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A NEW LOWER BOUND FOR A RAMSEY-TYPE PROBLEM

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Let $3 \le r < s$ be fixed integers and let G be a graph on n vertices not containing a complete graph on s vertices. The main aim of this paper is to provide a new lower bound on the size of the maximum subset of G without a copy of complete graph K_r. Our results substantially improve previous bounds of Krivelevich and Bollobás and Hind.

1. Introduction

The Ramsey number $R(s,t)$ is the smallest integer n such that every graph on n vertices contains either a clique of size s or an independent set of size t. The problem of determining or estimating Ramsey numbers is one of the central problems in Combinatorics and it received a considerable amount of attention, see, e.g., [6]. A more general function was first considered (for a special case) by Erdős and Gallai in [4]. Suppose $2 \le r < s < n$ are integers and let G be a K_s-free graph. Denote by $f_r(G)$ the maximum cardinality of a subset of vertices of G that contains no copy of K_r, and define

$$f_{r,s}(n) = \min f_r(G),$$

where the minimum is taken over all K_s-free graphs on n vertices.

By definition, it is easy to see that for $r = 2$, we have that $f_{2,s}(n) < t$ if and only if the Ramsey number $R(s,t)$ satisfies $R(s,t) > n$. Therefore

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the problem of determining the function $f_{r,s}(n)$ extends that of determining Ramsey numbers.

Erdős and Rogers [5] considered the case of fixed $s, r = s-1$ and n tending to infinity and proved that there exist a constant $\epsilon(s) = \Theta(1/(s^4 \ln s))$ and a K_s-free graph G of order n, such that every induced subgraph of G of order $n^{1-\epsilon(s)}$ contains a copy of K_{s-1}, i.e., $f_{s-1,s}(n) \le n^{1-\epsilon(s)}$. The next step came about thirty years later when Bollobás and Hind [2] improved the results of Erdős and Rogers and obtain the following bounds

$$n^{\frac{1}{s-r+1}} \le f_{r,s}(n) \le n^{\frac{s-3}{s-2}+\frac{2}{(s+1)(s-2)}+\epsilon}.$$

These bounds were improved again later by Krivelevich in [7], [8] and the currently best known bounds are

$$c_1 n^{\frac{1}{s-r+1}} (\ln \ln n)^{1-\frac{1}{s-r+1}} \le f_{r,s}(n) \le c_2 n^{\frac{r}{s+1}} (\ln n)^{\frac{1}{r-1}},$$

where c_1, c_2 are positive constants depending only on r and s. Most of these results were obtained using probabilistic method. Recently, two nice explicit constructions which provide upper bound on $f_{r,s}(n)$ were obtained by Alon and Krivelevich in [1].

As one can see, the upper bound on $f_{r,s}(n)$ attract a lot of attention and improved considerably during the last forty years. On the other hand not much progress was made on obtaining a good lower bounds. As was pointed out by Bollobás and Hind [2], no essentially nontrivial lower bound was known. In this paper we want to present first such bound, which give a significant improvement for all $r \ge 3$ and $s > r+1$. We will summarize our results in the next two theorems. The first one deals with the case $r = 3$.

Theorem 1.1. *For all integer $k \ge 0$,*

$$f_{3,3+k}(n) \ge \Omega\Big(n^{a_k} \ln^{b_k + o(1)} n\Big),$$

where sequences a_k and b_k are given by the following formulas

$$a_k = \frac{3}{2} \cdot \frac{1}{k + 11/6 - 1/3(-1/2)^k}, \qquad b_k = \frac{1}{2} \cdot \frac{k + 4/3 - 4/3(-1/2)^k}{k + 11/6 - 1/3(-1/2)^k}.$$

Next we present a table which compares quantitatively the results of Theorem 1.1 with the best previously known upper and lower bounds.

