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Hamiltonicity, independence number, and pancyclicity

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ABSTRACT

A graph on n vertices is called pancyclic if it contains a cycle of length ℓ for all $3 \leq \ell \leq n$. In 1972, Erdős proved that if G is a Hamiltonian graph on $n > 4k^4$ vertices with independence number k, then G is pancyclic. He then suggested that $n = \Omega(k^2)$ should already be enough to guarantee pancyclicity. Improving on his and some other later results, we prove that there exists a constant c such that $n > ck^{7/3}$ suffices.

1. Introduction

A *Hamilton cycle* of a graph is a cycle which passes through every vertex of the graph exactly once, and a graph is called *Hamiltonian* if it contains a Hamilton cycle. Determining whether a given graph is Hamiltonian is one of the central questions in graph theory, and there are numerous results which establish sufficient conditions for Hamiltonicity. For example, a celebrated result of Dirac asserts that every graph of minimum degree at least $\lceil n/2 \rceil$ is Hamiltonian. A graph is *pancyclic* if it contains a cycle of length ℓ for all $3 \leq \ell \leq n$. By definition, every pancyclic graph is Hamiltonian, but it is easy to see that the converse is not true. Nevertheless, these two concepts are closely related and many nontrivial conditions which imply Hamiltonicity also imply pancyclicity of a graph. For instance, extending Dirac's Theorem, Bondy [2] proved that every graph of minimum degree at least $\lceil n/2 \rceil$ either is the complete bipartite $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$, or is pancyclic. Moreover, in [3], he made a meta conjecture in this context which says that almost any non-trivial condition on a graph which implies that the graph is Hamiltonian also implies that the graph is pancyclic (there may be a simple family of exceptional graphs).

Let the *independence number* $\alpha(G)$ of a graph G be the order of a maximum independent set of G. A classical result of Chvátal and Erdős [4] says that every graph G whose vertex connectivity (denoted $\kappa(G)$) is at least as large as its independence number is Hamiltonian. Motivated by Bondy's metaconjecture, Amar et al. [1] obtained several results on the lengths of cycles in a graph G that satisfies the Chvátal–Erdős condition $\kappa(G) \geq \alpha(G)$, and conjectured that if such a graph G is not

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bipartite then either $G = C_5$, or G contains cycles of length ℓ for all $4 \le \ell \le n$ (Lou [9] made some partial progress towards this conjecture). In a similar context, Jackson and Ordaz [7] conjectured that that every graph G with $\kappa(G) > \alpha(G)$ is pancyclic. Keevash and Sudakov [8] proved that there exists an absolute constant c such that $\kappa(G) \ge c\alpha(G)$ is sufficient for pancyclicity.

In this paper, we study a relation between Hamiltonicity, pancyclicity, and the independence number of a graph. Such a relation was first studied by Erdős [5]. In 1972, he proved a conjecture of Zarins by establishing the fact that every Hamiltonian graph G on $n \ge 4k^4$ vertices with $\alpha(G) \le k$ is pancyclic (see also [6] for a different proof of a weaker bound). Erdős also suggested that the bound $4k^4$ on the number of vertices is probably not tight, and that the correct order of magnitude should be $\Omega(k^2)$. The following graph shows that this, if true, is indeed best possible. Let $K_1, \dots, K_k$ be disjoint cliques of size $k-2$, where each K_i has two distinguished vertices v_i and w_i. Let G be the graph obtained by connecting $v_i \in K_i$ and $w_{i+1} \in K_{i+1}$ by an edge (here addition is modulo k). One can easily show that this graph is Hamiltonian, has $k(k-2)$ vertices, and independence number k. However, this graph does not contain a cycle of length $k-1$ (thus is not pancyclic), since every cycle either is a subgraph of one of the cliques, or contains at least one vertex from each clique K_i. The former type of cycles have length at most $k-2$, and the latter type of cycles have length at least $2k$.

Recently, Keevash and Sudakov [8] improved Erdős' result and showed that $n > 150k^3$ already implies pancyclicity. Our main theorem further improves this bound.

