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Bandwidth theorem for random graphs

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ABSTRACT

A graph G is said to have *bandwidth* at most b , if there exists a labeling of the vertices by $1, 2, \dots, n$, so that $|i - j| \leq b$ whenever $\{i, j\}$ is an edge of G . Recently, Böttcher, Schacht, and Taraz verified a conjecture of Bollobás and Komlós which says that for every positive r, Δ, γ , there exists β such that if H is an n -vertex r -chromatic graph with maximum degree at most Δ which has bandwidth at most βn , then any graph G on n vertices with minimum degree at least $(1 - 1/r + \gamma)n$ contains a copy of H for large enough n . In this paper, we extend this theorem to dense random graphs. For bipartite H , this answers an open question of Böttcher, Kohayakawa, and Taraz. It appears that for non-bipartite H the direct extension is not possible, and one needs in addition that some vertices of H have independent neighborhoods. We also obtain an asymptotically tight bound for the maximum number of vertex disjoint copies of a fixed r -chromatic graph H_0 which one can find in a spanning subgraph of $G(n, p)$ with minimum degree $(1 - 1/r + \gamma)np$.

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1. Introduction

One of the central themes in extremal graph theory is the study of sufficient conditions which imply that a graph G contains a copy of a particular graph H . Two main interesting cases of this problem are when H has fixed order, and when it has size comparable or the same as graph G . The celebrated Erdős–Stone theorem [14] settled the first case, showing that sufficiently large graph G of

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n vertices and more than $(1 - \frac{1}{r-1} + o(1))\binom{n}{2}$ edges contains a copy of any r -chromatic graph H of fixed order.

In the second case, when the order of H is close to the order of G , the large number of edges is no longer sufficient to embed H because there might be isolated vertices in G . Therefore we need a lower bound on the minimum degree of G . The most well-known example of such a result is Dirac's theorem (see, e.g., [13]), which says that, if G is a graph on n vertices with minimum degree at least $\lfloor n/2 \rfloor$ then G contains a Hamilton cycle. Another example is a problem of packing vertex disjoint copies of a fixed graph H_0 in G . We say that G contains a perfect H_0 -packing if there are vertex disjoint copies of H_0 that cover all the vertices of G . For convenience, we may assume that the order of G is divisible by the order of H_0 . A classical theorem of Hajnal and Szemerédi [17] states that if G has minimum degree at least $(1 - 1/r)n$ then G contains a perfect packing of complete graphs K_r . More general packing problems have been studied in [2,28,24].

The r -th power of a graph G is the graph $G^{(r)}$ obtained from G by connecting every pair of vertices which have distance at most r in G . In particular, note that the $(r - 1)$ -st power of the n -cycle contains $\lfloor n/r \rfloor$ vertex disjoint copies of K_r . Pósa and Seymour [30] proposed a common generalization of Dirac's and Hajnal–Szemerédi's theorem. They conjectured that the same minimum degree bound $(1 - 1/r)n$ will force a graph G to have the $(r - 1)$ -st power of a Hamiltonian cycle in it. This conjecture has been open for quite a while until Komlós, Sárközy, and Szemerédi [23] proved it for large enough n . They used a combination of Szemerédi regularity lemma [34] and the so-called blow-up lemma [22]. We will discuss this technique in more detail later in the paper.

The above results might suggest that if G has minimum degree at least $(1 - 1/r + o(1))n$, then it contains a copy of any n -vertex r -chromatic graph H with bounded degree. However, the following example (see [6]) shows that some restrictions are necessary. Let H be a random bipartite graph with bounded maximum degree and parts of size $n/2$ and G be a graph formed by two cliques each of size $(1/2 + \gamma)n$ which share $2\gamma n$ vertices (for some small fixed $\gamma > 0$). Assume that H is embedded into G and look at the $(1/2 - \gamma)n$ vertices which come from one of the cliques and do not belong to their intersection. The only neighbors of these vertices in G are the $2\gamma n$ vertices in the intersection. But with high probability H contains no collection of $(1/2 - \gamma)n$ vertices which have at most $2\gamma n$ neighbors. Therefore we cannot embed H into G .

Thus to find a general theorem, we need some additional restriction on the graph H . A graph H is said to have *bandwidth* at most b , if there exists a labeling of the vertices by $1, 2, \dots, n$, so that $|i - j| \leq b$ whenever i, j form an edge. We denote by $bw(H) = b$ if b is the minimum integer such that H has bandwidth at most b . Bollobás and Komlós [19] conjectured that if H is an r -chromatic graph which has bounded degree and low enough bandwidth then one can embed it into a graph G with minimum degree at least $(1 - 1/r + o(1))n$. Note that the constant $1 - 1/r$ is the best constant we can expect for such an embedding result to hold. Indeed, assume that n is divisible by r and let G be the complete r -partite graph on n vertices whose partition classes are of size $n/r + 1, n/r - 1, n/r, \dots, n/r$. This graph has minimum degree $(1 - 1/r)n - 1$. Consider the graph H consisting of n/r vertex disjoint copies of K_r . It is clear that each copy of K_r must contain at least one vertex from each class of G and thus there can only be at most $n/r - 1$ such copies in G . Thus we cannot embed H into G .

Bollobás and Komlós' conjecture has been recently proved by Böttcher, Schacht, and Taraz [5,6]: for every positive r, Δ, γ , there exists β such that if H is an n -vertex r -chromatic graph with maximum degree at most Δ and bandwidth at most βn , then any graph G on n vertices with minimum degree at least $(1 - 1/r + \gamma)n$ contains a copy of H for large enough n (we will refer to this conjecture and theorem as the bandwidth conjecture and the bandwidth theorem from now on). There are a lot of graphs H satisfying the condition above. For example, r -th powers of cycles which have bandwidth $2r$, trees with constant maximum degree which have bandwidth at most $O(n/\log n)$ [10], and $n^{1/2}$ by $n^{1/2}$ square grids which have bandwidth $O(n^{1/2})$ are a few of those. For more examples, see Böttcher, Pruessmann, Taraz, and Würfl's [8] classification of bounded degree graphs with sublinear bandwidth. Moreover, the theorem proved in [6] is a strengthening of the bandwidth conjecture and also implies Dirac's theorem and Pósa–Seymour's conjecture asymptotically.

Most of the above mentioned results can also be viewed in the framework of *resilience* which we discuss next. A graph property is called *monotone increasing (decreasing)* if it is preserved under edge addition (deletion). Following [33], we define:

Definition 1. Let \mathcal{P} be a monotone increasing (decreasing) graph property.

- (i) The global resilience of G with respect to \mathcal{P} is the minimum number r such that by deleting (adding) r edges from G one can obtain a graph not having \mathcal{P} .
- (ii) The local resilience of a graph G with respect to \mathcal{P} is the minimum number r such that by deleting (adding) at most r edges at each vertex of G one can obtain a graph not having \mathcal{P} .

Intuitively, the question of determining resilience of a graph G with respect to a graph property \mathcal{P} is like asking, “How strongly does G possess \mathcal{P} ?”. Using this terminology, one can for example restate Dirac’s theorem as saying that K_n has local resilience $\lfloor n/2 \rfloor$ with respect to having a Hamilton cycle. In [33], Sudakov and Vu have initiated the systematic study of global and local resilience of random and pseudorandom graphs. The random graph model they considered is the binomial random graph $G(n, p)$, which denotes the probability space whose points are graphs with vertex set $[n] = \{1, \dots, n\}$ where each pair of vertices forms an edge randomly and independently with probability p . Given a graph property \mathcal{P} , we say that $G(n, p)$ possesses \mathcal{P} *asymptotically almost surely*, or a.a.s. for brevity, if the probability that $G(n, p)$ possesses \mathcal{P} tends to 1 as n tends to infinity. In the above mentioned paper, Sudakov and Vu studied the resilience of random graphs with respect to various properties such as Hamiltonicity, containing a perfect matching, increasing its chromatic number, and having a nontrivial automorphism (this result appeared in their earlier paper with Kim [18]). For example, they proved that if $p > \log^4 n/n$ then a.a.s. any subgraph of $G(n, p)$ with minimum degree $(1/2 + o(1))np$ is Hamiltonian. Note that this result can be viewed as a generalization of Dirac’s theorem mentioned above, since the complete graph is also a random graph $G(n, p)$ with $p = 1$. This connection is very natural and most of the resilience results for random and pseudorandom graphs can be viewed as a generalization of classical results from graph theory. For additional resilience type results, see, e.g. [3,4,9,12,15,26].

Using the above terminology, the bandwidth theorem says that the complete graph K_n has local resilience $(1/r + o(1))n$ with respect to containing spanning r -chromatic graphs H of low bandwidth and bounded degree. Böttcher, Kohayakawa, and Taraz [9] partially extended this result to random graphs by proving that for fixed $\eta, \gamma > 0, \Delta > 1$ there exist positive constants β and c such that if $p \geq c(\log n/n)^{1/\Delta}$ then a.a.s. every subgraph of $G(n, p)$ with minimum degree at least $(1/2 + \gamma)np$ contains a copy of any bipartite graph H with $(1 - \eta)n$ vertices, maximum degree Δ and bandwidth at most βn . They then posed a natural and interesting question [7], whether one can fully extend the bandwidth theorem to random graphs. More specifically, they suggested that it should be possible to extend the bandwidth theorem for spanning bipartite H in the regime of constant edge probability p . For this range of probabilities, there are well developed tools that we can use, and thus there are more hopes to understand the correct behavior of this problem. The reason we only focus on bipartite graphs is the following. Consider the problem of finding a triangle factor. A fixed vertex v in $G(n, p)$ a.a.s. has degree $(1 + o(1))np$ and has $(1 + o(1))np^2$ common neighbors with any other vertex. If we delete all the edges in the neighborhood of v , we destroy all the triangles containing v . On the other hand, the degree of any vertex in $G(n, p)$ will decrease by at most $O(np^2) \ll np$, and thus, it will still be greater than $(2/3 + \gamma)np$. This gives a subgraph of $G(n, p)$ with minimum degree at least $(2/3 + \gamma)np$ and no triangle factor. Since disjoint union of triangles has constant bandwidth, this simple observation shows that one cannot directly extend the bandwidth theorem in full generality.

In this paper we study the above mentioned question posed by Böttcher, Kohayakawa, and Taraz. We have the following two main contributions. First, we prove that for constant edge probability, it is possible to obtain a complete extension of the bandwidth theorem for spanning bipartite graphs H with bounded degree and sublinear bandwidth. We also suggest a natural minor restriction on non-bipartite graphs H , which makes possible an extension of bandwidth theorem to random graphs. More precisely, we show that having some vertices with independent neighborhoods in H is enough. Here is our main theorem.

Theorem 1.1. For fixed integers r, Δ , and reals $0 < p \leq 1$ and $\gamma > 0$, there exists a constant $\beta > 0$ such that a.a.s., any spanning subgraph G' of $G(n, p)$ with minimum degree $\delta(G') \geq (1 - 1/r + \gamma)np$ contains every n -vertex graph H which satisfies the following properties:

- (i) H is r -chromatic,
- (ii) has maximum degree at most Δ ,
- (iii) has bandwidth at most βn with respect to a labeling of vertices by $1, 2, \dots, n$, and
- (iv) for every interval $[a, a + \beta^2 n] \subset [1, n]$, there exists a vertex $v \in H$ such that $N_H(v)$ is an independent set.

