most $16d$. By Lemma 4.7, a.s. $|F(A)| \leq 4|A|$. By Lemmas 4.6, 4.4 and 4.5, a.s. the subgraph of G induced by $F(A)$ is $(2, 3)$-degenerate. Let $v_1, \ldots, v_{|F(A)|}$ be an ordering of $F(A)$ that respects the $(2, 3)$-degeneracy. Arbitrarily order the vertices not in $F(A)$ as $v_{|F(A)|+1}, \ldots, v_n$. Let $L_j = \{v_1, \ldots, v_j\}$. We claim that this is the desired ordering for the vertices of $G(n, d/n)$. Consider a vertex v_i. If $i \leq |F(A)|$, then v_i has the following three properties:

- there are at most two neighbors v_j of v_i with $j < i$,
- there are at most three subsets $S \subset L_i$ for which there is a neighbor v_h of v_i with $i < h \leq |F(A)|$ and $N(v_h) \cap L_i = S$, and
- every neighbor v_k of v_i with $k > |F(A)|$ has $N(v_k) \cap L_i = \{v_i\}$.

If $i > |F(A)|$, then v_i has maximum degree at most $16d$. Therefore, this ordering demonstrates that $G(n, d/n)$ is a.s. $(16d, 16d)$-degenerate, completing the proof. □

We next prove Lemma 4.11, which completes the proof of Theorem 1.2, and says that for $d \geq 300$ a.s. $G(n, d/n)$ is not $d^2/144$-arrangeable.

Lemma 4.9. *Let $p = d/n$ with $d \geq 300$. Almost surely every pair A, B of disjoint subsets of $G(n, p)$ of size at least $n/6$ have at least $p|A||B|/2$ edges between them.*

Proof. Let $s = n/6$. If there are disjoint subsets A, B each of size at least s which have less than $p|A||B|/2$ between them, then by averaging over all subsets $A' \subset A$ and $B' \subset B$ with $|A'| = |B'| = s$, there are subsets $A' \subset A$ and $B' \subset B$ with $|A'| = |B'| = s$ and the number of edges between A' and B' is at most $p|A'||B'|/2$.

Using the standard Chernoff bound (see page 306 of [4]) for s^2 independent coin flips each coming up heads with probability $p = d/n$, the probability that a fixed pair of disjoint subsets A', B' of $G(n, d/n)$ each of size s have less than $ps^2/2$ edges between them is at most

$$e^{-(ps^2/2)^2/(2ps^2)} = e^{-ps^2/8}.$$

The probability that no pair of disjoint subsets A', B' each of size s has less than $ps^2/2$ edges between them is at most

$$\binom{n}{s}\binom{n-s}{s} e^{-ps^2/8} \leq \left(\frac{en}{s}\right)^{2s} e^{-ps^2/8} = \left(6e \cdot e^{-d/96}\right)^{2s} = o(1).$$

So a.s. every pair of disjoint subsets A, B each with cardinality at least $n/6$ has at least $p|A||B|/2$ edges between them. □

Lemma 4.10. *In $G(n, d/n)$ with $d = o(n^{1/6})$, a.s. no pair of vertices have three common neighbors.*

Proof. A pair of vertices together with its three common neighbors form the complete bipartite graph $K_{2,3}$. Let us compute the expected number of $K_{2,3}$ in $G(n, d/n)$. We pick the vertices and then multiply by the probability that they form $K_{2,3}$. So the expected number of $K_{2,3}$ is

$$\binom{n}{2}\binom{n-3}{3}(d/n)^6 \leq n^5(d/n)^6 = o(1),$$

which implies the lemma. □

Lemma 4.11. *For $d \geq 300$, almost surely $G(n, d/n)$ is not $d^2/144$-arrangeable.*