Theorem 1.1. *There exists a constant c such that for every positive integer k, every Hamiltonian graph on $n \ge ck^{7/3}$ vertices with $\alpha(G) \le k$ is pancyclic.*

Suppose that one established the fact that for some function $f(k)$, every Hamiltonian graph on $n \ge f(k)$ vertices with $\alpha(G) \le k$ contains a cycle of length $n-1$. Then, by iteratively applying this result, one can easily see that for every constant $C \ge 1$, every graph on $n \ge Cf(k)$ vertices with $\alpha(G) \le k$ contains cycles of all length between $\frac{n}{C}$ and n. This simple observation was used in both [5,8], where they found cycles of length linear in n using this method, and then found cycles of smaller lengths using other methods. Thus the problem finding a cycle of length $n-1$ is a key step in proving pancyclicity. Keevash and Sudakov suggested that if one just is interested in this problem, then the bound between the number of vertices and independence number can be significantly improved. More precisely, they asked whether there is an absolute constant c such that every Hamiltonian graph on $n \ge ck$ vertices with independence number k contains a cycle of length $n-1$.

Despite the fact that this bound suggested by Keevash and Sudakov is only linear in k, even improving Erdős' original estimate of $n = \Omega(k^3)$ was not an easy task. Moreover, as we will explain in the concluding remarks, currently the bottleneck of proving pancyclicity lies in this step of finding a cycle of length $n-1$. Thus in order to prove Theorem 1.1, we partially answer Keevash and Sudakov's question for the range $n \ge ck^{7/3}$, and combine this result with tools developed in [8]. Therefore, the main focus of our paper will be to prove the following theorem.

Theorem 1.2. *There exists a constant c such that for every positive integer k, every Hamiltonian graph on $n \ge ck^{7/3}$ vertices with $\alpha(G) \le k$ contains a cycle of length $n-1$.*

In Section 2, we state a slightly stronger form of Theorem 1.2, and use it to deduce Theorem 1.1. Then in Sections 3 and 4, we prove the strengthened version Theorem 1.2. To simplify the presentation, we often omit floor and ceiling signs whenever these are not crucial and make no attempts to optimize absolute constants involved.

2. Pancyclicity

In order to prove Theorem 1.1, we use the following slightly stronger form of Theorem 1.2 whose proof will be given in the next two sections.

Theorem 2.1. *There exists a constant c such that for every positive integer k, every Hamiltonian graph on $n \ge ck^{7/3}$ with $\alpha(G) \le k$ contains a cycle of length $n-1$. Moreover, for an arbitrary fixed set of vertices W of size $|W| \le 20k^2$, we can find such a cycle which contains all the vertices of W.*

As mentioned in the Introduction, Theorem 2.1 will be used to find cycles of linear lengths. The following two results from [8, Theorem 1.3 and Lemma 3.2] allow us to find cycle lengths in the range not covered by Theorem 2.1.

Theorem 2.2. *If G is a graph with $\delta(G) \geq 300\alpha(G)$ then G contains a cycle of length ℓ for all $3 \leq \ell \leq \delta(G)/81$.*

Lemma 2.3. *Suppose G is a graph with independence number $\alpha(G) \leq k$ and $V(G)$ is partitioned into two parts A and B such that*

(i) *$G[A]$ is Hamiltonian.*
(ii) *$|B| \geq 9k^2 + k + 1$, and*
(iii) *every vertex in B has at least 2 neighbors in A.*

Then G contains a cycle of length ℓ for all $2k + 1 + \lfloor \log_2(2k + 1)\rfloor \leq \ell \leq |A|/2$.

Proof of Theorem 1.1. Note that the conclusion is immediate if $k = 1$. Thus we may assume that $k \geq 2$. Let c be the maximum of the constant coming from Theorem 2.1 and 300 and G be Hamiltonian graph on $n = 3ck^{7/3}$ vertices such that $\alpha(G) \leq k$. By repeatedly applying Theorem 2.1 with $W = \emptyset$, we can find cycles of length $ck^{7/3}$ to $3ck^{7/3}$.

Moreover, as we will see, by carefully using Theorem 2.1 in the previous step, we can prepare a setup for applying Lemma 2.3. Let C_1 be the cycle of length $n - 1$ obtained by Theorem 2.1, and let v_1 be the vertex not contained in C_1. We know that v_1 has at least 2 neighbors in C_1. Let W_1 be two arbitrary vertices out of them. By applying Theorem 2.1 with $W = W_1$, we can find a cycle C_2 of length $n - 2$ which contains W_1. Let v_2 be the vertex contained in C_1 but not in C_2, and let W_2 be the union of W_1 and two arbitrary neighbors of v_2 in C_2. We can repeat it $10k^2$ times (note that we maintain $|W| \leq 20k^2$), to obtain a cycle C_{10k^2} of length $n - 10k^2$, and vertices $v_1, \ldots, v_{10k^2}$ so that each v_i has at least 2 neighbors in the cycle C_{10k^2}. Since $10k^2 \geq 9k^2 + k + 1$, by Lemma 2.3, G contains a cycle of length ℓ for all $2k + 1 + \lfloor \log_2(2k + 1)\rfloor \leq \ell \leq (n - 10k^2)/2$.