In particular, the theorem holds for any bipartite H which has bounded degree and sublinear bandwidth. Thus it positively answers the above mentioned question of Böttcher, Kohayakawa, and Taraz for dense random graphs. Note that for non-bipartite graphs, we only require constant number of vertices with independent neighborhoods.

Another main contribution of this paper is an extension of the classical extremal results on H_0 -packings in graphs with large minimum degree to the setting of random graphs. The above theorem implies that if H_0 is a fixed r -chromatic graph having a vertex not contained in a triangle and $\gamma > 0$ is any fixed constant, then a.a.s. every spanning $G' \subset G(n, p)$ with minimum degree at least $(1 - 1/r + \gamma)np$ contains a perfect H_0 -packing. This suggests the following natural question. Let H_0 be a fixed r -chromatic graph whose every vertex belongs to some triangle. What is the maximum number of vertex disjoint copies of H_0 that one can find in a spanning subgraph G' of $G(n, p)$ with $\delta(G') \geq (1 - 1/r + \gamma)np$? We proved the following result, which gives a rather accurate answer to this question.

Theorem 1.2. *Let H_0 be an r -chromatic graph whose every vertex is contained in a triangle. Then there exist constants $c = c(r)$ and $C = C(r)$ such that for any fixed $0 < p \leq 1$ and $0 < \gamma \leq 1/(2r)$, the random graph $G(n, p)$ a.a.s. has the following properties.*

- (i) *There exists a spanning subgraph G' with minimum degree $\delta(G') \geq (1 - 1/r + \gamma)np$ such that at least $\lfloor cp^{-2} \rfloor$ vertices of G' are not contained in a copy of H_0 .*
- (ii) *For every spanning subgraph $G' \subset G$ which has minimum degree $\delta(G') \geq (1 - 1/r + \gamma)np$, at least $n - Cp^{-2}$ vertices of G' can be covered by vertex disjoint copies of H_0 .*

The rest of this paper is organized as follows. In Section 2 we collect some known results which we need later to prove our main theorem. In Section 3 we state several important lemmas, and outline the proof of the main theorem using these lemmas. In Section 4 we prove the lemmas given in Section 3. In Section 5 we provide a detailed proof of Theorem 1.1 by using the tools developed in previous sections and other known results. As an application of the main theorem, in Section 6, we study the packing problem in random graphs (Theorem 1.2). The last section contains some concluding remarks and open problems.

To simplify the presentation, we often omit floor and ceiling signs whenever these are not crucial and make no attempts to optimize absolute constants involved. We also assume that the order n of all graphs tends to infinity and therefore is sufficiently large whenever necessary. Throughout the paper, whenever we refer, for example, to a function with subscript as $f_{3.1}$, we mean the function f defined in Lemma/Theorem 3.1.

Notation. $G = (V, E)$ denotes a graph with vertex set V and edge set E . $\Delta(G), \delta(G), \chi(G)$ denote the maximum degree, the minimum degree, and the chromatic number of G respectively. In the following, we will use v for a vertex and X for an arbitrary set. Let $N(X)$ be the collection of all vertices which are adjacent to at least one vertex in X . If $X = \{v\}$ is a singleton set we denote its neighborhood by $N(v)$. Let $N^{(0)}(v) := \{v\}$ and $N^{(k)}(v)$ be the vertices at distance exactly k from v . Note that $N^{(1)}(v) = N(v)$. Similarly define $N^{(k)}(X)$ to be the vertices at distance exactly k from the set X , where the distance of a vertex v from a set X is defined as the minimum number t such that $N^{(t)}(v) \cap X \neq \emptyset$. The degree of a vertex is defined as $\deg(v) := |N(v)|$. The neighborhood of a vertex in a set is defined as $N(v, X) := N(v) \cap X$ and the degree of a vertex in a set is defined as $\deg(v, X) := |N(v, X)|$. We denote by $E(X)$ the set of edges in the induced subgraph $G[X]$ and by $e(X) := |E(X)|$ its size. Similarly, for two sets X and Y , we denote by $E(X, Y)$ the set of ordered pairs $(x, y) \in E$ such that $x \in X$ and $y \in Y$, also $e(X, Y) := |E(X, Y)|$. Note that $e(X, X) = 2e(X)$. By $d(X, Y) := e(X, Y)/|X||Y|$ we denote the density of the pair. If we have several graphs, then the graph we are currently working with will be stated as a subscript. For example $N_G^{(k)}(v)$ is the k -th neighborhood of v in graph G .

We also utilize the following standard asymptotic notation. For two functions $f(n)$ and $g(n)$, write $f(n) = \Omega(g(n))$ if there exists a constant C such that $\liminf_{n \rightarrow \infty} f(n)/g(n) \geq C$. If there is a subscript such as in Ω_ε this means that the constant C may depend on ε . We write $f(n) = o(g(n))$ or $f(n) \ll g(n)$ if $\limsup_{n \rightarrow \infty} f(n)/g(n) = 0$. Also, $f(n) = O(g(n))$ if there exists a positive constant $C > 0$ such that $\limsup_{n \rightarrow \infty} f(n)/g(n) \leq C$. Throughout the paper \log denotes the natural logarithm.

2. Preliminaries

In this section, we collect several known results to be used later in the proof of the main theorem.

The following well-known concentration result (see, for example [1, Appendix A]) will be used several times throughout the proof. We denote by $Bi(n, p)$ a binomial random variable with parameters n and p .

Theorem 2.1 (Chernoff inequality). *If $X \sim Bi(n, p)$ and $\lambda \leq np$, then*

$$P(|X - np| \geq \lambda) \leq e^{-\Omega(\lambda^2/(np))}.$$

Our approach in proving the main theorem is to use the regularity lemma and the blow-up lemma. These powerful tools developed by Szemerédi [34], and Komlós, Sárközy and Szemerédi [22], respectively, have been successively applied to solve several embedding results (e.g., [27]). Here we state these facts without proof. Readers may consult [21,19] for more detailed discussions on these topics.

Let $G = (V, E)$ be a graph and $\varepsilon > 0$ be fixed. A disjoint pair of sets $X, Y \subset V$ is called an ε -regular pair in G if all $A \subset X, B \subset Y$ such that $|A| \geq \varepsilon|X|, |B| \geq \varepsilon|Y|$ satisfy $|d(X, Y) - d(A, B)| \leq \varepsilon$. An ε -regular pair (X, Y) is called (d, ε) -regular, if it has density at least d . A vertex partition V_0, \dots, V_k is called an ε -regular partition of G if (i) $|V_0| \leq \varepsilon n$, (ii) V_i have equal size for $i \geq 1$, and (iii) (V_i, V_j) is ε -regular in G for all but at most εk^2 pairs $1 \leq i < j \leq k$. The regularity lemma states that every large enough graph admits a regular partition. Here we state it in a stronger form which can be found in [21].

Lemma 2.2 (Regularity lemma). *For every integer t and real $\varepsilon > 0$, there exists $n_0 = n_0(t, \varepsilon)$ and $T = T(t, \varepsilon)$ such that for every graph G on $n \geq n_0$ vertices and $d \in [0, 1]$, there exists a subgraph $G' \subset G$ with an ε -regular partition V_0, \dots, V_k of G' satisfying the following properties.*

- (i) $t \leq k \leq T$,
- (ii) $\deg_{G'}(v) > \deg_G(v) - (d + \varepsilon)n$ for all $v \in V$,
- (iii) $e(G'[V_i]) = 0$ for all $i \geq 1$,
- (iv) every pair (V_i, V_j) ($1 \leq i < j \leq k$) either is ε -regular in G' with density at least d or has no edges between them.

Let V_0, \dots, V_k be an ε -regular partition of G . Then we define the reduced graph R with parameters (d, ε) as the graph on the vertex set $[k]$ with edges $\{i, j\} \in E(R)$ if and only if (V_i, V_j) is (d, ε) -regular. In this case, we also say that V_0, \dots, V_k is (d, ε) -regular on R in G . Furthermore, if $G' \subset G$ and V_0, \dots, V_k satisfy (i), (ii), (iii), (iv) of Lemma 2.2, we say that V_0, \dots, V_k is a pure (d, ε) -regular partition of G' . The following lemma establishes the fact that the reduced graph inherits the minimum degree condition.

Lemma 2.3. *Let $0 < p \leq 1$ and $\alpha, \gamma > 0$ be fixed. There exists $\varepsilon_0 = \varepsilon_0(p, \alpha, \gamma)$ such that for all $\varepsilon \leq \varepsilon_0$ and $d > 0$, the following a.s. holds. Given a graph $G = G(n, p)$, let V_0, V_1, \dots, V_k be a pure (d, ε) -regular partition of a subgraph $G' \subset G$, and R be its reduced graph. If G' has minimum degree at least $(\alpha + \gamma)np$, then R has minimum degree at least $(\alpha + 3\gamma/4)k$.*

Proof. Let $m := |V_i|$. Since $|V_0| \leq \varepsilon n$, we have the bound $m \geq (1 - \varepsilon)n/k$. Thus by Chernoff inequality, a.s. $e_{G'}(V_i, V_j) \leq e_G(V_i, V_j) \leq (1 + \varepsilon)m^2 p$ for all $i, j \geq 1$. From the definition of a pure (d, ε) -regular

partition, we know that if $\{i, j\} \notin E(R)$, then the pair (V_i, V_j) has no edges between them. Thus for a vertex $i \in V(R)$,

$$\begin{aligned} e_{G'}(V_i, V \setminus V_0) &= \sum_{j=1}^k e_{G'}(V_i, V_j) \leq (k - \deg_R(i)) \cdot 0 + \deg_R(i)(1 + \varepsilon)m^2p \\ &= \deg_R(i)(1 + \varepsilon)m^2p. \end{aligned}$$

On the other hand, by the minimum degree condition of G' and the fact $e_{G'}(V_i) = 0$,

$$e_{G'}(V_i, V \setminus V_0) \geq \left(\sum_{v \in V_i} \deg_{G'}(v) \right) - e_{G'}(V_i, V_0) \geq (\alpha + \gamma)np|V_i| - \varepsilon n|V_i|.$$

Combine the bounds, divide each side by m^2p and use the bound $n > mk$ to get, $(\alpha + \gamma - \varepsilon/p)k \leq (1 + \varepsilon)\deg_R(i)$. By selecting ε small enough, we have $\deg_R(i) \geq (\alpha + 3\gamma/4)k$. \square

With respect to embedding small subgraphs, regular pairs behave like random graphs. Thus, merely knowing the structure of the reduced graph already tells us plenty of information about the original graph and the subgraphs that it contains. The following lemma is a formal description of this intuition. A *graph homomorphism* between two graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ is a map $f : V_1 \rightarrow V_2$ such that $(f(v), f(w)) \in E_2$ if $(v, w) \in E_1$. We say that G_1 is homomorphic to G_2 if there is a homomorphism from G_1 to G_2 .