	$f_{3,4}(n)$	$f_{3,5}(n)$	$f_{3,6}(n)$	$f_{3,s}(n)$, for large s
Old lower bound	$\Omega\left(n^{\frac{1}{2}}(\ln\ln n)^{\frac{1}{2}}\right)$	$\Omega\left(n^{\frac{1}{3}}(\ln\ln n)^{\frac{2}{3}}\right)$	$\Omega\left(n^{\frac{1}{4}}(\ln\ln n)^{\frac{3}{4}}\right)$	$\Omega\left(n^{\frac{1}{s-2}}(\ln\ln n)^{\frac{s-3}{s-2}}\right)$
New lower bound	$\Omega\left(n^{\frac{1}{2}}\ln^{\frac{1}{2}+o(1)}n\right)$	$\Omega\left(n^{\frac{2}{5}}\ln^{\frac{2}{5}+o(1)}n\right)$	$\Omega\left(n^{\frac{4}{13}}\ln^{\frac{6}{13}+o(1)}n\right)$	$\Omega\left(n^{\frac{3+o(1)}{2s}}\ln^{\frac{1}{2}+o(1)}n\right)$
Best upper bound	$\Omega\left(n^{\frac{3}{5}}\ln^{\frac{1}{2}}n\right)$	$\Omega\left(n^{\frac{1}{2}}\ln^{\frac{1}{2}}n\right)$	$\Omega\left(n^{\frac{3}{7}}\ln^{\frac{1}{2}}n\right)$	$\Omega\left(n^{\frac{3}{s+1}}\ln^{\frac{1}{2}}n\right)$

The second theorem treats the general case of $r \geq 4$.

Theorem 1.2. *Let $r \geq 4$ be a fixed integer and let $k \geq 0$. Then*

$$f_{r,r+k}(n) \geq \Omega\left(n^{a_k(r)}\right),$$

where $a_k(r)$ satisfies the following recurrence relation

$$a_i = 1, \qquad -(r-2) \leq i \leq 0 \quad \textit{and} \quad \frac{1}{a_{k+1}} = 1 + \frac{1}{r-1}\sum_{i=0}^{r-2}\frac{1}{a_{k-i}}, \quad \forall k \geq 0.$$

Moreover, $a_k(r)$ has the following asymptotic behavior,

$$a_k(r) = \frac{r}{2}\cdot\frac{1}{k+(5r-4)/6+h_k(r)}, \quad \textit{where} \quad \lim_{k\to\infty} h_k(r) = 0.$$

For example we obtain that for $s=r+2$, $f_{r,s}(n) \geq \Omega(n^{\frac{r-1}{2r-1}})$, improving the previous lower bound of $\Omega(n^{1/3}(\ln\ln n)^{2/3})$. In addition, the above theorem closes considerably the gap between lower and upper bounds and together with result of Krivelevich [8] implies the following corollary on behavior of $f_{r,s}(n)$.

Corollary 1.3. *For a fixed integer $r \geq 3$ and $s > r$,*

$$\Omega\left(n^{\frac{r}{2s}+O(1/s^2)}\right) \leq f_{r,s}(n) \leq O\left(n^{\frac{r}{s+1}}(\ln n)^{\frac{1}{r-1}}\right).$$

It is also worth to mention here that using a similar argument as in proof of Theorem 1.1, one can also improve the result of Theorem 1.2 by logarithmic factor. For the sake of clarity of presentation we will leave the details of this tedious computations to the interested reader.

The rest of this short paper is organized as follows. In the next section we obtain bounds on the largest triangle-free subset in K_s-free graphs. We consider this case separately since for $r=3$ we get a relatively simple recursive relation, which we can solve explicitly. We treat the general case when $r>3$ in Section 4, where we provide the proof of Theorem 1.2. Last section of the paper is devoted to concluding remarks and relevant open problems.

Finally we close this section with some conventions and notation. An *r-uniform hypergraph* H is an ordered pair $H=(V,E)$, where V is a finite non-empty set (the set of *vertices*), and E is a collection of distinct r-subsets of V (the set of *edges*). Thus a 2-uniform hypergraph is just a graph. A subset $I\subseteq V(H)$ is called *independent* if I does not contain any edge of H. The maximal size of an independent set in H is called the *independence number* of H and is denoted by $\alpha(H)$. Given an r-uniform hypergraph $H=(V,E)$ and a subset $T\subset V$, we denote by

$$N_H(T) = \{\cup e \mid T \subset e, e \in E(H)\} - T.$$

We will frequently write simply $N(T)$, when it is clear from the context what hypergraph we are talking about. Note that if H is graph, then $N(T)$ is simply the set of vertices of H adjacent to all the vertices in T. We denote by ln the natural logarithm. Throughout the paper, we omit the floor and ceilings signs for the sake of convenience.