Now we find all the remaining cycle lengths. From the graph G, pick one by one, a vertex of degree less than $ck^{4/3}$, and remove it together with its neighbors. Note that since the picked vertices form an independent set in G, at most k vertices will be removed. Therefore, when there are no more vertices to pick, at least $3ck^{7/3} - k \cdot (ck^{4/3} + 1) > ck^{7/3}$ vertices remain, and the induced subgraph of G on these vertices will be of minimum degree at least $ck^{4/3}$. Since $ck^{4/3} \geq 300k \geq 300\alpha(G)$, by Theorem 2.2, G contains a cycle of length ℓ for all $3 \leq \ell \leq (c/81)k^{4/3}$.

By noticing the inequalities $(n - 10k^2)/2 = (3ck^{7/3} - 10k^2)/2 \geq ck^{7/3}$ and $(c/81)k^{4/3} \geq 2k + 1 + \lfloor \log_2(2k + 1)\rfloor$ we can see the existence of cycles of all possible lengths. □

3. A structural lemma

In Sections 3 and 4, we will prove Theorem 2.1. Given a Hamiltonian graph on n vertices, one can easily see that there are many ways one can find a cycle of length $n - 1$, if certain 'chords' are present in the graph. Our strategy is to find such chords that are 'nicely' arranged. In particular, in this section, we consider pairs of chords and the way they cross each other in order to deduce some structure of our graph. Then in the next section, we prove the main theorem by considering certain triples of chords, which we call semi-triangles.

Throughout this section, let G be a fixed graph on $n \geq 80k^2$ vertices such that $\alpha(G) \leq k$, and let W be a fixed set of vertices such that $|W| \leq 20k^2$. Note that the bound on the number of vertices is weaker than that of Theorem 2.1. The results developed in this section still hold under this weaker bound, and we only need the stronger bound $n \geq ck^{7/3}$ in the next section. Since our goal is to prove the existence of a cycle of length $n - 1$, assume to the contrary that G does not contain a cycle of length $n - 1$. Under these assumptions, we will prove a structural lemma on the graph G which will immediately imply a slightly weaker form of Theorem 2.1 where the bound on the number of vertices is replaced by $\Omega(k^{5/2})$. In the next section, we will apply this structural lemma more carefully to prove Theorem 2.1.

One of the main ingredients of the proof is the following proposition proved by Erdős [5], whose idea has its origin in [4].

Proposition 3.1. *For all $1 \le i \le 12k$, G does not have a cycle of length $n - i$ containing W for which all the vertices not in this cycle have degree at least $13k$.*

Proof. Assume that C is the vertex set of a cycle given as above and let $X = V(G) \setminus C$. We will show that there exists a cycle of length $|C|+1$ which contains C. By repeatedly applying the same argument, we can show the existence of a cycle of length $n - 1$. Since this contradicts our hypothesis, we can conclude that G cannot contain a cycle as above.

Consider a vertex $x \in X$. Since $|X| \le 12k$ and $\mathrm{d}(x) \ge 13k$, we know that the number of neighbors of x in C is at least k. Without loss of generality, let $C = \{1, 2, \dots, n - i\}$, and assume that the vertices are labeled in the order in which they appear on the cycle. Let $w_1, \dots, w_k$ be distinct neighbors of x in C. Then since G has independence number less than k, there exists two vertices $w_i - 1, w_j - 1$ which are adjacent (subtraction is modulo $n - i$). Then G contains a cycle $x, w_i, w_i + 1, \dots, w_j - 1, w_i - 1, w_i - 2, \dots, w_j, x$ of length $n - i + 1$. $\square$

In view of Proposition 3.1, we make the following definition.

Definition 3.2. Let a *contradicting cycle* be a cycle containing W, of length $n - i$ for some $1 \le i \le 12k$, for which all the vertices not in this cycle have degree at least $13k$.