Theorem 2.4. *For any fixed graph H and $d > 0$, there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$, there is an n_0 with the following property. Let G be a graph on $n \geq n_0$ vertices, V_0, \dots, V_k be an ε -regular partition of G , and R be its reduced graph with parameters (d, ε) . If H is homomorphic to R , then G contains a copy of H .*

It is well known that the regularity lemma together with this embedding lemma implies the following generalization of Erdős–Stone theorem to random graphs $G(n, p)$ when $p \in (0, 1]$ is fixed (see, e.g., [16] for discussion of the case $p \ll 1$). Recently, by using a different approach, Conlon and Gowers [11], and Schacht [29] independently extended this result to the range $p \ll 1$, but we do not need this stronger form for our purpose.

Theorem 2.5. *For any fixed $\gamma > 0$, $0 < p \leq 1$ and a graph H , $G = G(n, p)$ satisfies the following with probability $1 - e^{-\Omega(n^2p)}$. Any subgraph $G' \subset G$ with $e(G') \geq (1 - 1/(\chi(H) - 1) + \gamma)n^2p/2$ contains a copy of H .*

In fact, we need the following seemingly stronger result which directly follows from Theorem 2.5 by taking the union bound.

Corollary 2.6. *For any fixed $\alpha, \gamma > 0$, $0 < p \leq 1$ and a graph H , $G = G(n, p)$ satisfies the following with probability $1 - e^{-\Omega(n^2p)}$. For any subset $W \subset V$ of size $|W| \geq \alpha n$, every subgraph $G' \subset G[W]$ with $e(G') \geq (1 - 1/(\chi(H) - 1) + \gamma)|W|^2p/2$ contains a copy of H .*

The theorems above illustrate the strength of regularity in finding fixed size subgraphs. On the other hand, the blow-up lemma, which we will introduce next, exemplifies the strength of regularity in embedding graphs which are as large as G . Before we state the theorem we must define the concept of *super-regularity*. Let $G = (V, E)$ be a graph and $d, \varepsilon > 0$. Then a pair of disjoint sets $X, Y \subset V$ is called (d, ε) -super-regular in G if it is (i) (d, ε) -regular in G , and (ii) $\forall x \in X, \deg(x, Y) \geq d|Y|$ and $\forall y \in Y, \deg(y, X) \geq d|X|$. As in the regularity case, given a partition V_0, \dots, V_k of G we define the (d, ε) -super-regular reduced graph R to be the graph on the vertex set $[k]$ with edges $\{i, j\} \in E(R)$ if and only if (V_i, V_j) forms a (d, ε) -super-regular pair in G . We may also say that V_0, \dots, V_k is (d, ε) -super-regular on R in G . The following version of the blow-up lemma was used in [5] and [6].

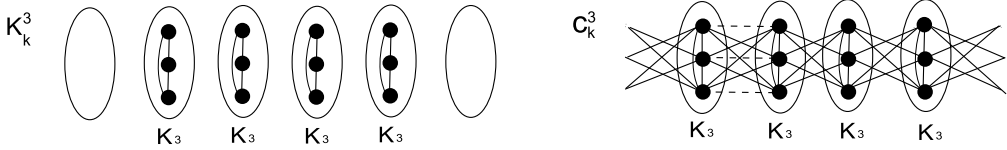


Fig. 1. K_k^3 and C_k^3 .

Theorem 2.7 (Blow-up lemma). For any positive d, Δ, c and r , there exist $\varepsilon = \varepsilon(d, \Delta, c, r)$ and $\alpha = \alpha(d, \Delta, c, r)$ such that the following is true. Let n_1, n_2, \dots, n_r be arbitrary integers and consider the following two graphs over the vertex set $V = V_1 \cup \dots \cup V_r$ with $|V_i| = n_i$ for all $1 \leq i \leq r$.

- (i) In G_0 , each pair (V_i, V_j) forms a complete bipartite graph, and
- (ii) in G_1 , each pair (V_i, V_j) forms a (d, ε) -super-regular pair.

Then any graph $H = (W_1 \cup \dots \cup W_r, E_H)$ with $\Delta(H) \leq \Delta$ and $|W_i| = n_i$ ($\forall i \in [r]$) which can be embedded into G_0 so that all the vertices of W_i get mapped into V_i ($\forall i \in [r]$) can be embedded into G_1 in the same way.

Moreover, assume that we are given subsets $W'_i \subset W_i$ such that $|W'_i| \leq \alpha \cdot \min_{j \in [r]} |W_j|$, and for each $w \in W'_i$, a set $C_w \subset V_i$ such that $|C_w| \geq c|V_i|$. Then there exists an embedding of H into G such that every vertex $w \in W'_1 \cup \dots \cup W'_k$ is mapped into a vertex in C_w .

3. Outline of the proof

The setting of Theorem 1.1 can be briefly stated as following. We have a host graph $G' \subset G(n, p)$ with large minimum degree, a graph H with certain restrictions, and we want to embed H into G' . Hence, with this setting in mind, in the future discussion, G' will always stand for the host graph, and H will stand for the graph that we want to embed.

To prove Theorem 1.1, we adapt several lemmas from the proof of the bandwidth theorem given in [6]. In this section, we will provide the statement of the lemmas, and outline the proof of the main theorem by using these lemmas. The statement of these lemmas might seem quite technical, so to understand the intuition which lies behind the lemmas, it will be useful to keep in mind that the final part of the proof will be an application of the blow-up lemma given in Theorem 2.7.

First lemma, which is a variant of ‘Lemma for G' ’ (Lemma 6 in [6]), prepares the graph G' so that we have many regular and super-regular pairs. Before stating the lemma, we introduce some graphs. The graphs C_k^r and K_k^r are defined as following (see Fig. 1). C_k^r is a graph over the vertex set $[k] \times [r]$ such that $(i_1, j_1), (i_2, j_2)$ is connected by an edge if (i) $i_1 = i_2$ and $j_1 \neq j_2$, or (ii) $|j_2 - j_1| = 1$ and $j_1 \neq j_2$. K_k^r is a graph over the same vertex set $[k] \times [r]$ consisting of k disjoint copies of K_r , each of which lies on the vertices $\{i\} \times [r]$. Note that $K_k^r \subset C_k^r$ by construction.

An integer partition $(n_{i,j})_{1 \leq i \leq k, 1 \leq j \leq r}$ of n is called r -equitable if $|n_{i,j} - n_{i,j'}| \leq 1$ for all $1 \leq i \leq k$ and $1 \leq j, j' \leq r$.

Lemma 3.1 (Lemma for G). For every integer $r \geq 2, 0 < p \leq 1$ and $\gamma > 0$ there exist $d = d(r, p, \gamma) > 0$ and $\varepsilon_0 = \varepsilon_0(r, p, \gamma) > 0$ such that for every positive $\varepsilon \leq \varepsilon_0$ there exist $b_0 = b_0(r, p, \gamma, \varepsilon), \xi_0 = \xi_0(r, p, \gamma, \varepsilon) > 0$, and $K_0 = K_0(r, p, \gamma, \varepsilon)$ such that, $G = G(n, p)$ a.a.s. satisfies the following. For every spanning subgraph $G' \subset G$ with $\delta(G') \geq (1 - 1/r + \gamma)np$ there exist a spanning subgraph $G'' \subset G'$ with $\delta(G'') \geq (1 - 1/r + 4\gamma/5)np$, a set B of size at most b_0 , an r -equitable integer partition $(m_{i,j})_{1 \leq i \leq k, 1 \leq j \leq r}$ of $n - |B|$, sets $(V_{i,j}^*)_{1 \leq i \leq k, 1 \leq j \leq r}$, and a graph R on vertex set $[k] \times [r]$ with $k \leq K_0$ such that

- (i) $K_k^r \subset C_k^r \subset R$ and $\delta(R) \geq (1 - 1/r + \gamma/2)kr$,
- (ii) $\forall 1 \leq i \leq k, 1 \leq j \leq r, m_{i,j} \geq (1 - \varepsilon)n/(kr)$,
- (iii) $\forall 1 \leq i \leq k, 1 \leq j \leq r, m_{i,j} \geq |V_{i,j}^*| \geq (1 - \varepsilon)m_{i,j}$,
- (iv) $(V_{i,j}^*)_{1 \leq i \leq k, 1 \leq j \leq r}$ is (d, ε) -regular on R in G'' ,

such that for every choice of $(n_{i,j})_{1 \leq i \leq k, 1 \leq j \leq r}$ with $m_{i,j} - \xi_0 n \leq n_{i,j} \leq m_{i,j} + \xi_0 n$ and $\sum_{i,j} n_{i,j} \leq n - |B|$, there exists a partition $(V_{i,j})_{1 \leq i \leq k, 1 \leq j \leq r}$ of $V \setminus B$ with

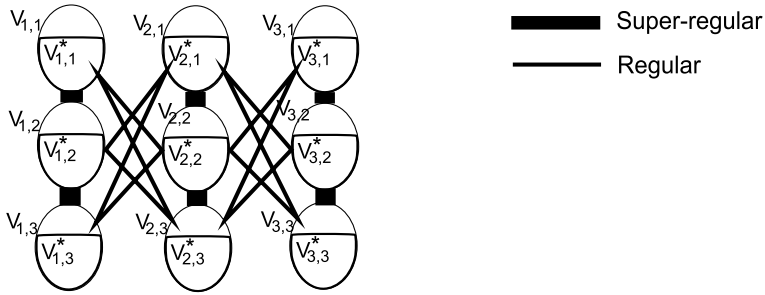


Fig. 2. Lemma for G . $V_{i,j}^*$ are fixed and $V_{i,j}$ are flexible in size.

- (a) $|V_{i,j}| \geq n_{i,j}$, $V_{i,j}^* \subset V_{i,j}$, $\forall 1 \leq i \leq k, 1 \leq j \leq r$,
- (b) $(V_{i,j})_{1 \leq i \leq k, 1 \leq j \leq r}$ is (d, ε) -regular on R in G'' , and
- (c) $(V_{i,j}^*)_{1 \leq i \leq k, 1 \leq j \leq r}$ is (d, ε) -super-regular on K_k^r in G'' .

Heuristically, given a graph G' , this lemma returns some set B and a ‘temporary’ vertex partition of $V \setminus B$ with parts of size $m_{i,j}$ for some integer partition $(m_{i,j})_{1 \leq i \leq k, 1 \leq j \leq r}$ of $n - |B|$. The vertex partition is flexible in the sense that given any other integer partition $(n_{i,j})_{1 \leq i \leq k, 1 \leq j \leq r}$ which is close to $(m_{i,j})_{i,j}$, we can change the partition slightly so that the new partition $(V_{i,j})_{i,j}$ has size $|V_{i,j}| = n_{i,j}$ for all i, j . Moreover, each partition $V_{i,j}$ has an underlying ‘core’ set $V_{i,j}^*$ which always remains where they were regardless of the given $(n_{i,j})_{i,j}$ (see Fig. 2). The main difference between this lemma and ‘Lemma for G' ’ in [6] is the set B whose existence is unavoidable due to the inherent randomness of G' , and the ‘core’ sets $V_{i,j}^*$ which are there to help controlling the set B . Note that $|B|$ is bounded by some constant b_0 which does not depend on n .