2. Large triangle-free subgraphs

In this section we obtain results on the largest triangle-free subset in K_s-free graphs. To do so we need a lower bound on the size of the maximum independent set in 3-uniform hypergraphs. Let us first recall some terminology. A *2-cycle* in a r-uniform hypergraph is a pair of edges with intersection of size at least two. We call a hypergraph H to be *uncrowded* if it does not contains any 2-cycles. The following proposition is a corollary of more general result on the size of independent set in uncrowded hypergraphs, obtained by Duke, Lefmann and Rödl [3].

Proposition 2.1. *Let H be an 3-uniform hypergraph on n vertices and with m edges. Let $(m/n)^{1/2}\le t$ and suppose there exist an $\epsilon>0$ such that the number of edges containing any fixed pair of vertices of H is at most $t^{1-\epsilon}$. Then H contains an independent set of size*

$$\alpha(H) \ge \Omega\left(\frac{n}{t}\ln^{1/2} t\right).$$

Next we need the result of Shearer [10] on the size of independent set in K_s-free graphs.

Proposition 2.2. *Let G be a K_s-free ($s \geq 4$ fixed) graph on n vertices with maximum degree d. Then the independence number of G has size at least*

$$\alpha(G) \geq \Omega\left(\frac{n}{d}\Big(\frac{\ln d}{\ln\ln d}\Big)\right).$$

Having finished all the necessary preparations, we are now ready to complete the proof of our first result.

Proof of Theorem 1.1. We will prove the statement of the theorem by induction on k. First note that, by definition, the sequences a_k, b_k are the unique solutions of the following recurrence relations

$$a_0 = 1, \qquad a_1 = 1/2, \qquad \frac{1}{a_{k+1}} = 1 + \frac{1}{2}\left(\frac{1}{a_k} + \frac{1}{a_{k-1}}\right)$$

and

$$b_0 = 0, \qquad b_1 = 1/2, \qquad \frac{b_{k+1}}{a_{k+1}} = \frac{1}{2} + \frac{1}{2}\left(\frac{b_k}{a_k} + \frac{b_{k-1}}{a_{k-1}}\right).$$

Let G be a graph on n vertices, not containing a clique of size $k+3$. If $k=0$, then clearly G is triangle free itself. Hence $f_{3,3}=n$ and $a_0=1$, $b_0=0$. Next suppose $k=1$, i.e., G is K_4-free. If G contains a vertex v of degree $d(v) \geq d = n^{1/2}\ln^{1/2} n$, then by definition, the neighborhood $N(v)$ induces a triangle-free subgraph of G of size $n^{1/2}\ln^{1/2} n$. Otherwise, if all vertices of G have degree less than d, then by Proposition 2.2 it contains an independent set of size

$$\Omega\left(\frac{n}{d}\Big(\frac{\ln d}{\ln\ln d}\Big)\right) = \Omega\Big(n^{1/2}\ln^{1/2+o(1)} n\Big).$$

This implies that $a_1 = 1/2$ and $b_1 = 1/2$.

Next we assume that the assertion of the theorem is true for all $k' \leq k$ and will prove it for $k+1$. Let G be graph not containing a clique of size $(k+1)+3 = k+4$ and let v be a vertex of maximal degree in G. Note that the graph induce by $N(v)$ has no K_{k+3} subgraph. Therefore if $d(v) \geq n^{\frac{a_{k+1}}{a_k}}(\ln n)^{\frac{b_{k+1}-b_k}{a_k}}$, then by induction hypothesis $G[N(v)]$ contains a triangle-free subset of size

$$\begin{aligned}\Omega\Big(d^{a_k}(v)\big(\ln d(v)\big)^{b_k+o(1)}\Big) &= \Omega\Big(n^{a_{k+1}}(\ln n)^{b_{k+1}-b_k}\big(\ln d(v)\big)^{b_k+o(1)}\Big)\\ &= \Omega\Big(n^{a_{k+1}}(\ln n)^{b_{k+1}+o(1)}\Big).\end{aligned}$$

Here we use the fact that $\ln d(v) = \Omega(\ln n)$.

