



# A remark on Hamilton cycles with few colors

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**Abstract:** Akbari, Etesami, Mahini, and Mahmoody conjectured that every proper edge-coloring of  $K_n$  with  $n$  colors contains a Hamilton cycle with  $\leq O(\log n)$  colors. They proved that there is always a Hamilton cycle with  $\leq 8\sqrt{n}$  colors. In this note we improve this bound to  $O(\log^3 n)$ .

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An edge-coloring of  $K_n$  is proper if no two edges of the same color share a vertex. The smallest number of colors needed to properly edge-color a graph  $G$  is called the chromatic index of  $G$ , denoted by  $\chi'(G)$ . It is well known that  $\chi'(K_n) = n - 1$  if  $n$  is even and  $\chi'(K_n) = n$  if  $n$  is odd.

Akbari, Etesami, Mahini, and Mahmoody [1] investigated cycles in properly edge-colored complete graphs. Specifically they looked for Hamilton cycles in properly colored complete graphs which have either many or few colors (a Hamilton cycle in a graph is one which passes through every vertex.) When looking for Hamilton cycles with *few* colors it is natural to bound the total number of colors in the properly colored  $K_n$ . Otherwise, if one looks at a properly colored  $K_n$  with  $\binom{n}{2}$  colors, then every Hamilton cycle trivially has exactly  $n$  colors. Because of this Akbari et al. looked at properly colored  $K_n$  with  $\chi'(K_n)$  colors. They made the following conjecture about how few colors one can have on a Hamilton cycle in such a coloring.

CONJECTURE 1. (Akbari, Etesami, Mahini, and Mahmoody [1]). Every properly edge-colored  $K_n$  with  $\chi'(K_n)$  colors has a Hamilton cycle with  $\leq O(\log n)$  colors.

To see that there are proper  $\chi'(K_n)$ -edge-colored  $K_n$  with no Hamilton cycles with less than  $\log n$  colors, consider a coloring of the edges of the complete graph with vertex set  $\mathbb{Z}_2^k$ , where the edge  $ij$  is colored by color  $i+j$ . Indeed, any Hamilton cycle of this graph contains 0 and any other vertex  $i$  is a sum of the colors of the edges on the path from 0 to  $i$ . Thus the number of edge colors must be at least  $\dim(\mathbb{Z}_2^k) = k$ .

Towards Conjecture 1, Akbari et al. proved that every properly  $\chi'(K_n)$ -edge-colored  $K_n$  has a Hamilton cycle with  $8\sqrt{n}$  colors [1]. In this note we explain how to improve this to  $O(\log^3 n)$ .

THEOREM 1. *For any sufficiently large  $n$ , any properly edge-colored  $K_n$  with  $\chi'(K_n)$  colors contains a Hamilton cycle with at most  $O(\log^3 n)$  colors.*

To prove Theorem 1 we select a set of  $\log^3 n$  colors at random and show that, with high probability, the subgraph consisting of these colors is Hamiltonian. The Hamiltonicity of this subgraph follows from the proof of Theorem 10 in Christofides and Markstrom [4]. First we show that, with high probability, all eigenvalues of this graph except the first one are small in absolute value, so that the graph is pseudo-random. Then a result of Krivelevich and Sudakov [5] implies that such graphs are Hamiltonian.

Given a  $d$ -regular graph  $H$  with vertex set  $\{1, \dots, n\}$ , let  $A$  be the corresponding adjacency matrix, i. e. an  $n \times n$  matrix such that  $A_{i,j} = 1$  if  $ij \in E(H)$  and  $A_{i,j} = 0$  otherwise. Let  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$  be the eigenvalues of  $A$ . Then we have  $d = \lambda_1$  and  $|\lambda_i| \leq d$  for all  $i \in [n]$ . We define  $\lambda(H) = \max_{i=2}^n |\lambda_i|$  and note that the smaller  $\lambda(H)$  is, the more pseudo-random our graph  $H$  is. For any  $\lambda > 0$ , we say that  $H$  is an  $(n, d, \lambda)$ -graph if  $\lambda(H) \leq \lambda$ .

THEOREM 2 (Krivelevich and Sudakov [5]). *Let  $H$  be an  $(n, d, \lambda)$ -graph. If  $n$  is large enough and*

$$\lambda \leq \frac{(\log \log n)^2}{1000 \log n (\log \log \log n)} d,$$

*then  $H$  is Hamiltonian.*

Thus it will suffice to show that by choosing  $d = \log^3 n$  colors at random, the graph  $H$  we obtain is an  $(n, d, \frac{2 \log \log n}{\log n} d)$ -graph with high probability. Actually,  $H$  will only be regular when  $n$  is even and so we first prove the result for even  $n$ .

LEMMA 3. *For any sufficiently large even  $n$ , any properly edge-colored  $K_n$  with  $n - 1$  colors contains a Hamilton cycle with at most  $\log^3 n$  colors.*

The case of odd  $n$  can then be obtained from the above lemma as follows.

PROOF OF THEOREM 1. The case of even  $n$  is given by Lemma 3. So let  $n$  be odd and let a proper  $n$  edge-coloring of  $K_n$  be given. Note that each vertex  $u$  has a color  $c(u)$  which is not used by any of the edges incident to it. Moreover, since there are  $n$  colors, each color  $a$  must appear  $(n - 1)/2$  times and hence there is some vertex  $u$  such that  $c(u) = a$ . Thus if we add a new vertex  $v$  and for each  $u \in K_n$  color the edge  $uv$  with  $c(u)$ , we obtain a proper  $n$  edge-coloring of  $K_{n+1}$ . By Lemma 3, it has a Hamilton cycle with at most  $\log^3(n + 1)$  colors, which forms a Hamilton path on  $K_n$ . By connecting the endpoints of the path (possibly using an extra color), we obtain a Hamilton cycle with at most  $\log^3(n + 1) + 1$  colors.  $\square$

The fact that  $\lambda(H)$  is small will follow from an operator Hoeffding inequality for Hilbert spaces, obtained by Christofides and Markstrom [3] (extending the work of [2]). To this end, let  $V$  be a Hilbert space of dimension  $d$  and  $S(V)$  be the set of self adjoint operators on  $V$ . For any  $A, B \in S(V)$ , we define  $A \leq B$  iff  $B - A$  is positive semidefinite and define  $[A, B] = \{C \in S(V) : A \leq C \leq B\}$ . We let  $\|A\| = \sup_{|v|=1} |Av|$  denote the operator norm, where  $|v| = \sqrt{\langle v, v \rangle}$ .