Assume for the sake of argument, that the graph H which we want to embed into G' consists of vertex disjoint copies of C_4 , and $r = 2$, and n is divisible by 4. Provide the graph G' to Lemma 3.1, and get as output an integer partition $(m_{i,j})_{1 \leq i \leq k, 1 \leq j \leq 2}$ and a set B . If B was empty, then the rest of the argument can go as following. Find an integer partition $(n_{i,j})_{1 \leq i \leq k, 1 \leq j \leq 2}$ which is close to $(m_{i,j})_{1 \leq i \leq k, 1 \leq j \leq 2}$, and satisfies $n_{i,1} = n_{i,2}$ with both $n_{i,1}, n_{i,2}$ being an even integer for all $1 \leq i \leq k$. By Lemma 3.1, we can obtain a partition $(V_{i,j})_{1 \leq i \leq k, 1 \leq j \leq 2}$ of $V \setminus B = V$ such that $|V_{i,j}| = n_{i,j}$ for all i, j . Then apply the blow-up lemma on each copy of K_2 in K_k^2 separately, to find vertex disjoint copies of C_4 in the graph.

To cover the case when B is not empty, we need to slightly modify this argument. As a first step, find copies of C_4 which only use vertices from B and $(V_{i,j}^*)_{1 \leq i \leq k, 1 \leq j \leq 2}$. Assume that there are no remaining vertices in B after finding some copies of C_4 (this part is not trivial but assume that we can do this), and by doing so we have used $\delta_{i,j}$ vertices from each set $V_{i,j}^*$. Then $n - \sum_{i,j} \delta_{i,j} - |B|$ is divisible by 4 and hence we can find an integer partition $(n_{i,j})_{1 \leq i \leq k, 1 \leq j \leq 2}$ of it which satisfies $n_{i,1} = n_{i,2}$ with both of $n_{i,1}, n_{i,2}$ being an even integer for all $1 \leq i \leq k$. If this integer partition were also close to $(m_{i,j})_{1 \leq i \leq k, 1 \leq j \leq 2}$, then by Lemma 3.1, we can obtain a partition $(V_{i,j})_{1 \leq i \leq k, 1 \leq j \leq 2}$ of $V \setminus B$ such that $|V_{i,j}| = n_{i,j} + \delta_{i,j}$ for all i, j . Recall that the copies of C_4 which we have already found use $\delta_{i,j}$ vertices from each set $V_{i,j}^*$, and thus also from $V_{i,j}$. Therefore the remaining number of vertices in $V_{i,j}$ after disregarding these copies of C_4 is exactly $n_{i,j}$. Also note that deleting constant number of vertices from each part does not destroy super-regularity. Now apply the blow-up lemma to the remaining vertices and find vertex disjoint copies of C_4 which cover all the vertices of V .

The strategy of embedding a general graph H is not too different from this. However, a general graph H can have more complicated structure than vertex disjoint copies of C_4 , and requires some preprocessing before being embedded into G' . In the next lemma, we use the bound on the bandwidth to map H ‘nicely’ onto the $[k] \times [r]$ grid. This lemma is a variant of ‘Lemma for H ’ (Lemma 8 in [6]), and can be derived from it without much difficulty.

Lemma 3.2 (Lemma for H). Let $r, k \geq 1$ be integers and let $\beta, \xi > 0$ satisfy $\beta \leq \xi^2 / (3026r^3)$. Let R be a graph over the vertex set $[k] \times [r]$ such that $\delta(R) > (r - 1)k$ and $K_k^r \subset C_k^r \subset R$. Let H be a graph on n vertices with maximum degree Δ , and assume that

- (i) H has a labeling of bandwidth at most βn and has chromatic number at most r .
- (ii) For every interval $[a, a + \beta^2 n] \subset [1, n]$, there exists a vertex $v \in [a, a + \beta^2 n]$ such that $N_H(v)$ is an independent set.
- (iii) $(m_{i,j})_{1 \leq i \leq k, 1 \leq j \leq r}$ is an r -equitable integer partition of n with $m_{i,j} \geq 200\beta n$ for every $1 \leq i \leq k$ and $1 \leq j \leq r$.

Then there exists a mapping $f : V(H) \rightarrow [k] \times [r]$ and a set of special vertices $X \subset V(H)$ with the following properties.

- (a) $|X| \leq kr\xi n$,
- (b) the sets $W_{i,j} := f^{-1}(i, j)$ have size $m_{i,j} - \xi n \leq |W_{i,j}| \leq m_{i,j} + \xi n$ for every i and j ,
- (c) for every edge $\{u, v\} \in E(H)$ we have $\{f(u), f(v)\} \in E(R)$,
- (d) if $\{u, v\} \in E(H)$ and, moreover, u and v are both in $V(H) \setminus X$, then $\{f(u), f(v)\} \in E(K_k^r)$,
- (e) $\forall 1 \leq i \leq k, \exists$ at least β^{-1} vertices $w \in (\bigcup_{1 \leq j \leq r} W_{i,j}) \setminus (\bigcup_{l=0}^3 N_H^{(l)}(X))$ whose neighborhood $N_H(w)$ forms an independent set.

Proof. The process of finding a map f which satisfies (a), (b), (c), and (d) can be found in the proof of Lemma 8 in [6]. We claim that (e) is also a byproduct of their proof. It suffices to verify that for all $1 \leq i \leq k$, there exists an interval of length at least $2\beta n$ in the set $\bigcup_{j=1}^r W_{i,j} \setminus (\bigcup_{s=0}^3 N_H^{(s)}(X))$, since by condition (ii) this will give at least β^{-1} vertices in this set which have independent neighborhoods. The stronger lower bound of $m_{i,j} \geq 200\beta n$ that we imposed on top of the conditions of Lemma 8 in [6] guarantees that such an interval always exists. We omit the details. \square

Let G' be a given graph and use Lemma 3.1 to get a set B , a ‘temporary’ partition of $V \setminus B$ which we can adjust (see the discussion following Lemma 3.1), and an integer partition $(m_{i,j})_{1 \leq i \leq k, 1 \leq j \leq r}$. To simplify the explanation, assume for a moment that the set B is empty. Use this integer partition $(m_{i,j})_{i,j}$ as an input to Lemma 3.2, and we get a partition $(W_{i,j})_{i,j}$ of the vertex set of H , such that the integer partition $(|W_{i,j}|)_{i,j}$ is close to $(m_{i,j})_{i,j}$. Thus by Lemma 3.1, we can get a partition $(V_{i,j})$ of $V(G)$ such that $|V_{i,j}| = |W_{i,j}|$ for all i, j .

Ideally, we want all the pairs $(V_{i,j}, V_{i',j'})$ to be super-regular. But in reality, the super-regular pairs are only guaranteed over K_k^r , and the set X in Lemma 3.2 is designed to overcome this difficulty. Observe that all the edges of H which are not incident to X correspond to K_k^r in the homomorphic image (property (d) of Lemma 3.2). Thus if we can find an embedding of vertices of X first, so that its neighborhood $Y := N(X)$ is only ‘mildly’ restricted, then we can extend this embedding by using the version of the blow-up lemma as in Theorem 2.7. The next lemma, which is Lemma 9 in [6], can be used to embed X so that the number of the possible images of each vertex $y \in Y$ is still large enough.

Lemma 3.3. For every integer $\Delta \geq 2$ and every $d \in (0, 1]$ there exist constants $c = c(\Delta, d)$ and $\varepsilon_0 = \varepsilon_0(\Delta, d)$ such that for every positive $\varepsilon \leq \varepsilon_0$ the following is true.

Let R be a graph over the vertex set $V(R) = [k] \times [r]$ and G be a graph on n vertices with $V(G) = \bigcup_{1 \leq i \leq k, 1 \leq j \leq r} V_{i,j}$, such that $|V_{i,j}| \geq (1 - \varepsilon)n / (kr)$ for all $1 \leq i \leq k, 1 \leq j \leq r$ and as a partition, $(V_{i,j})$ is (d, ε) -regular on R . Furthermore, let Γ be a graph with $V(\Gamma) = X \cup Y$ and $f : V(\Gamma) \rightarrow V(R) = [k] \times [r]$ be a mapping with $\{f(a), f(a')\} \in E(R)$ for all $\{a, a'\} \in E(\Gamma)$.

If $|V(\Gamma)| \leq \varepsilon_0 n / (kr)$ and $\Delta(\Gamma) \leq \Delta$, then there exists an injective mapping $g : X \rightarrow V(G)$ with $g(x) \in V_{f(x)}$ for all $x \in X$ such that for all $y \in Y$ there exist sets $C_y \subset V_{f(y)} \setminus g(X)$ such that

- (i) g is a graph homomorphism of $\Gamma[X]$ to G ,
- (ii) for all $y \in Y$ we have $C_y \subset N_G(g(x))$ for all $x \in N_{\Gamma}(y) \cap X$, and
- (iii) $|C_y| \geq c|V_{f(y)}|$ for every $y \in Y$.

4. Technical lemmas

In this section we prove Lemma 3.1 by using the following useful statement. This statement hints where the set B in Lemma 3.1 comes from.

Lemma 4.1. *Let $0 < p \leq 1$ be fixed and T be an integer. Then for every $\varepsilon > 0$, there exists a constant $b_0 = b_0(p, T, \varepsilon)$ such that $G = G(n, p)$ a.s. satisfies the following. For arbitrary subsets V_1, \dots, V_T of the vertex set V with $|V_i| \geq \varepsilon n$ for all $1 \leq i \leq T$, there exists a set B of size at most b_0 such that for all $v \in V \setminus B$, we have $\deg(v, V_i) \in [(1 - \varepsilon)|V_i|p, (1 + \varepsilon)|V_i|p]$ for all $1 \leq i \leq T$.*

Proof. Let b' be a constant to be chosen later. As a first step, we fix a set $W \subset V$ of size at least εn , and analyze the probability of there being b' vertices v such that $\deg(v, W) \notin [(1 - \varepsilon)|W|p, (1 + \varepsilon)|W|p]$. Let B be a set of size b' and assume that for all $v \in B$, we have $\deg(v, W) < (1 - \varepsilon)|W|p$. Then by definition, $e(B, W) < |B| \cdot (1 - \varepsilon)|W|p$. We estimate the probability of this event. Note that B is a set of constant size and W has size $|W| \geq \varepsilon n$, thus it suffices to bound the probability of $e(B, W \setminus B) < |B| \cdot (1 - \varepsilon/2)|W \setminus B|p$. Since $e(B, W \setminus B)$ has expectation $|B||W \setminus B|p$ and is a sum of independent binomial random variables, we can use Chernoff inequality to get,

$$P(e(B, W \setminus B) < (1 - \varepsilon/2)|B||W \setminus B|p) \leq e^{-\Omega_\varepsilon(b'np)}.$$

Thus for a fixed set B of size b' and W of size at least εn , the probability that all the vertices $v \in B$ have $\deg(v, W) < (1 - \varepsilon)|W|p$ is $e^{-\Omega_\varepsilon(b'np)}$. Take the union bound of this event over all choices of B and W and we can conclude that the probability of there existing such sets B and W in G is at most $\binom{n}{b'} \cdot 2^n \cdot e^{-\Omega_\varepsilon(b'np)} = o(1)$ as long as $b' = b'(\varepsilon, p)$ is large enough. In other words, a.s. every set W of size $|W| \geq \varepsilon n$ has at most b' vertices v such that $\deg(v, W) < (1 - \varepsilon)|W|p$.