Thus Proposition 3.1 is equivalent to saying that G does not contain a contradicting cycle (under the assumption that G does not contain a cycle of length $n - 1$). By considering several cases, we will show that there always exists a contradicting cycle, from which we can deduce a contradiction on our assumption that there is no cycle of length $n - 1$ in G. The next simple proposition will provide a set-up for this argument.

Proposition 3.3. *G contains at most $13k^2$ vertices of degree less than $13k$.*

Proof. Assume that there exists a set U of at least $13k^2 + 1$ vertices of degree less than $13k$, and let $G' \subset G$ be the subgraph of G induced by U. Take a vertex of G' of degree less than $13k$, remove it and all its neighbors from G', and repeat the process. This produces an independent set of size at least $\lceil (13k^2 + 1)/13k \rceil = k + 1$ which is a contradiction. $\square$

Assume that we are given a Hamilton cycle of G. Place the vertices of G on a circle in the plane according to the order they appear in the Hamilton cycle and label the vertices by elements in $[n]$ accordingly. Consider the $40k^2$ intervals $[1 + (i-1)\lfloor \frac{n}{40k^2} \rfloor, i\lfloor \frac{n}{40k^2} \rfloor]$ for $i = 1, \dots, 40k^2$ consisting of consecutive vertices on the cycle. Take the intervals which only consist of vertices not in W of degree at least $13k$. Let t be the number of such intervals and let $I_1, I_2, \dots, I_t$ be these intervals (see Fig. 1). By Proposition 3.3, the number of intervals which contain a vertex from W or of degree less than $13k$ is at most $|W| + 13k^2$, and therefore

$$t \ge 40k^2 - 13k^2 - |W| \ge 7k^2.$$

For each interval I_j, let I'_j be the set of first at most $k + 1$ odd vertices in it (thus I'_j is the set of all odd vertices in I_j if $|I_j| \le 2(k + 1)$). If there exists an edge inside I'_j then since I'_j lies in an interval of length at most $2k + 2$, we can find a contradicting cycle. Therefore I'_j is an independent set of size at least $\min\{k + 1, \lfloor \frac{n}{80k^2} \rfloor\}$. However, since the independence number of the graph is at most k, the first case $|I'_j| = k + 1$ gives us a contradiction. Therefore, we may assume that $|I'_j| \le k$, and thus I'_j lies in an interval of length at most $2k$.

Consider an auxiliary graph H on the vertex set $[t]$ so that i, j are adjacent if and only if there exists an edge between I'_i and I'_j. Furthermore, color the edges of H into three colors according to the following rule (see Fig. 1).

(i) Red if there exists $x_1, x_2 \in I'_i, y_1, y_2 \in I'_j$ such that $x_1 < x_2, y_1 < y_2$ and x_1 is adjacent to y_1, and x_2 is adjacent to y_2.
(ii) Blue if not colored red, and there exists $x_1, x_2 \in I'_i, y_1, y_2 \in I'_j$ such that $x_1 < x_2, y_1 < y_2$ and x_1 is adjacent to y_2, and x_2 is adjacent to y_1.
(iii) Green if not colored red nor blue.

A red edge in the graph H will give a cycle $x_1 - y_1 - x_2 - y_2 - x_1$, see Fig. 2. The length of the cycle is at least $n - 4k$ since each I'_i lies in an interval of length at most $2k$, and is at most $n - 2$since there

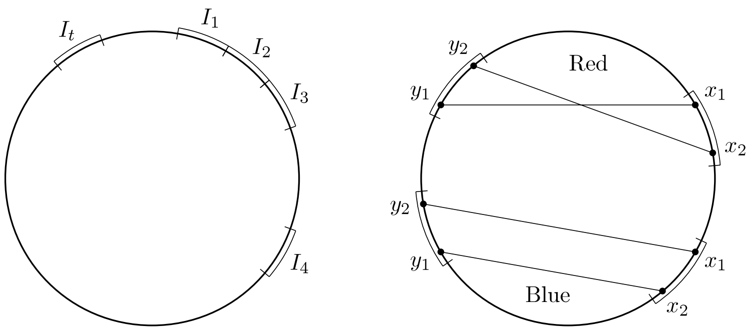

Fig. 1. Intervals I_i, and the type of edges in the graph H.