Also note that for every edge (u,v) in G, the set of vertices $N(u,v)$ contains no K_{k+2}. Thus if there exist an edge (u,v) such that $|N(u,v)| \geq$

$n^{\frac{a_{k+1}}{a_{k-1}}}(\ln n)^{\frac{b_{k+1}-b_{k-1}}{a_{k-1}}}$, then again by induction hypothesis $G[N(u,v)]$ contains an induced triangle-free subgraph of size

$$\begin{aligned}\Omega\Big(|N(u,v)|^{a_{k-1}}(\ln|N(u,v)|)^{b_{k-1}+o(1)}\Big)\\ &= \Omega\Big(n^{a_{k+1}}(\ln n)^{b_{k+1}-b_{k-1}}(\ln|N(u,v)|)^{b_{k-1}+o(1)}\Big)\\ &= \Omega\Big(n^{a_{k+1}}(\ln n)^{b_{k+1}+o(1)}\Big).\end{aligned}$$

Finally, suppose that

$$\forall v \in V(G), \quad d(v) \le n^{\frac{a_{k+1}}{a_k}}(\ln n)^{\frac{b_{k+1}-b_k}{a_k}}$$

and

$$\forall (u,v) \in E(G), \quad |N(u,v)| \le n^{\frac{a_{k+1}}{a_{k-1}}}(\ln n)^{\frac{b_{k+1}-b_{k-1}}{a_{k-1}}}.$$

Let H be a 3-uniform hypergraph whose vertices are the vertices of G and whose edges are all copies of K_3 contained in graph G. Clearly by definition an independent set in H corresponds to induced triangle-free subgraph of G. Denote by

$$t = \sqrt{n^{\frac{a_{k+1}}{a_k}}(\ln n)^{\frac{b_{k+1}-b_k}{a_k}} \cdot n^{\frac{a_{k+1}}{a_{k-1}}}(\ln n)^{\frac{b_{k+1}-b_{k-1}}{a_{k-1}}}}.$$

Then an easy calculation shows that the number m of the triangles in graph G is at most

$$m \le \frac{n}{6} \cdot \max_v d(v) \cdot \max_{(u,v)\in E(G)} |N(u,v)| = \frac{n}{6}\, t^2.$$

Also note that $a_{k+1}/a_k > a_{k+1}/a_{k-1}$, since it is straightforward to check that the sequence a_k is strictly decreasing. Therefore there exist an $\epsilon > 0$ such that the maximum number of triangles containing any fixed pair of vertices of G is at most $n^{\frac{a_{k+1}}{a_{k-1}}}(\ln n)^{\frac{b_{k+1}-b_{k-1}}{a_{k-1}}} \le t^{1-\epsilon}$. Now by Proposition 2.1, the hypergraph H contains an independent set (i.e., triangle free subset of G) of size

$$\begin{aligned}\Omega\left(\frac{n}{t}\ln^{1/2} t\right) &= \Omega\left(n^{1-\frac{a_{k+1}}{2}\left(\frac{1}{a_k}+\frac{1}{a_{k-1}}\right)}(\ln n)^{\frac{1}{2}-\frac{b_{k+1}-b_k}{2a_k}-\frac{b_{k+1}-b_{k-1}}{2a_{k-1}}}\right)\\ &= \Omega\Big(n^{a_{k+1}}\ln^{b_{k+1}} n\Big).\end{aligned}$$

Here we use the following equalities, which easily follow from recurrence relations of a_k and b_k.

$$a_{k+1} = 1 - \frac{a_{k+1}}{2}\left(\frac{1}{a_k} + \frac{1}{a_{k-1}}\right) \qquad \text{and}$$

$$b_{k+1} = \frac{1}{2}\left(1 - b_{k+1}\Big(\frac{1}{a_k} + \frac{1}{a_{k-1}}\Big) + \frac{b_k}{a_k} + \frac{b_{k-1}}{a_{k-1}}\right).$$

This completes the proof of induction step and the proof of the theorem. ∎

3. Bounds on $f_{r,s}(n)$, for $r \geq 4$

In this section we obtain a lower bound on the size of maximal K_r-free subgraph of K_s-free graph for all $s > r \geq 4$. In order to clarify the presentation we will only make efforts to obtain the best possible exponent of n in Theorem 1.2. Our bounds can be easily improved by a logarithmic factor for all particular values r and s using Proposition 2.2 and results on uncrowded hypergraphs, similarly as we did in previous section. We omit these details here.