Given subsets V_1, \dots, V_T of size at least εn , the previous observation implies that there are at most $b'T$ vertices which have $\deg(v, V_i) < (1 - \varepsilon)|V_i|p$ for some $1 \leq i \leq T$, and similarly at most $b'T$ vertices which have $\deg(v, V_i) > (1 + \varepsilon)|V_i|p$ for some $1 \leq i \leq T$. Therefore by setting $b_0 = 2b'T$, we can derive the conclusion of the lemma. \square

The proof of Lemma 3.1 consists of two steps. The first step is to show the existence of a ‘temporary’ partition $(U_{i,j})_{1 \leq i \leq k, 1 \leq j \leq r}$ which has size $m_{i,j} := |U_{i,j}|$ for all i, j (see the discussion following the statement of Lemma 3.1). Once this partition is constructed, we select sets $V_{i,j}^*$ arbitrarily within the ‘temporary’ set $U_{i,j}$, and for a given integer partition $(n_{i,j})$, modify the partition slightly without moving the vertices in $V_{i,j}^*$ to make the sizes of the partition as desired.

The two lemmas below establish stability results for regular and super-regular pairs. They basically say that regularity can be changed into super-regularity by small perturbation (Lemma 4.2), and regularity and super-regularity are stable under small perturbation (Lemma 4.3). These can be found in [6, Propositions 13 and 14].

Lemma 4.2. *Fix $\varepsilon, d > 0$. For any graph G and ε -regular partition V_1, \dots, V_k with (d, ε) -reduced graph R , let S be a subgraph of R with $\Delta(S) \leq \Delta$. Then for each vertex i of S , we can find a set $V'_i \subset V_i$ of size $(1 - \varepsilon\Delta)|V_i|$ such that for every edge $\{i, j\} \in E(S)$ the pair (V'_i, V'_j) is $(d - \varepsilon(\Delta + 1), \varepsilon/(1 - \varepsilon\Delta))$ -super-regular. Moreover, for every edge $\{i, j\}$ of the original reduced graph R , the pair (V'_i, V'_j) is still $(d - \varepsilon(\Delta + 1), \varepsilon/(1 - \varepsilon\Delta))$ -regular.*

Lemma 4.3. *Let (A, B) be a (d, ε) -regular pair and let (\hat{A}, \hat{B}) be a pair such that $|\hat{A} \Delta A| \leq \hat{\alpha}|\hat{A}|$ and $|\hat{B} \Delta B| \leq \hat{\beta}|\hat{B}|$ for some $0 \leq \hat{\alpha}, \hat{\beta} \leq 1$. Then, (\hat{A}, \hat{B}) is a $(\hat{d}, \hat{\varepsilon})$ -regular pair with $\hat{d} := d - 2(\hat{\alpha} + \hat{\beta})$ and $\hat{\varepsilon} := \varepsilon + 3(\hat{\alpha}^{1/2} + \hat{\beta}^{1/2})$. If, moreover, (A, B) is (d, ε) -super-regular and each vertex v in \hat{A} has at least $d|\hat{B}|$ neighbors in \hat{B} and each vertex v in \hat{B} has at least $d|\hat{A}|$ neighbors in \hat{A} , then (\hat{A}, \hat{B}) is $(\hat{d}, \hat{\varepsilon})$ -super-regular.*

Next lemma is an immediate corollary of the bandwidth theorem proved in [6] (which is Theorem 1 there).

Lemma 4.4. Given an integer $r \geq 1$ and a constant $\gamma > 0$, any sufficiently large graph G on n vertices with minimum degree $(1 - 1/r + \gamma)n$ contains a copy of C_m^r with $m = \lfloor n/r \rfloor$.

Now we are ready to prove Lemma 3.1.

Proof of Lemma 3.1. Given $r \geq 2$, p, γ , choose $d \leq \gamma p/90$ and let $\varepsilon_0 = \min\{\varepsilon_{2.3}(p, 1 - 1/r, \gamma), d/(12r)\}$. Assume that an $\varepsilon \leq \varepsilon_0$ is given, and let $\varepsilon' = \gamma p \varepsilon^6/(1152r)$ and $d' = d + 2\varepsilon$. Let $t = \max\{4r/\gamma, 1/(\varepsilon')\}$ and $T = T_{2.2}(t, \varepsilon')$. Let $b_0 = b_{4.1}(p, T, \varepsilon')$.

Let $G = G(n, p)$ and let $G' \subset G$ be a subgraph with $\delta(G') \geq (1 - 1/r + \gamma)np$. By using the degree form of the regularity lemma (Lemma 2.2), we obtain a graph $G'' \subset G'$ and a pure (d', ε') -regular partition $(U_i)_{0 \leq i \leq s}$ of G'' with reduced graph R and $t \leq s \leq T$. From now on we will only consider the graph G'' , unless mentioned otherwise. Remove at most $r - 1$ parts and put them into the set U_0 so that we can assume $s = kr$ for some integer k . Note that by Lemma 2.2(ii),

$$\begin{aligned} \delta(G'') &\geq \delta(G') - (d' + \varepsilon')n \geq (1 - 1/r + \gamma)np - (d' + \varepsilon')n \\ &\geq (1 - 1/r + 4\gamma/5)np, \end{aligned}$$

and thus by Lemma 2.3 we have $\delta(R) \geq (1 - 1/r + \gamma/2)s$. Let $m := |U_i|$ and note $|U_0| \leq \varepsilon'n + (r - 1)\frac{n}{t} \leq r\varepsilon'n$, so we have $ms = mkr \leq n \leq mkr/(1 - r\varepsilon')$.

By Lemma 4.4, R contains a copy of C_k^r . Thus we may assume that R is a graph over the vertex set $[k] \times [r]$ with $K_k^r \subset C_k^r \subset R$. Rename the parts U_x as $U_{i,j}$ according to this new vertex set of R to get a vertex partition $U_0 \cup \bigcup_{1 \leq i \leq k, 1 \leq j \leq r} U_{i,j}$. Then by applying Lemma 4.2 with $S = K_k^r$ and $\Delta = r - 1$, one can obtain a new partition $U'_0 \cup \bigcup_{1 \leq i \leq k, 1 \leq j \leq r} U'_{i,j}$ which is $(d' - \varepsilon', \varepsilon'/(1 - \varepsilon'))$ -super-regular on K_k^r , $(d' - \varepsilon', \varepsilon'/(1 - \varepsilon'))$ -regular on R , and $|U'_{i,j}| = (1 - \varepsilon')m$. Since all the discarded vertices of $U_{i,j}$ are collected into U'_0 , we have $|U'_0| \leq |U_0| + \varepsilon'mkr^2 \leq \varepsilon'rn + \varepsilon'mkr^2 \leq 2\varepsilon'rn$. Applying Lemma 4.1 to the sets $U'_{i,j}$, we get a set B such that for all $v \in V \setminus B$, $\deg_{G''}(v, U'_{i,j}) \leq \deg_G(v, U'_{i,j}) \leq (1 + \varepsilon')mp$ for all $i \in [k], j \in [r]$. Remove all the vertices of B belonging to $U'_{i,j}$ for $i \in [k], j \in [r]$, and put it into U'_0 , and then remove some more vertices from each partition so that the number of vertices in each part is the same for all i, j . Since B is a set of constant size, asymptotically the effect of this process is negligible and we may use the same bounds on the size of the sets as before.

We would like to spread the vertices in the exceptional set $U'_0 \setminus B$ into $(U'_{i,j})_{1 \leq i \leq k, 1 \leq j \leq r}$ while keeping the r -equitable property of the partition, regularity on R and super-regularity on K_k^r . For a vertex $u \in U'_0 \setminus B$ call an index i good if u has at least $d'm$ neighbors in each $U'_{i,j}$ for all $j \in [r]$. Let g_u be the number of good indices for u , and let $U'_i = \bigcup_{1 \leq j \leq r} U'_{i,j}$. Note that if i is a good index for u , then we can add u to any part of U'_i without destroying the super-regularity on K_k^r . By the definition of B , for $u \in V \setminus B$ and arbitrary $i \in [k], j \in [r]$, $\deg_{G''}(u, U'_{i,j}) \leq (1 + \varepsilon')mp$ and so $\deg_{G''}(u, U'_i) \leq (1 + \varepsilon')rmp$ in general. However, if i is not a good index for u , then u can only have at most $d'm$ neighbors in one of the parts, and we have the bound $\deg_{G''}(u, U'_i) \leq (1 + \varepsilon')(r - 1)mp + d'm$. Thus we have

$$\deg_{G''}(u, U'_1 \cup \dots \cup U'_k) \leq g_u(1 + \varepsilon')rmp + (k - g_u)((1 + \varepsilon')(r - 1)mp + d'm).$$

On the other hand, since G'' has minimum degree at least $(1 - 1/r + 4\gamma/5)np$, and $|U'_0| \leq 2\varepsilon'rn$, we have,

$$\begin{aligned} \deg_{G''}(u, U'_1 \cup \dots \cup U'_k) &= \deg_{G''}(u, V \setminus U'_0) \geq (1 - 1/r + 4\gamma/5)np - 2\varepsilon'rn \\ &\geq (1 - 1/r + 3\gamma/4)np. \end{aligned}$$

Combine these bounds to get,

$$(1 - 1/r + 3\gamma/4)np \leq g_u(1 + \varepsilon')rmp + (k - g_u)((1 + \varepsilon')(r - 1)mp + d'm).$$

Using the fact $mkr \leq n$, we can divide the left-hand side by np , and right-hand side by $mkrp$ to get,

$$\begin{aligned}
 1 - 1/r + 3\gamma/4 &\leq \frac{g_u}{k}(1 + \varepsilon') + \left(1 - \frac{g_u}{k}\right) \left(\left(1 - \frac{1}{r}\right)(1 + \varepsilon') + \frac{d'}{rp} \right) \\
 &\leq \frac{g_u}{kr}(1 + \varepsilon') + \left(1 - \frac{1}{r}\right) + \varepsilon' + \frac{d'}{rp},
 \end{aligned}$$

which implies $g_u \geq \gamma kr/2$. Pick a vertex in $U'_0 \setminus B$ one by one, and assign one of its good index to it as follows. Always pick the index which has been assigned the least number of vertices so far. In this way, we can assign an index to every vertex $U'_0 \setminus B$ so that each index gets assigned at most $2|U'_0|/(\gamma kr)$ vertices. By using the fact $|U'_0| \leq 2\varepsilon'rn$ and $n \leq mkr/(1 - r\varepsilon')$ we get,

$$\frac{2|U'_0|}{\gamma kr} \leq \frac{4\varepsilon'rn}{\gamma kr} \leq \frac{4\varepsilon'mkr^2}{(1 - r\varepsilon')\gamma kr} \leq \frac{8\varepsilon'r}{\gamma}m \leq \frac{\varepsilon^6}{144}m =: \alpha m.$$