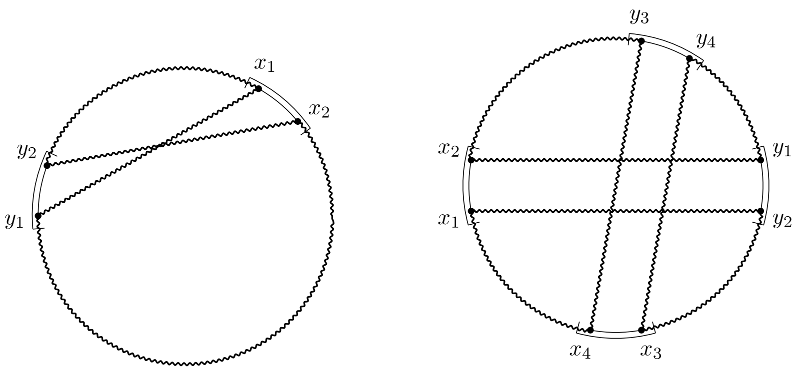

Fig. 2. Contradicting cycles for a red edge, and two crossing blue edges in H.

always exist vertices between x_1, x_2 and between y_1, y_2. Moreover, the cycle contains the set W since W does not intersect the intervals I_i. Therefore it is a contradicting cycle. Thus we may assume that there does not exist red edges in H.

Consider the following drawing of the subgraph of H induced by the blue edges. First place all the vertices of the graph G on the cycle along the given order. A vertex of H, which corresponds to an interval I_i, will be placed on the circle in the middle of the interval I_i. Draw a straight line between I_i and I_j if there is a blue edge. Assume that there exists a crossing in this drawing. Then this gives a situation as in Fig. 2 which gives the cycle $x_1 - y_2 - x_3 - y_4 - y_1 - x_2 - y_3 - x_4 - x_1$. This cycle has length at least $n - 4 \cdot 2k \geq n - 8k$ and at most $n - 4$, hence is a contradicting cycle.

Therefore, the subgraph of H induced by blue edges form a planar graph. This implies that there exists a subset of $[t]$ of size at least $t/5$ which does not contain any blue edge (note that here we use the fact that every planar graph is 5-colorable). By slightly abusing notation, we will only consider these intervals, and relabel the intervals as $I_1, \ldots, I_s$ where $s \geq t/5 > k^2$.

Lemma 3.4. *Let $a_1, \ldots, a_p \in [s]$ be distinct integers and let $X_{a_i} \subset I'_{a_i}$ for all i. Then $X_{a_1} \cup \cdots \cup X_{a_p}$ contains an independent set of size at least*

$$\sum_{i=1}^{p} |X_{a_i}| - \frac{p(p-1)}{2}.$$

Proof. The proof of this lemma relies on a fact about green edges in the auxiliary graph H. Assume that there exists a green edge between i and j in H. Then by the definition, since the edge is neither red nor blue, we know that there is no matching in G of size 2 between I'_i and I'_j. Therefore there exists a vertex v which covers all the edges between I'_i and I'_j.

Now consider the following process of constructing an independent set J. Take $J = \emptyset$ in the beginning. At step i, add X_{a_i} to the set J. By the previous observation, for $j < i$, all the edges between X_{a_i} and X_{a_j} can be deleted by removing at most one vertex (either from X_{a_i} or X_{a_j}). Therefore $J \cup X_{a_i}$ can be made into an independent set by removing at most $i - 1$ vertices. By iterating the process, we can obtain an independent set of size at least $\sum_{i=1}^{p} \left(|X_{a_i}| - (i-1)\right) \geq \sum_{i=1}^{p} |X_{a_i}| - \frac{p(p-1)}{2}$. □

Remark. As mentioned before, this lemma already implies a weaker version of Theorem 2.1 where the bound is replaced by $n = 240k^{5/2}$. To see this, assume that we have a graph on at least $240k^{5/2}$ vertices. Take $X_i = I'_i$ for $i = 1, \ldots, \lceil k^{1/2} \rceil$ in this lemma and notice that $|I'_i| \geq \min\{k+1, \lfloor 3k^{1/2} \rfloor\}$. As we have seen before, $|I'_i| = k + 1$ cannot happen. On the other hand, $|I'_i| = \lfloor 3k^{1/2} \rfloor$ implies the existence of an independent set of size at least

$$\lfloor 3k^{1/2} \rfloor \cdot \lceil k^{1/2} \rceil - \frac{\lceil k^{1/2} \rceil (\lceil k^{1/2} \rceil - 1)}{2} \geq (3k^{1/2} - 1)k^{1/2} - \frac{k + k^{1/2}}{2} > k,$$

which gives a contradiction.