In the proof of Theorem 1.2 we will need the following well known Turán-type estimate on the size of maximum independent set in r-uniform hypergraphs. We include its short proof for the sake of completeness.

Lemma 3.1. *Let $H = (V, E)$ be an r-uniform hypergraph on n vertices and with $m \geq n/r$ edges. Then H contains an independent set of size*

$$\alpha(H) \geq \Omega\left(\frac{n^{\frac{r}{r-1}}}{m^{\frac{1}{r-1}}}\right).$$

Proof. Choose a random subset V_0 of V by taking each $v \in V$ into V_0 independently and with probability $p = (n/(rm))^{\frac{1}{r-1}}$. Define random variables X, Y by letting X be the number of vertices in V_0 and letting Y be the number of edges spanned by V_0. Then by linearity of expectation there exists a set V_0, for which

$$X - Y \geq np - mp^r = \Omega\left(\frac{n^{\frac{r}{r-1}}}{m^{\frac{1}{r-1}}}\right).$$

Fix such a set V_0 and for every edge e spanned by V_0 delete from V_0 an arbitrary vertex of e. This produces an independent set of size guaranteed by assertion of the lemma. ∎

Next we need the following simple estimate on the number of edges in hypergraph H, which we will use later in the proof.

Lemma 3.2. *Let $H=(V,E)$ be an r-uniform hypergraph on n vertices. Then the number m of edges of H is bounded by*

$$m \le O\left(n \cdot \prod_{t=1}^{r-1} \max_{T\subset V, |T|=t} |N_H(T)|\right).$$

Proof. We prove this statement by induction on r. If $r=2$, then H is a graph on n vertices and $\max_{T\subset V,|T|=1}|N(T)|$ is just maximum degree of H. In that case the result is obviously true. Next suppose the result is true for any r-uniform hypergraph and let $H=(V,E)$ be an $(r+1)$-uniform hypergraph. Let v be a vertex of maximum degree in H, then it is easy to see that the number of edges in H is at most $m \le nd(v)/(r+1)$. Define a new hypergraph H' with the vertex set $N_H(v)$ and the edge set $E(H')=\{e-\{v\}|v\in e, e\in E(H)\}$. Then, by definition, this hypergraph is r-uniform, has $d(v)$ edges and by induction hypothesis we obtain that

$$d(v) \le O\left(|N_H(v)| \cdot \prod_{t=1}^{r-1} \max_{T'\subset N_H(v), |T'|=t} |N_{H'}(T')|\right).$$

Now to complete the proof note that, by definition, for every $T'\subset N_H(v)$ the set of vertices $N_{H'}(T')$ equals to the set $N_H(T)$ for $T=T'\cup\{v\}$. Therefore we finally obtain

$$m \le O(nd(v)) \le O\left(n\cdot |N_H(v)| \cdot \prod_{t=1}^{r-1} \max_{T'\subset N_H(v), |T'|=t} |N_{H'}(T')|\right)$$
$$\le O\left(n \cdot \prod_{t=1}^{r} \max_{T\subset V, |T|=t} |N_H(T)|\right).$$

This completes the proof of the lemma. ∎

Proof of Theorem 1.2. Let a_k be a sequence which satisfies that

$$(1)\quad a_i = 1, \quad -(r-2)\le i\le 0 \quad \text{and} \quad \frac{1}{a_{k+1}} = 1 + \frac{1}{r-1}\sum_{i=0}^{r-2}\frac{1}{a_{k-i}}, \quad \forall k\ge 0,$$

and let G be a graph on n vertices, not containing a clique of size $r+k$. We prove that G contains a K_r-free subset of size $\Omega(n^{a_k})$ by induction on k.

If $k=0$, then G itself is K_r-free. Therefore $f_{r,r}(n)=n$ and $a_0=1$. Next suppose that our statement is true for all $k'\le k$ and let G be a graph not containing a clique of size $r+(k+1)$.