For each index i , spread the vertices of U'_0 assigned to it as evenly as possible into $U'_{i,j}$ for $j \in [r]$ so that the resulting partition $(U''_{i,j})_{1 \leq i \leq k, 1 \leq j \leq r}$ is r -equitable. Recall that (i) all the vertices assigned to an index have degrees at least $d'm$ in every part belonging to that index, and (ii) $(U'_{i,j})_{1 \leq i \leq k, 1 \leq j \leq r}$ was $(d' - \varepsilon'r, \varepsilon/(1 - \varepsilon'r))$ -super-regular on K_k^r and $(d' - \varepsilon'r, \varepsilon/(1 - \varepsilon'r))$ -regular on R . Furthermore, the sets $U'_{i,j}$ had size $(1 - \varepsilon'r)m$, and $|U''_{i,j} \Delta U'_{i,j}| \leq \lceil \alpha m/r \rceil \leq \alpha m \leq \alpha |U'_{i,j}|/(1 - \varepsilon'r) \leq 2\alpha |U'_{i,j}|$. Thus by Lemma 4.3 we know that $(U''_{i,j})_{1 \leq i \leq k, 1 \leq j \leq r}$ is $(d' - \varepsilon'r - 8\alpha, \varepsilon'/(1 - \varepsilon'r) + 6\sqrt{2}\alpha^{1/2})$ -super-regular on K_k^r and $(d' - \varepsilon'r - 8\alpha, \varepsilon'/(1 - \varepsilon'r) + 6\sqrt{2}\alpha^{1/2})$ -regular on R . By the choice of the parameters, we have,

$$\begin{aligned}
 d' - \varepsilon'r - 8\alpha &\geq d + 2\varepsilon - \frac{\gamma p \varepsilon^6}{1152} - \frac{\varepsilon^6}{18} \geq d + \varepsilon, \quad \text{and} \\
 \frac{\varepsilon'}{1 - \varepsilon'r} + 6\sqrt{2}\alpha^{1/2} &\leq 2\varepsilon' + \frac{\varepsilon^3}{\sqrt{2}} \leq \varepsilon^3.
 \end{aligned}$$

Therefore $(U''_{i,j})$ is $(d + \varepsilon, \varepsilon^3)$ -super-regular on K_k^r and $(d + \varepsilon, \varepsilon^3)$ -regular on R .

Let $m_{i,j} := |U''_{i,j}|$ and note that this satisfies

$$m_{i,j} \geq |U'_{i,j}| \geq (1 - \varepsilon'r)m \geq \frac{(1 - \varepsilon'r)^2 n}{kr} \geq \frac{(1 - \varepsilon)n}{kr}$$

for all $i \in [k], j \in [r]$. Then fix an arbitrary set $V^*_{i,j} \subset U''_{i,j}$ of size $(1 - 3\varepsilon^3r)m_{i,j}$ for all $i \in [k], j \in [r]$ and note that $(1 - 3\varepsilon^3r)m_{i,j} \geq (1 - \varepsilon)m_{i,j}$ so that (iv) holds. Since $|U''_{i,j} \Delta V^*_{i,j}| = 3\varepsilon^3r|U''_{i,j}|$, by Lemma 4.3, the partition $(V^*_{i,j})$ will be $(d + \varepsilon - 12\varepsilon^3r, \varepsilon^3 + 6\sqrt{3}(\varepsilon^3r)^{1/2})$ -regular on R , and in particular (d, ε) -regular on R . This concludes the first part of Lemma 3.1 where given a graph G' , we obtain a subgraph G'' , a set B , sets $(V^*_{i,j})$ which are (d, ε) -regular on R , and an r -equitable integer partition $(m_{i,j})$ of $n - |B|$.

It remains to show that given another integer partition $(n_{i,j})$, we can find a partition $(V_{i,j})$ of $V \setminus B$ with $|V_{i,j}| \geq n_{i,j}$ for all i, j . This partition will be obtained from the partition $(U''_{i,j})$ by *pushing around the vertices*. This is a process of moving vertices from one partition to another while keeping regularity and super-regularity of pairs. For example, say that we want to move one vertex from $U''_{1,1}$ to $U''_{2,1}$. Then by the regularity of $(U''_{i,j})$ on C_k^r , there exists a vertex $u \in U''_{1,1}$ which has high degree in all the sets $U''_{2,j}$ for $2 \leq j \leq r$. Moving this vertex to $U''_{2,1}$ will not destroy the regularity and super-regularity of pairs. One must observe that the proof in [6] allows to fix a set $V^*_{i,j}$ of size $(1 - 3\varepsilon^3r)m_{i,j}$ and always choose a vertex outside of it to push around (this follows from the $(d + \varepsilon, \varepsilon^3)$ -regularity on R). After the process of pushing around the vertices is done, the size of the sets $U''_{i,j}$ will change, and thus affect the super-regularity and regularity between parts. This is where we want to choose $\xi_0 = \xi_0(r, p, \gamma, \varepsilon)$ to be small enough. By doing so, we can make sure that the sets $U''_{i,j}$ change only by some small amount, and since $(U''_{i,j})$ is $(d + \varepsilon, \varepsilon^3)$ -super-regular on K_k^r and $(d + \varepsilon, \varepsilon^3)$ -regular on R ,

the resulting partition $(V_{i,j})$ will still be (d, ε) -super-regular and regular on K_k^r and R , respectively. For further details, we refer the reader to the proof of Lemma 6 in [6]. \square

5. Main theorem

In this section we prove the main theorem.

Theorem 5.1. *For fixed integers r, Δ , and reals $0 < p \leq 1$ and $\gamma > 0$, there exists a constant $\beta > 0$ such that a.a.s., any spanning subgraph G' of $G(n, p)$ with minimum degree $\delta(G') \geq (1 - 1/r + \gamma)np$ contains every n -vertex graph H which satisfies the following properties.*

- (i) H is r -chromatic,
- (ii) has maximum degree at most Δ ,
- (iii) has bandwidth at most βn with respect to a labeling of vertices by $1, 2, \dots, n$, and
- (iv) for every interval $[a, a + \beta^2 n] \subset [1, n]$, there exists a vertex $v \in H$ such that $N_H(v)$ is an independent set.

Proof. First we will adjust the parameters. We may assume that $r \geq 2$, since the case $r = 1$ is trivial. Given r, Δ, p, γ , take $d = d_{3.1}(r, p, \gamma)$, $c = \min\{c_{3.3}(\Delta, d/2), (d/8)^\Delta\}$, and $\alpha = \alpha_{2.7}(d/2, \Delta, c, r)$. Then let

$$\varepsilon = \frac{1}{2} \min \left\{ \varepsilon_{2.7} \left(\frac{d}{2}, \Delta, c, r \right), \varepsilon_{3.3} \left(\Delta, \frac{d}{2} \right), \varepsilon_{3.1}(r, p, \gamma), \frac{dp}{6r\Delta}, \left(\frac{d}{8} \right)^\Delta \right\},$$

$b_0 = b_{3.1}(r, p, \gamma, \varepsilon)$, $K_0 = K_{3.1}(r, p, \gamma, \varepsilon)$, and

$$\xi = \frac{1}{2} \min \left\{ \xi_{3.1}(r, p, \gamma, \varepsilon), \frac{(1 - \varepsilon)\alpha\varepsilon^2c}{144\Delta(K_0r)^2} \right\}.$$

Finally, choose $\beta \leq \min\{\xi^2/(6052r^3), 1/(b_0\Delta^5)\}$.

Lemma 3.1 applied to G' provides us a subgraph $G'' \subset G'$, a graph R over the vertex set $[k] \times [r]$ with $k \leq K_0$, a set B with $|B| = b \leq b_0$, sets $(V_{i,j}^*)_{1 \leq i \leq k, 1 \leq j \leq r}$, and an r -equitable integer partition $(m_{i,j})_{1 \leq i \leq k, 1 \leq j \leq r}$ satisfying (i)–(iv). Given this partition $(m_{i,j})$, apply Lemma 3.2 to H and get a partition $W_{i,j}$ of H satisfying (a)–(e) of the lemma. Since an embedding of H into G'' is also an embedding into G' , by abusing notation, we will denote G' for the graph G'' . Note that by doing this, we can only guarantee $\delta(G') \geq (1 - 1/r + 4\gamma/5)np$.

To control the set B , we will find vertices of H which can be mapped into the set B . Note that for this step, the set B contained in $V(G')$ comes first, and then we look at H to decide which of its vertices can be mapped into B . Considering the fact that we are trying to embed a particular given graph H into G' , this step might seem somewhat peculiar.

Claim 5.2. *There exists a set $Z \subset V(H) \setminus (\bigcup_{s=0}^2 N^{(s)}(X))$, and a one-to-one graph homomorphism $g : Z \rightarrow V(G')$ which satisfies the following properties.*

- (i) $B \subset g(Z) \subset B \cup (\bigcup_{i,j} V_{i,j}^*)$,
- (ii) for $W_B = g^{-1}(B)$, $Z = W_B \cup N_H(W_B)$,
- (iii) for $w \in N_H^{(2)}(W_B)$, assume that $w \in W_{i,j}$. Then there exists a set $C_w \subset V_{i,j}^* \setminus g(Z)$ of size $|C_w| \geq 2cm_{i,j}$ which is contained in the common neighborhood of all vertices in $g(N_H(w) \cap Z)$.

The proof of this claim will be given later. Once we apply this claim, we obtain a partial embedding g of H which embeds the vertices Z , and constrains the image of every vertex $w \in N_H(Z) \setminus Z$ to some set C_w . Moreover, the set B is covered by the image of this map.

Next, we adjust the partition of G' in order to embed the remaining vertices of H . The goal is to obtain a partition in which the sets $V_{i,j}$ have size $n_{i,j} = |W_{i,j} \setminus Z| + |V_{i,j}^* \cap g(Z)|$, where the first term comes from the number of remaining vertices to be mapped and the second term comes

from the vertices which have already been mapped to $V_{i,j}^*$. Let $\delta_{i,j} = |V_{i,j}^* \cap g(Z)|$, and note that $\sum_{1 \leq i \leq k, 1 \leq j \leq r} \delta_{i,j} \leq |g(Z)| = |Z| \leq (\Delta + 1)b_0$ by part (ii) of Claim 5.2. Since Δ, b_0 are constants and $m_{i,j}$ is linear in n for all i, j ,

$$n_{i,j} \leq |W_{i,j}| + \delta_{i,j} \leq (1 + \xi)m_{i,j} + (\Delta + 1)b_0 \leq (1 + 2\xi)m_{i,j}, \quad \text{and}$$

$$n_{i,j} \geq |W_{i,j}| - |Z| \geq |W_{i,j}| - (\Delta + 1)b_0 \geq (1 - \xi)m_{i,j} - (\Delta + 1)b_0 \geq (1 - 2\xi)m_{i,j}.$$

Therefore $n_{i,j} \in [(1 - \xi_{3.1})m_{i,j}, (1 + \xi_{3.1})m_{i,j}]$. Moreover, we have

$$\sum_{i,j} n_{i,j} = \sum_{i,j} |W_{i,j} \setminus Z| + |V_{i,j}^* \cap g(Z)| = \left(\sum_{i,j} |W_{i,j}| \right) - |Z| + (|g(Z)| - |B|) = n - b.$$

Thus we can use Lemma 3.1 to obtain a partition $(V_{i,j})_{1 \leq i \leq k, 1 \leq j \leq r}$ of the vertices $V \setminus B$ such that $|V_{i,j}| = n_{i,j}$ for all i, j , $(V_{i,j})$ is (d, ε) -regular on R , and (d, ε) -super-regular on K_k^r . Then since $g(Z) \subset V_{i,j}^* \subset V_{i,j}$, by defining $V'_{i,j} = V_{i,j} \setminus g(Z)$, we have, $|V'_{i,j}| = n_{i,j} - \delta_{i,j} = |W_{i,j} \setminus Z|$. Note that we removed only at most constant number of vertices from $V_{i,j}$ to obtain $V'_{i,j}$. Thus by Lemma 4.3, $(V'_{i,j})_{1 \leq i \leq k, 1 \leq j \leq r}$ is $(d - \varepsilon, 2\varepsilon)$ -regular on R and $(d - \varepsilon, 2\varepsilon)$ -super-regular on K_k^r . Let $V' := \bigcup_{1 \leq i \leq k, 1 \leq j \leq r} V'_{i,j}$. Since $d - \varepsilon \geq d/2$, we may assume that the partition $(V'_{i,j})$ is $(d/2, 2\varepsilon)$ -regular and super-regular, respectively.