4. Proof of Theorem 2.1

In this section, we will prove Theorem 2.1 which says that there exists a constant c such that every Hamiltonian graph on $n \geq ck^{7/3}$ with $\alpha(G) \leq k$ contains a cycle of length $n - 1$. We will first focus on proving the following relaxed statement: there exists k_0 such that for $k \geq k_0$, every Hamiltonian graph on $n \geq 960k^{7/3}$ vertices with $\alpha(G) \leq k$ contains a cycle of length $n - 1$. Note that for the range $k < k_0$, since there exists a constant c' such that $c'k^{7/3} \geq 240k^{5/2}$, by the remark at the end of the previous section, the bound $n \geq c'k^{7/3}$ will imply pancyclicity. Therefore by taking $\max\{960, c'\}$ as our final constant, the result we prove in this section will in fact imply Theorem 2.1. By relaxing the statement as above, we may assume that k is large enough. This will simplify many calculations. In particular, it allows us to ignore the floor and ceiling signs in this section.

Now we prove the above relaxed statement using the tools we developed in the previous section. Assume that $n \geq 960k^{7/3}$ and k is large enough. Recall that we have independent sets $I'_1, \ldots, I'_s$ such that $s > k^2$ and $|I'_i| \geq \lfloor \frac{n}{80k^2} \rfloor \geq 12k^{1/3}$ for all i. For each i, let M_i and L_i be the smaller $|I'_i|/2$ vertices and larger $|I'_i|/2$ vertices of I'_i in the cycle order given in the previous section, and call them as the *main set* and *leftover set*, respectively. Note that M_i and L_i both have size at least $6k^{1/3}$. For a vertex v, call a set M_j (or an index j) as a *neighboring main set* of v if v contains a neighbor in M_j.

Lemma 4.1. *There exists a subcollection of indices $S \subset [s]$ such that the following holds. For every $i \in S$, the set M_i contains at least $3k^{1/3}$ vertices which each have at least k neighboring main sets whose indices lie in S.*

Proof. In order to find the set of indices S described in the statement, consider the process of removing the main sets which do not satisfy the condition one by one. If the process ends before we run out of sets, then the remaining indices will satisfy the condition.

Let $J = \emptyset$. Pick the first set M_i which has been removed. It contains at most $3k^{1/3}$ vertices which have at least k neighboring main sets. Since there are at least $6k^{1/3}$ vertices in M_i, we can pick $3k^{1/3}$ vertices in M_i which have less than k neighboring main sets and add them to J. For each such vertex added to J, remove all the neighboring main sets of it. In this way, at each step we will increase the size of J by $3k^{1/3}$ and remove at most $1 + (k-1) \cdot 3k^{1/3}$ main sets. Now pick the first main sets among the remaining ones, and repeat what we have done to further increase J.

Assume that in the end, there are no remaining sets (if this is not the case, then we have found our set S). Note that J is an independent set by construction, and since $s > k^2$, the size of it will be at least

$$3k^{1/3} \cdot \frac{k^2}{1 + 3k^{1/3} \cdot (k-1)} > k.$$

This gives a contradiction and concludes the proof since the independence number of the graph is at most k. □

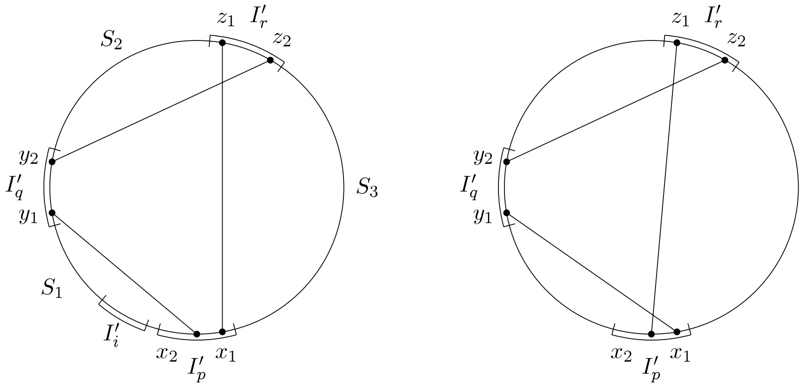

Fig. 3. Semi-triangles of Type A and Type B respectively.

From now on we will only consider sets which have indices in S. Let a *semi-triangle* be a sequence of three indices (p, q, r) in S which lies in clockwise order on the cycle, and satisfies either one of the following two conditions (see Fig. 3).