First, consider the case when $k\ge r-2$. Let T be a subset of vertices of G which form a clique of size $1\le t\le r-1$. Then, by definition, the subgraph of

G induced by the set $N(T)$ contains no clique of size $r+(k+1-t)$. Therefore if $|N(T)| \geq n^{a_{k+1}/a_{k+1-t}}$, then by induction hypothesis $G[N(T)]$ contains a K_r-free set of size

$$\Omega\Big(|N(T)|^{a_{k+1-t}}\Big) = \Omega\Big(n^{a_{k+1}}\Big).$$

Thus we can assume that $|N_G(T)| \leq n^{a_{k+1}/a_{k+1-t}}$ for every clique T in G of size $1 \leq t \leq r-1$.

Let H be a r-uniform hypergraph whose vertices are the vertices of G and whose edges are all copies of K_r contained in graph G. Clearly, by definition, an independent set in H corresponds to induced K_r-free subgraph of G and also $|N_H(T)| \leq |N_G(T)|$ for any clique T in G of size at most $r-1$. Denote by m the number of edges in H. Then by Lemma 3.2 we can bound this number by

$$m \leq O\left(n \cdot \prod_{t=1}^{r-1} \max_{T \subset V, |T|=t} |N_H(T)|\right) \leq O\left(n \cdot \prod_{t=1}^{r-1} \max_{T \text{ is a clique}, |T|=t} |N_G(T)|\right)$$
$$\leq O\left(n^{1+\sum_{t=1}^{r-1} \frac{a_{k+1}}{a_{k+1-t}}}\right).$$

We may also assume that $m \geq n/r$, since otherwise it is easy to see that H contains an independent set of size $\Omega(n)$. Thus, we can apply Lemma 3.1 to show that the hypergraph H contains an independent set of size

$$\Omega\left(\frac{n^{\frac{r}{r-1}}}{m^{\frac{1}{r-1}}}\right) = \Omega\left(n^{1-\frac{a_{k+1}}{r-1}\sum_{t=1}^{r-1}\frac{1}{a_{k+1-t}}}\right) = \Omega\left(n^{1-\frac{a_{k+1}}{r-1}\sum_{i=0}^{r-2}\frac{1}{a_{k-i}}}\right).$$

This completes the proof of the first case, since from the recurrence relation (1) it is easy to see that

$$a_{k+1} = 1 - \frac{a_{k+1}}{r-1}\sum_{i=0}^{r-2}\frac{1}{a_{k-i}}\,.$$

Next consider the case when $k < r-2$. Then, similarly as above, by induction hypothesis we can assume that $|N_G(T)| \leq n^{a_{k+1}/a_{k+1-t}}$ for every clique T in G, but now only of size $1 \leq t \leq k+1$. On the other hand any clique T in G of size larger than $k+1$ contains a sub-clique T_1 of size equal to $k+1$. Since by definition, $N_G(T) \subset N_G(T_1)$ and also $a_i = a_0 = 1$ for $-(r-2) \leq i \leq -1$, we obtain that nevertheless

$$|N_G(T)| \leq |N_G(T_1)| \leq n^{a_{k+1}/a_0} = n^{a_{k+1}/a_{k+1-t}},$$

for all $k+1 \le t \le r-1$. Thus

$$|N_G(T)| \le n^{a_{k+1}/a_{k+1-t}},$$

for every clique T in G of size $1 \le t \le r-1$. Now we can finish the proof of induction step in the same way as in the first case.

Finally we complete the proof of Theorem 1.2 by proving the following lemma about asymptotic behavior of the sequence $x_k = 1/a_k$.

Lemma 3.3. *Let r be a fixed integer and let x_k be a sequence satisfying the following recurrence relation*

$$x_i = 1, \quad -(r-2) \le i \le 0 \quad \text{and} \quad x_{k+1} = 1 + \frac{1}{r-1}\sum_{i=0}^{r-2} x_{k-i}, \quad \forall k \ge 0. \tag{2}$$