We would like to find an embedding of the remaining vertices of H so that $W_{i,j} \setminus Z$ gets mapped to $V'_{i,j}$ for all i, j , and every vertex $w \in N(Z) \setminus Z$ gets mapped to a vertex in C_w . Recall that X is a subset of $V(H)$ obtained in Lemma 3.2 and $|X \cup N(X)| \leq (\Delta + 1)kr\xi n \leq (\varepsilon_{3.3}/(kr))n$. Apply Lemma 3.3 with the set X and $Y = N(X) \setminus X$ to embed the vertices X into V' so that (i)–(iii) of Lemma 3.3 hold. Now we have a new set of constraints, namely, every $y \in Y$ has a set C_y which it has to be mapped to. Since $Z \subset V(H) \setminus (\bigcup_{s=0}^2 N^{(s)}(X))$, the set Y and $N(Z) \setminus Z$ are disjoint, thus the constraints coming from the vertices Y and the ones coming from $N(Z) \setminus Z$ will not interfere with each other. Extend the map g which embedded the vertices Z so that g is an embedding of $X \cup Z$. Let $V''_{i,j} := V'_{i,j} \setminus g(X) = V_{i,j} \setminus g(X \cup Z)$ and $V'' = \bigcup_{i,j} V''_{i,j}$. Then by $m_{i,j} \geq (1 - \varepsilon)n/(kr)$ from Lemma 3.1 and $|V_{i,j}| = n_{i,j} \geq (1 - 2\xi)m_{i,j}$,

$$|V_{i,j} \setminus V''_{i,j}| \leq |X| + |Z| \leq kr\xi n + (\Delta + 1)b_0 \leq \frac{2\xi k^2 r^2 m_{i,j}}{1 - \varepsilon} \leq \frac{2\xi k^2 r^2}{(1 - 2\xi)(1 - \varepsilon)} |V_{i,j}|$$

$$\leq \frac{\varepsilon^2}{36} |V_{i,j}|.$$

Recall that the partition $(V_{i,j})$ was (d, ε) -regular on R and (d, ε) -super-regular on K_k^r . Consequently, by Lemma 4.3 with $\hat{\alpha} = \hat{\beta} = \varepsilon^2/36$, the partition $(V''_{i,j})$ is $(d - \varepsilon^2/9, \varepsilon + \varepsilon)$ -regular on R and $(d - \varepsilon^2/9, \varepsilon + \varepsilon)$ -super-regular on K_k^r . We may assume that $(V''_{i,j})$ is $(d/2, 2\varepsilon)$ -regular and super-regular, respectively.

Let f be the graph homomorphism of H to R given in Lemma 3.2. Since we finished embedding X , by (d) of Lemma 3.2, the homomorphic image under f of all the remaining edges of H corresponds to K_k^r in the graph R . Thus once we check that the parameters are chosen correctly, we can apply the blow-up lemma, Theorem 2.7, to each of the partition $(V''_{i,j})_{1 \leq j \leq r}$ for fixed $i \in [k]$ separately, to find an embedding of the remaining vertices $V(H) \setminus (X \cup Z)$ which is consistent with the map g .

In the remaining part of the proof, we verify that the parameters are chosen so that we can apply the blow-up lemma. The previously embedded vertices constrain the possible images of vertices in $N_H(Z) \setminus Z$ and $Y = N_H(X) \setminus X$. For a vertex $w \in N_H(Z) \setminus Z$, by Claim 5.2, the image of w was constrained to a set $C_w \subset V_{i,j}^*$ of size at least $2cm_{i,j}$ for some i, j . Among these vertices, some could have been used for the sets X , but the number of remaining vertices in C_w is still at least

$$2cm_{i,j} - |X| \geq 2cm_{i,j} - kr\xi n \geq 2cm_{i,j} - \frac{(kr)^2 \xi}{(1 - \varepsilon)} m_{i,j} \geq 2cm_{i,j} - \frac{c}{4} m_{i,j} \geq cn_{i,j} \geq c|V''_{i,j}|,$$

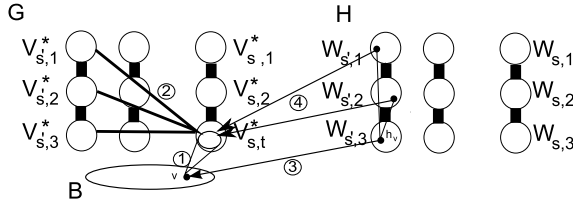


Fig. 3. Assigning vertices to B . Number indicates the logical order.

where we used $m_{i,j} \geq (1 - \epsilon)n/(kr)$ from Lemma 3.1(ii), and $n_{i,j} \leq (1 + 2\xi)m_{i,j}$ which we established above, and $n_{i,j} = |V_{i,j}| \geq |V''_{i,j}|$. For a vertex $y \in Y$, the size of the set C_y is at least $c|V'_{i,j}| \geq c|V''_{i,j}|$ for corresponding i, j by Lemma 3.3.

Moreover, by the choice of ξ depending on α , we have $|N_H(X)| \leq \Delta|X| \leq \Delta kr\xi n \leq (\alpha/4)m_{i,j} \leq (\alpha/2)n_{i,j}$ for arbitrary i, j , and so the size of Y is less than $(\alpha/2) \min_{i,j} n_{i,j}$. Also, $N(Z) \setminus Z$ has size at most $|N^{(2)}(W_B)| \leq b_0\Delta^2$ which is a constant. Thus there are at most $\alpha \min_{i,j} n_{i,j}$ vertices inside V'' whose images are constrained. Finally, note that we picked $2\epsilon \leq \epsilon_{2.7}(d/2, \Delta, c, r)$, so that $(d/2, 2\epsilon)$ -super-regularity over K_k^r suffices for the application of the blow-up lemma, Theorem 2.7. Once we apply the blow-up lemma, we can find a mapping which embeds all the remaining vertices of H , and when combined with the previous mappings, forms a graph homomorphism of H into G . \square

Proof of Claim 5.2. For a vertex $v \in B$, since $n = |B| + \sum_{i,j} m_{i,j}$,

$$\begin{aligned} \left| v \setminus \left(\bigcup_{1 \leq i \leq k, 1 \leq j \leq r} V_{i,j}^* \right) \right| &= n - \sum_{i,j} |V_{i,j}^*| = |B| + \sum_{i,j} (m_{i,j} - |V_{i,j}^*|) \\ &\leq b_0 + \sum_{i,j} \epsilon m_{i,j} \leq b_0 + \epsilon n, \end{aligned}$$

and v has at least

$$\delta(G') - (b_0 + \epsilon n) \geq (1 - 1/r)np - 2\epsilon n = (1 - 1/r - 2\epsilon p^{-1})np$$

neighbors in $\bigcup_{i,j} V_{i,j}^*$. By the fact $\sum_{i,j} m_{i,j} \leq n$, this implies that there exists an index (s, t) such that v has at least $(1 - 1/r - 2\epsilon p^{-1})m_{s,t}p \geq \frac{1}{3}m_{s,t}p$ neighbors in $V_{s,t}^*$. Since R has rk vertices and $\delta(R) > (r - 1)k$ by Lemma 3.1(i), by pigeonhole principle there exists an index $s' \in [k]$ such that (s, t) is adjacent to (s', j) in R for all $j \in [r]$. By property (e) of Lemma 3.2 there exists at least $1/\beta$ vertices in $(\bigcup_{1 \leq j \leq r} W_{s',j}) \setminus (\bigcup_{l=0}^3 N^{(l)}(X))$ which have independent neighborhoods. Since $|B|\Delta^5 \leq b_0\Delta^5 \leq 1/\beta$, we can assign one such vertex h_v to each $v \in B$ so that the vertices h_v have distance at least 5 to each other in H (we want them to be far apart from each other so that later they do not constrain the same set of vertices). Thus we have assigned $g(h_v) = v$ (see Fig. 3).

Fix a vertex $v \in B$. By the choice of the indices, $(V_{s,t}^*, V_{s',j}^*)$ is a (d, ϵ) -regular pair for all $j \in [r]$. Therefore, there are at most $r\epsilon|V_{s,t}^*|$ vertices in $V_{s,t}^*$ which have at most $(d - \epsilon)|V_{s',j}^*|$ neighbors in at least one of the sets $V_{s',j}^*$ for $j \in [r]$. Since v has at least $(1/3)m_{s,t}p$ neighbors in $V_{s,t}^*$, and

$$\frac{1}{3}m_{s,t}p - r\epsilon|V_{s,t}^*| \geq \frac{p}{3}|V_{s,t}^*| - r\epsilon|V_{s,t}^*| > \frac{p}{4}|V_{s,t}^*|, \tag{1}$$

we can find one vertex $v_1 \in N_{G'}(v)$ which has at least $(d - \epsilon)|V_{s',j}^*|$ neighbors in $V_{s',j}^*$ for all $j \in [r]$. We will show by induction that there are Δ vertices v_1, \dots, v_Δ which have many common neighbors in the sets $V_{s',j}^*$ for all $j \in [r]$. Assume that for some $k \leq \Delta - 1$, we have found $v_1, \dots, v_k \in N_{G'}(v)$ which have at least $(d - \epsilon)^k|V_{s',j}^*|$ common neighbors in $V_{s',j}^*$ for all $j \in [r]$. For a fixed j , since $(d - \epsilon)^k|V_{s',j}^*| \geq \epsilon|V_{s',j}^*|$ (recall that we chose $\epsilon \leq (d/8)^\Delta$), by the ϵ -regularity of pairs, there are at most $\epsilon|V_{s,t}^*|$ vertices in $V_{s,t}^*$ which have less than $(d - \epsilon)^{k+1}|V_{s',j}^*|$ neighbors in $V_{s',j}^* \cap \bigcap_{i=1}^k N_{G'}(v_i)$.

Thus when we consider all the indices, there would be at most $r\varepsilon|V_{s,t}^*|$ such ‘bad’ vertices. By (1), since $(p/4)|V_{s,t}^*| - k > 0$, we can pick a vertex $v_{k+1} \in N_{G'}(v)$ not equal to v_1, \dots, v_k so that the size of the common neighborhood of v_1, \dots, v_{k+1} in $V_{s',j}^*$ is at least $(d - \varepsilon)^{k+1}|V_{s',j}^*|$ for all $j \in [r]$. In the end, we will find v_1, \dots, v_Δ as promised.