(i) Type A: there exists $x_1, x_2 \in I'_p, y_1, y_2 \in I'_q, z_1, z_2 \in I'_r$ such that $x_1 < x_2, y_1 < y_2, z_1 < z_2$ and $\{x_1, z_1\}, \{x_2, y_1\}, \{y_2, z_2\} \in E(G)$. Moreover, there exists at least one set I'_i with $i \in S$ in the arc starting at p and ending at q (traverse clockwise).

(ii) Type B: there exists $x_1, x_2 \in I'_p, y_1, y_2 \in I'_q, z_1, z_2 \in I'_r$ such that $x_1 < x_2, y_1 < y_2, z_1 < z_2$ and $\{x_1, y_1\}, \{x_2, z_1\}, \{y_2, z_2\} \in E(G)$.

Note that (p, q, r) being a semi-triangle does not necessarily imply that (q, r, p) is also a semi-triangle. Semi-triangles are constructed so that 'chords' intersect in a predescribed way. This arrangement of chords will allow us to find contradicting cycles, once we are given certain semi-triangles in our graph. As an instance, one can see that a semi-triangle of Type B contains a cycle $x_1-y_1-x_2-z_1-y_2-z_2-x_1$, see Fig. 3. Recall that each set I'_j lies in a consecutive interval of length at most $2k$, and thus the length of the cycle is at least $n - 6k$. Moreover, since each set I'_j is defined as the set of odd vertices in I_j, the length of the cycle is at most $n - 3$ (it must miss vertices between x_1 and x_2, y_1 and y_2, and z_1 and z_2). Finally, since all the intervals I_j do not intersect W, the cycle is a contradicting cycle. Therefore we may assume that no such semi-triangle exists. We will later see that one can find a contradicting cycle even if Type A semi-triangles intersect in a certain way.

The next lemma shows that the graph G contains many semi-triangles of Type A.

Lemma 4.2. *Let M_p be a fixed main set, and let $S' \subset S$ be a set of indices such that at least $k^{1/3}$ vertices in M_p have at least $k/3$ neighboring main sets in S'. Then there exists a semi-triangle (p, q_1, q_2) of Type A such that $q_1, q_2 \in S'$.*

Proof. Let M_p and S' be given as in the statement. Among the sets M_x with indices in S', let M_i be the closest one to M_p in the clockwise direction. To make sure that we get a semi-triangle of Type A, we will remove i from S' and only consider the set $S'' = S' \setminus \{i\}$. Thus we will have $k/3 - 1$ neighboring main sets in S'' for each of the given vertices.

Arbitrarily select $k^{1/3}$ vertices in M_p which have at least $k/3 - 1$ neighboring main sets in S''. Since for large k we have $k^{1/3} \cdot k^{1/3} \leq (k/3) - 1$, we can assign $k^{1/3}$ neighboring main sets to each selected vertex so that the assigned sets are distinct for different vertices. Then for a selected vertex $v \in M_p$, let J_v be the union of the leftover sets L_x corresponding to the $k^{1/3}$ main sets M_x assigned to v. Since each set L_x has size at least $6k^{1/3}$, by Lemma 3.4, J_v contains an independent set of size at least $k^{1/3} \cdot 6k^{1/3} - k^{2/3}/2 \geq (11/2)k^{2/3}$. Denote this independent set by J'_v.

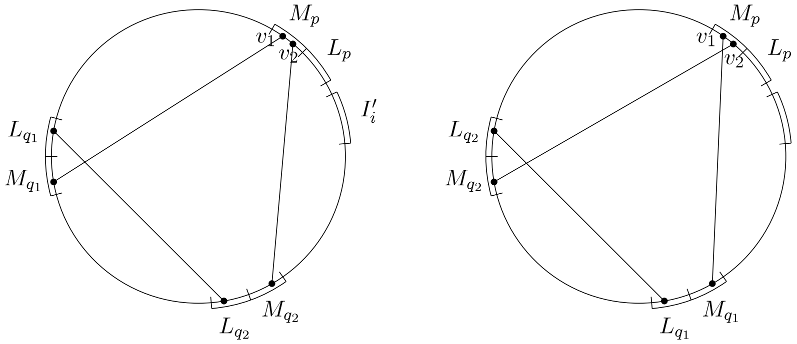

Fig. 4. Constructing semi-triangles, Type A and B, respectively.