Then

$$\lim_{k\to\infty} x_k - \left(\frac{2}{r}k + \frac{5r-4}{3r}\right) = 0.$$

Proof. From the basic theory of solutions of recurrence relations (see, e.g., [9]) we know that the sequence x_k should have the following form

$$x_k = ck + c_1{\alpha_1}^k + \cdots + c_{r-1}{\alpha_{r-1}}^k,$$

where ck also satisfies (2) and $\alpha_i, 1 \le i \le r-1$ are the $r-1$ complex roots of the following equation

$$(r-1)\alpha^{r-1} = \alpha^{r-2} + \alpha^{r-3} + \cdots + 1. \tag{3}$$

Moreover, if root α_i has multiplicity one then c_i is a constant and it is a polynomial in k of degree at most $t-1$ if α_i has multiplicity t. Substituting ck into (2) one can easily check that $c = 2/r$. Taking absolute value on both sides of (3) and assuming that $|\alpha| \ge 1$ we obtain that

$$(r-1)|\alpha|^{r-1} = |\alpha^{r-2} + \alpha^{r-3} + \cdots + 1| \le |\alpha|^{r-2} + \cdots + 1 \le (r-1)|\alpha|^{r-1},$$

with only possibility for equality when $\alpha = 1$. Therefore we obtain that all the roots of (3) except the $\alpha_1 = 1$ are in absolute value strictly less than 1. It is also easy to see that $\alpha_1 = 1$ is a root of (3) with multiplicity one and the rest α_i, $i > 1$ satisfy the equality

$$\frac{(r-1)\alpha^{r-1} - \sum_{j=0}^{r-2}\alpha^j}{\alpha - 1} = \sum_{j=0}^{r-2}(j+1)\alpha^j = 0\,. \tag{4}$$

Let $f(\alpha) = (r-1)\alpha^{r-1} - \sum_{j=0}^{r-2}\alpha^j$ and $f'(\alpha) = (r-1)^2\alpha^{r-2} - \sum_{j=0}^{r-3}(j+1)\alpha^j$. Using (4) we obtain that if $\alpha \ne 1$ is a common root of f and f', then it satisfies

$[(r-1)^2+(r-1)]\alpha^{r-2}=0$ and thus $\alpha=0$. This implies that polynomials f and f' do not have common roots and therefore all $\alpha_i, i\geq 2$ have multiplicity one and all $c_i, i\geq 2$ are constants not depending on k. Denote by $h_k=\sum_{i=2}^{r-1} c_i{\alpha_i}^k$. Then $x_k=(2/r)k+c_1+h_k$ and since for all $i>1$, $|\alpha_i|<1$, we get that h_k tends to zero when k tends to infinity. In addition, (4) implies that for all $k\geq -(r-2)$ the sequence h_k satisfies

$$\sum_{j=0}^{r-2}(j+1)h_{k+j}=\sum_{j=0}^{r-2}(j+1)\sum_{i=2}^{r-1}c_i{\alpha_i}^{k+j}=\sum_{i=2}^{r-1}c_i{\alpha_i}^k\sum_{j=0}^{r-2}(j+1){\alpha_i}^j=0.$$

To finish the proof we need to compute c_1. To do so consider the following expression

$$\begin{aligned}\frac{r(r-1)}{2}&=\sum_{j=0}^{r-2}(j+1)x_{j-(r-2)}\\&=\frac{2}{r}\sum_{j=0}^{r-2}(j+1)(j-(r-2))+c_1\sum_{j=0}^{r-2}(j+1)+\sum_{j=0}^{r-2}(j+1)h_{j-(r-2)}\\&=\frac{2}{r}\left(\sum_{j=0}^{r-2}(j^2+j)-(r-2)\sum_{j=0}^{r-2}(j+1)\right)+\frac{r(r-1)}{2}c_1\\&=-\frac{(r-1)(r-2)}{3}+\frac{r(r-1)}{2}c_1.\end{aligned}$$

Solving this equation we obtain that $c_1=\frac{5r-4}{3r}$. This completes the proof of the lemma and the proof of Theorem 1.2. ∎

4. Concluding remarks

In this paper we obtain new lower bounds on the size of the largest K_r-free subset of graph, which does not contain a copy of a complete graph on s vertices. Our results improve substantially best previously known bounds for all $r\geq 3$ and $s>r+1$. Nevertheless, it is easy to see that the gap between lower and upper bounds in Corollary 1.3 is still relatively large and it would be very interesting to close it.

An additional intriguing problem is to decide if for every $0<\delta<1$, the value of $f_{s-1,s}(n)$ is greater than $n^{1-\delta}$ for sufficiently large s. Note that the exponent of n in the current lower bound is only $1/2$ and our method is not sufficient to improve it.

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