Proof. Suppose for contradiction that $G(n, d/n)$ is $d^2/144$-arrangeable, so there is a corresponding ordering $v_1, \ldots, v_n$ of its vertices. Let $V_1 = \{v_i\}_{i \leq n/3}$, $V_2 = \{v_i\}_{n/3 < i \leq 2n/3}$, and $V_3 = \{v_i\}_{2n/3 < i \leq n}$. Delete from V_3 all vertices that have fewer than $\frac{d}{n}|V_1|/2 = d/6$ neighbors in V_1, and let V_3' be the remaining subset of V_3. Note that a.s. $|V_3'| \geq n/3 - n/6 = n/6$ since otherwise $V_3 \setminus V_3'$ and V_1 each has cardinality at least $n/6$ and have fewer than $\frac{d}{n}|V_3 \setminus V_3'||V_1|/2$ edges between them, which would contradict Lemma 4.9. Delete from V_2 all vertices that have fewer than $\frac{d}{n}|V_3'|/2$ neighbors in V_3', and let V_2' be the remaining subset of V_2. Note that $|V_2'| \geq n/3 - n/6 = n/6$ since otherwise $V_2 \setminus V_2'$ and V_3' each has cardinality at least $n/6$ and have fewer than $\frac{d}{n}|V_2 \setminus V_2'||V_3'|/2$ edges between them, which would contradict Lemma 4.9. Pick any vertex $v \in V_2'$. Vertex v has at least $\frac{d}{n}|V_3'|/2 \geq d/12$ neighbors in V_3'. Let $U = \{u_1, \ldots, u_r\}$ denote a set of $r = d/12$ neighbors of v in V_3'. Let $d(u_i)$ denote the number of neighbors of u_i in V_1 and $d(u_i, u_j)$ denote the number of common neighbors of u_i and u_j in V_1. Note that for each $u_i \in U$, $d(u_i) \geq d/6$. Also, by Lemma 4.10, $d(u_i, u_j) \leq 2$ for distinct vertices u_i, u_j. By the inclusion–exclusion principle, the number of vertices in V_1 adjacent to at least one vertex in U is at least

$$\sum_{1 \leq i \leq r} d(u_i) - \sum_{1 \leq i < j \leq r} d(u_i, u_j) \geq rd/6 - \binom{r}{2} 2 > d^2/144.$$

Hence, a.s. $G(n, d/n)$ is not $d^2/144$-arrangeable. $\square$

5. Concluding remarks

In this paper we proved that for fixed d, the Ramsey number of the random graph $G(n, d/n)$ is a.s. linear in n. More precisely, we showed that there are constants c_1, c_2 such that a.s.

$$2^{c_1 d} n \leq r\,(G(n, d/n)) \leq 2^{c_2 d \log^2 d} n.$$

We think that the upper bound can be further improved and that the Ramsey number of the random graph $G(n, d/n)$ is a.s. at most $2^{cd} n$ for some constant c.

There are many results demonstrating that certain parameters of random graphs are highly concentrated (see, e.g., the books [5,16]). Probably the most striking example of this phenomena is a recent result of Achlioptas and Naor [1]. Extending earlier results from [23,20], they demonstrate that for fixed $d > 0$, a.s. the chromatic number of the random graph $G(n, d/n)$ is k_d or $k_d + 1$, where k_d is the smallest integer k such that $d < 2k \log k$. We do not think the Ramsey numbers of random graphs are nearly as highly concentrated. However, it would be interesting to determine if there is a constant $c_d > 0$ for each $d > 1$ such that the random graph $G(n, d/n)$ a.s. has Ramsey number $(c_d + o(1))\, n$.

Acknowledgements

The first author's research is supported by an NSF Graduate Research Fellowship and a Princeton Centennial Fellowship. The second author's research is supported in part by NSF CAREER award DMS-0812005 and by a USA-Israeli BSF grant.