THEOREM 4 (Operator Hoeffding [3, 4]). *Let  $V$  be a Hilbert space of dimension  $n$  and let  $0 = X_0, X_1, \dots, X_d$  be a martingale taking values in  $S(V)$ , such that  $X_i - X_{i-1} \in [-\frac{1}{2}I, \frac{1}{2}I]$  for all  $i \in [d]$ . Then for  $0 < t < 1/2$ ,*

$$\mathbf{P} [\|X_d - \mathbf{E}[X_d]\| \geq dt] \leq 2n \exp(-2dt^2).$$

PROOF OF LEMMA 3. Given a proper edge-coloring  $c : E(K_n) \rightarrow [n - 1]$ , let  $d = \log^3 n$  and let  $c_1, \dots, c_d$  be a sequence of colors chosen independently and uniformly at random from  $[n - 1]$ . For each  $l \in [d]$ , let  $A^{(l)}$  be the adjacency matrix of color  $c_l$  with probability  $\frac{n-1}{n}$  and  $A^{(l)} = I$  with probability  $1/n$ . Let  $A = A^{(1)} + \dots + A^{(d)}$  and note that the probability that all  $d$  colors are distinct and

all  $A^{(l)} \neq I$  is  $\frac{n-1}{n} \frac{n-2}{n} \dots \frac{n-d}{n} = 1 - o(1)$ . In this case,  $A$  is the adjacency matrix of a simple,  $d$ -regular graph  $H$ .

Furthermore, we claim that  $\lambda(H) = \|A - \frac{d}{n}J\|$  where  $J$  is the all 1's matrix. Indeed, letting  $v_1, \dots, v_n$  be an orthonormal basis of eigenvectors of  $A$  ( $v_1$  is the all 1's vector), with corresponding eigenvalues  $\lambda_1 \geq \dots \geq \lambda_n$ , we have  $(A - \frac{d}{n}J)v_1 = 0$  and  $(A - \frac{d}{n}J)v_i = \lambda_i v_i$  for all  $i \geq 2$ .

Also note that for each  $l \in [d]$ , the eigenvalues  $\mu_1 \geq \dots \geq \mu_n$  of  $A^{(l)}$  satisfy  $\mu_1 = 1$  and  $|\mu_i| \leq 1$  for  $i \geq 2$ . Moreover,  $v = (1, \dots, 1)$  is an eigenvector of  $\mu_1$  and hence  $Y_l = \frac{1}{2}(A^{(l)} - \frac{1}{n}J)$  has eigenvalues  $0, \mu_2/2, \dots, \mu_n/2$ , so that  $Y_l \in [-\frac{1}{2}I, \frac{1}{2}I]$ . Moreover, we have  $E[Y_l] = 0$  which implies that  $X_i = Y_1 + \dots + Y_i$  is a martingale. Note also that  $X_d = \frac{1}{2}(A - \frac{d}{n}J)$  and  $X_i \in S(\mathbb{R}^n)$  for all  $i$ . Thus, we can apply Theorem 4 with  $t = \log \log n / \log n$  to conclude that

$$\mathbb{P} \left[ \frac{1}{2} \lambda(H) \geq dt \right] = \mathbb{P} [\|X_d\| \geq dt] \leq 2n \exp(-2dt^2) = 2n^{1-2(\log \log n)^2} \rightarrow 0$$

as  $n \rightarrow \infty$ . Thus for  $n$  large enough, we have that with high probability

$$\lambda(H) \leq 2dt \leq \frac{(\log \log n)^2}{1000 \log n (\log \log \log n)} d,$$

and so in this case, we may apply Theorem 2 to conclude that  $H$  has a Hamilton cycle.  $\square$

*REMARK.* One can deduce the Hamiltonicity of the random set of  $\log^3 n$  colors directly from the statement of Theorem 10 in [4]. We choose not to do so for the convenience of the reader and since the proof of Christofides and Markstrom in [4] needs a version of Theorem 2 that works for multi-graphs with self-loops. Although such a theorem should have a very similar proof to that in [5], it does not appear in the literature. We avoid this issue by observing that the graph we obtain is simple with high probability.

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## Bibliography

1. **S. Akbari, O. Etesami, H. Mahini, M. Mahmoody**, *On rainbow cycles in edge coloured complete graphs*, Australas. J. Combin. **37** (2007), 33–42.

2. **R. Ahlswede, A. Winter**, *Strong Converse for Identification via Quantum Channels*, IEEE Transactions on Information Theory **48** (2003), 569–579.
3. **D. Christofides, K. Markstrom**, *Expansion properties of random Cayley graphs and vertex transitive graphs*, Random Structures & Algorithms **1** (2008), 88–100.
4. **D. Christofides, K. Markstrom** *Random Latin square graphs*, Random Structures & Algorithms **41** (2012), 47–65.
5. **M. Krivelevich, B. Sudakov** *Sparse pseudo-random graphs are Hamiltonian*, J. Graph Theory, **42** (2003), 17–33.

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