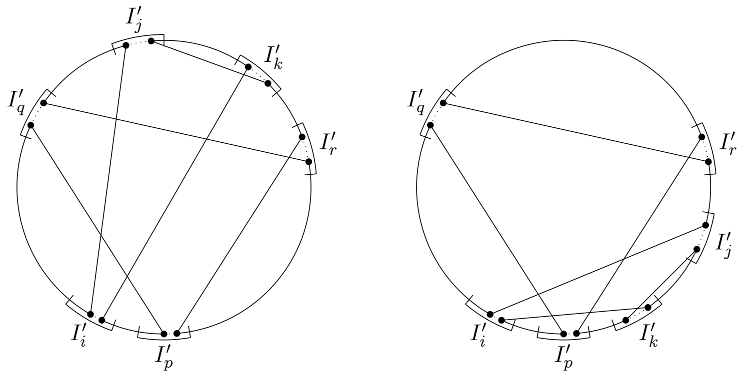

Fig. 5. Two overlapping semi-triangles which give a contradicting cycle.

Since the sets J'_v are disjoint for different vertices, we have $|\bigcup_{v\in M_p} J'_v| \geq (11/2)k^{2/3} \cdot k^{1/3} \geq k+1$. Therefore, by the restriction on the independence number there exists an edge between J'_{v_1} and J'_{v_2} for two distinct vertices v_1 and v_2 (the edge cannot be within one set J'_v since J'_v is an independent set for all v). Let M_{q_1} be the main set in which the neighborhood of v_1 lies in, and similarly define M_{q_2} so that there exists an edge between L_{q_1} and L_{q_2}. Depending on the relative position of M_{q_1}, M_{q_2} and M_p on the cycle, the edge $\{v_1, v_2\}$ will give rise to a semi-triangle of Type A or B, see Fig. 4 (note that the additional condition for semi-triangle of Type A is satisfied because we removed the index i in the beginning). Since we know that there does not exist a semi-triangle of Type B, it should be a semi-triangle of Type A. $\square$

In particular, Lemma 4.2 implies the existence of a semi-triangle (p, q, r) of Type A. Let the *length* of a Type A semi-triangle be the number of sets I'_i with $i \in S$ in the arc that starts at p and ends at q (traverse clockwise). Among all the semi-triangles of Type A consider the one which has minimum length and let this semi-triangle be (p, q, r). By definition, every semi-triangle has length at least 1, and thus we know that there exists an index in S in the arc starting at p and ending at q (traverse clockwise). Let $i \in S$ be such an index which is closest to p (see Fig. 5).

Now consider the set of indices $S_1, S_2, S_3 \subset S$ such that S_1 is the set of indices between p and q, S_2 is the set of indices between q and r, and S_3 is the set of indices between r and p along the circle, all in clockwise order (see Fig. 3). By the pigeonhole principle and how we constructed the indices S in Lemma 4.1, there exists at least one set out of S_1, S_2, S_3 such that at least $k^{1/3}$ vertices of M_i have at least $k/3$ neighboring main sets inside it.

If this set is S_1, then by Lemma 4.2 there exists a Type A semi-triangle which is completely contained in the arc between p and q, and thus has smaller length than the semi-triangle (p, q, r). Since this is impossible, we may assume that the set mentioned above is either S_2 or S_3. In either of the cases, by Lemma 4.2 we can find a Type A semi-triangle (i, j, k) which together with (p, q, r) will give a contradicting cycle, see Fig. 5 (recall that each set I_i' lies in a consecutive interval of length at most $2k$, and thus the length of this cycle is at least $n - 12k$ and at most $n - 6$). This shows that the assumption we made at the beginning on G not containing a cycle of length $n - 1$ cannot hold. Therefore we have proved Theorem 2.1.

5. Concluding remarks

In this paper we proved that there exists an absolute constant c such that if G is a Hamiltonian graph with $n \geq ck^{7/3}$ vertices and $\alpha(G) \leq k$, then G is pancyclic. The main ingredient of the proof was Theorem 1.2, which partially answers a question of Keevash and Sudakov, and tells us that under the same condition as above, G contains a cycle of length $n - 1$. It seems very likely that if one can answer Keevash and Sudakov's question, even for $n = \Omega(k^2)$, then one can also resolve Erdős' question, by using a similar approach to that of Section 2 (see Theorem 2.1, which is a strengthened version of Theorem 1.2).

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