References

[1] D. Achlioptas, A. Naor, The two possible values of the chromatic number of a random graph, Ann. Math. 162 (2005) 1333–1349.
[2] M. Ajtai, J. Komlós, E. Szemerédi, Topological complete subgraphs in random graphs, Studia Sci. Math. Hungar. 14 (1979) 293–297.
[3] N. Alon, M. Krivelevich, B. Sudakov, Turán numbers of bipartite graphs and related Ramsey-type questions, Combin. Probab. Comput. 12 (2003) 477–494.
[4] N. Alon, J.H. Spencer, The Probabilistic Method, 2nd ed., Wiley, New York, 2000.
[5] B. Bollobás, Random Graphs, 2nd ed, in: Cambridge Studies in Advanced Mathematics, vol. 73, Cambridge University Press, Cambridge, 2001.
[6] S.A. Burr, P. Erdős, On the magnitude of generalized Ramsey numbers for graphs, in: Infinite and Finite Sets I, in: Colloquia Mathematica Soc. Janos Bolyai, vol. 10, North-Holland, Amsterdam, London, 1975, pp. 214–240.
[7] G. Chen, R.H. Schelp, Graphs with linearly bounded Ramsey numbers, J. Combin. Theory Ser. B 57 (1993) 138–149.
[8] V. Chvátal, V. Rödl, E. Szemerédi, W.T. Trotter Jr., The Ramsey number of a graph with bounded maximum degree, J. Combin. Theory Ser. B 34 (1983) 239–243.

[9] D. Conlon, A new upper bound for diagonal Ramsey numbers, Ann. Math. 170 (2009) (in press).
[10] J. Fox, B. Sudakov, Induced Ramsey-type theorems, Adv. Math. 219 (2008) 1771.
[11] J. Fox, B. Sudakov, Density theorems for bipartite graphs and related Ramsey-type results, Combinatorica (in press).
[12] W.T. Gowers, A new proof of Szemerédi's theorem for arithmetic progressions of length four, Geom. Funct. Anal. 8 (1998) 529–551.
[13] R. Graham, V. Rödl, A. Ruciński, On graphs with linear Ramsey numbers, J. Graph Theory 35 (2000) 176–192.
[14] R. Graham, V. Rödl, A. Ruciński, On bipartite graphs with linear Ramsey numbers, Combinatorica 21 (2001) 199–209.
[15] R. Graham, B. Rothschild, J. Spencer, Ramsey Theory, 2nd ed., Wiley, New York, 1990.
[16] S. Janson, T. Łuczak, A. Ruciński, Random Graphs, Wiley, New York, 2000.
[17] A. Kostochka, V. Rödl, On graphs with small Ramsey numbers, J. Graph Theory 37 (2001) 198–204.
[18] A. Kostochka, V. Rödl, On graphs with small Ramsey numbers II, Combinatorica 24 (2004) 389–401.
[19] A. Kostochka, B. Sudakov, On Ramsey numbers of sparse graphs, Combin. Probab. Comput. 12 (2003) 627–641.
[20] T. Łuczak, A note on the sharp concentration of the chromatic number of random graphs, Combinatorica 11 (1991) 295–297.
[21] J. Nešetřil, P. Ossona de Mendez, Fraternal augmentations, arrangeability and linear Ramsey numbers, European J. Combin. 30 (7) (2009) 1696–1703.
[22] V. Rödl, R. Thomas, Arrangeability and clique subdivisions, in: R. Graham, J. Nešetřil (Eds.), The mathematics of Paul Erdős, II, in: Algorithms Combin., vol. 14, Springer, Berlin, 1997, pp. 236–239.
[23] E. Shamir, J. Spencer, Sharp concentration of the chromatic number on random graphs $G_{n,p}$, Combinatorica 7 (1987) 121–129.
[24] B. Sudakov, Few remarks on the Ramsey–Turán-type problems, J. Combin. Theory Ser. B 88 (2003) 99–106.
[25] B. Sudakov, Large K_r-free subgraphs in K_s-free graphs and some other Ramsey-type problems, Random Struct. Algorithms 26 (2005) 253–265.