Arbitrarily embed the neighbors of h_v into v_i one by one. Since H has maximum degree at most Δ and the neighborhood of h_v is an independent set, this embedding is a graph homomorphism (note that we heavily rely on the fact that $N_H(h_v)$ is an independent set). Repeat it for other vertices of B . Since B is a set of constant size, and in (1) we have $(p/4)|V_{s,t}^*| - \Delta|B| > 0$, this can be done for every vertex in B even if they share the same set $V_{s,t}^*$. Moreover, for two vertices $v, v' \in B$, their preimages $h_v = g^{-1}(v)$ and $h_{v'} = g^{-1}(v')$ were chosen to be at distance at least 5 apart from each other. Thus there will be no edges between the neighborhood of h_v and the neighborhood of $h_{v'}$. Consequently, once we find a map as above for all the vertices in the neighborhood of $W_B := g^{-1}(B)$, it will become a graph homomorphism of $H[W_B \cup N(W_B)]$ to G' (in fact, here we only need $h^{-1}(v)$ and $h^{-1}(v')$ to be at distance 4 apart). Let $Z = W_B \cup N(W_B)$.

For v and h_v as above, pick a vertex $w \in N_H^{(2)}(h_v)$. By the fact $h_v \in (\bigcup_{j=1}^r W_{s',j}) \setminus (\bigcup_{l=0}^3 N^{(l)}(X))$ and the property of the set X saying that edges not incident to X only lie on K_k^r in the homomorphic image of H into R , we know that $w \in W_{s',t'}$ for some $t' \in [r]$. Therefore by the condition on the size of the common neighbors that we imposed on the images of $N_H(h_v)$, there exists a set C_w of size at least $(d - \varepsilon)^\Delta |V_{s',t'}^*| \geq 4c|V_{s',t'}^*|$ inside $V_{s',t'}^*$ whose every element is a possible image of w . Here we rely on the fact that the vertices in W_B are at distance at least 5 apart from each other, since this implies that all the neighbors of w in Z are solely contained in $N_H(h_v)$, and thus all the vertices in C_w are indeed possible images of w . Even if we discard the elements of $g(Z)$ from C_w , since $|Z| \leq (\Delta + 1)|B|$ is a constant and $|V_{s',t'}^*|$ is linear in n , the size of the set C_w will be at least $2c|V_{s',t'}^*|$. \square

Equipped with this theorem, we can prove an embedding result for general graphs H which does not satisfy the condition of having enough vertices with independent neighborhood. The following corollary states that as long as the order of H is slightly smaller than that of G , we can still find a copy of H in subgraphs of $G(n, p)$. The necessity of H being smaller than $G(n, p)$ will be discussed in the next section.

Corollary 5.3. *For all integers r, Δ , and reals $0 < p \leq 1, \gamma > 0$, there exists a constant $\beta > 0$ such that the following holds. Let H be an r -chromatic graph on at most $n - 1/\beta^2$ vertices with $\Delta(H) \leq \Delta$ and bandwidth at most βn . Then $G = G(n, p)$ a.a.s. satisfies the following. Let $G' \subset G$ be a spanning subgraph with $\delta(G') \geq (1 - 1/r + \gamma)np$, then G' contains a copy of H .*

Proof. Let $\beta' = \beta_{5,1}$ and $\beta = \beta'/2$. Assume that H is a graph with exactly $n - 1/\beta^2$ vertices which satisfies the condition above and label the vertices as $1, \dots, n - 1/\beta^2$ so that the bandwidth is at most βn . We will construct a new graph H' containing H which satisfies the condition of Theorem 5.1 with parameter β' as following. Insert an isolated vertex at the end of every interval $[(\beta^2 n - 1)k + 1, (\beta^2 n - 1)(k + 1)]$. Clearly, H is still r -chromatic, since we added an independent set. Moreover, since we added at most $1/\beta^2$ new vertices, H' has at most n vertices and bandwidth at most $\beta n + 1/\beta^2 \leq \beta' n$. By the fact that all the new vertices are isolated, for every $[a, a + \beta^2 n] \subset [1, n]$, there exists a vertex with independent neighborhood. Since $\beta' \geq \beta$, this also holds with β replaced by β' . Therefore we can apply Theorem 1.1 to find a copy of H' in G which also gives us a copy of H in G . \square

6. Packing problem

Throughout this section let H_0 be a fixed graph on h vertices with chromatic number r . We will investigate the following problem: “For a fixed $0 < p \leq 1$, when does every spanning $G' \subset G(n, p)$ with $\delta(G') \geq (1 - 1/r + \gamma)np$ a.a.s. have a perfect H_0 -packing?”. Our goal is to extend the results of Alon and Yuster [2], Komlós, Sárközy, and Szemerédi [24] to random graphs. It is clear that n must be a multiple of h but is there any additional necessary condition?

For the simplicity of later exposition, before proceeding further, we establish several properties of random graphs that will be used later.

Lemma 6.1. *Let $0 < p \leq 1$ be fixed, and C, α be positive constants. Then $G = G(n, p)$ satisfies the following properties with probability $1 - e^{-\Omega_{\alpha,c,p}(n)}$.*

- (i) Every vertex v has degree $\deg(v) \in [(1 - \alpha)np, (1 + \alpha)np]$.
- (ii) Every pair of distinct vertices $v, w \in V$ has between $(1 - \alpha)np^2$ and $(1 + \alpha)np^2$ common neighbors.
- (iii) For all $X, Y \subset V$ of size $|X|, |Y| = \Omega(n)$, $e(X, Y) \in [(1 - \alpha)|X||Y|p, (1 + \alpha)|X||Y|p]$. In particular, $e(X) = e(X, X)/2 \in [(1 - \alpha)|X|^2p/2, (1 + \alpha)|X|^2p/2]$.
- (iv) For every set X of size $|X| \leq Cp^{-2}$, there are at most $e^{-\Omega_{\alpha}(|X|p)}n$ vertices $v \in V \setminus X$ which have $\deg(v, X) \notin [(1 - \alpha)|X|p, (1 + \alpha)|X|p]$.
- (v) For every set X of size $|X| \leq Cp^{-2}$, there are at most $e^{-\Omega_{\alpha}(|X|p^2)}n^2p$ edges $\{v, w\}$ in $G[V \setminus X]$ such that v and w have fewer than $(1 - \alpha)|X|p^2$ common neighbors in X .

Proof. (i), (ii), (iii) follow directly from Chernoff inequality and taking union bounds. We omit the details. Let X be a fixed set of size $|X| \leq Cp^{-2}$. To prove (iv), note that by Chernoff inequality, the probability of a single vertex $v \in V \setminus X$ having $\deg(v, X) \notin [(1 - \alpha)|X|p, (1 + \alpha)|X|p]$ is $e^{-\Omega_{\alpha}(|X|p)}$. Thus the expected number of such vertices in $V \setminus X$ is $e^{-\Omega_{\alpha}(|X|p)}n$. Since these events for different vertices are mutually independent of each other, we can apply Chernoff inequality once more to conclude that with probability $1 - e^{-\Omega_{\alpha,c,p}(n)}$, there are at most $2e^{-\Omega_{\alpha}(|X|p)}$ vertices $v \in V \setminus X$ which have $\deg(v, X) \notin [(1 - \alpha)|X|p, (1 + \alpha)|X|p]$. And since there are at most $\sum_{k \leq Cp^{-2}} \binom{n}{k}$ choices for X , we can take the union bound to derive the conclusion for all choices of X .

To prove (v), first expose the edges between X and $V \setminus X$ and call a pair of vertices $\{v, w\} \in V \setminus X$ bad if v and w have fewer than $(1 - \alpha)|X|p^2$ common neighbors in X . We will bound the number of bad pairs of vertices by bounding the number of pairs $\{v, w\}$ where (a) v has too few neighbors in X or (b) v has enough neighbors but w does not have enough common neighbors with v in X . To bound (a), by (iv) with $\alpha/2$ instead of α , we know that there are at most $e^{-\Omega_{\alpha}(|X|p)}n$ vertices $v \in V \setminus X$ which have less than $(1 - \alpha/2)|X|p$ neighbors in X . Even if we assume that all the pairs which contain these vertices are bad, there will be at most $e^{-\Omega_{\alpha}(|X|p)}n^2$ such pairs. Then to bound (b), assume that $v \in V \setminus X$ has more than $(1 - \alpha/2)|X|p$ neighbors in X . Then by Chernoff inequality, any $w \in V \setminus (X \cup \{v\})$ has at least $(1 - \alpha/3)|X|p \geq (1 - \alpha)|X|p^2$ neighbors in X with probability $1 - e^{-\Omega_{\alpha}(|X|p^2)}$. Since for distinct vertices in $V \setminus (X \cup \{v\})$ these events are independent, by using Chernoff inequality again, with probability $1 - e^{-\Omega_{\alpha,c,p}(n)}$, there will be at most $e^{-\Omega_{\alpha}(|X|p^2)}n$ vertices $w \in V \setminus (X \cup \{v\})$ such that v and w have fewer than $(1 - \alpha)|X|p^2$ common neighbors in X . By taking the union bound over all vertices $v \in V \setminus X$, we can conclude that with probability $1 - e^{-\Omega_{\alpha,c,p}(n)}$, the contribution from (b) is $e^{-\Omega_{\alpha}(|X|p^2)}n^2$. Thus there are at most $e^{-\Omega_{\alpha}(|X|p^2)}n^2$ bad pairs in $V \setminus X$.

Now expose the edges within $V \setminus X$. By Chernoff inequality, with probability $1 - e^{-\Omega_{\alpha,c,p}(n^2p)}$, at most $e^{-\Omega_{\alpha}(|X|p^2)}n^2p$ bad pairs will form an edge. Since $n^2p \gg n$, all the required events happen with probability $1 - e^{-\Omega_{\alpha,c,p}(n)}$. Since there are at most $\sum_{k \leq Cp^{-2}} \binom{n}{k}$ choices for X , we can take the union bound to derive the conclusion for all choices of X . \square

Coming back to our main question of this section regarding perfect packing in subgraphs of $G(n, p)$, a simple observation combined with Theorem 1.1 shows that n being a multiple of h is sufficient for certain graphs. More precisely, this condition is sufficient if H_0 contains a vertex having independent neighborhood. To see this, let $n = hn'$ for some integer n' and let H be a graph consisting of n' vertex disjoint copies of H_0 . Then H has bandwidth at most h and chromatic number r . Moreover, since each copy of H_0 has a vertex with independent neighborhood, it is clear that the conditions of Theorem 1.1 hold. Therefore a.a.s. H can be embedded into every spanning subgraph $G' \subset G(n, p)$ with $\delta(G') \geq (1 - 1/r + \gamma)np$. This result can be formally stated as following.

Proposition 6.2. *Let H_0 be an r -chromatic graph which has a vertex not contained in a triangle. Then for any $0 < p \leq 1$ and $\gamma > 0$, a.a.s. any spanning subgraph $G' \subset G(n, p)$ with minimum degree $(1 - 1/r + \gamma)np$ contains a perfect H_0 -packing.*

In particular, if H_0 is a bipartite graph then a.a.s. every G' contains a perfect H_0 -packing. One might suspect that the same result holds for every graph H_0 , but unfortunately this is not true. The following proposition shows that perfect packing is impossible for every graph which does not satisfy the condition of having a vertex with independent neighborhood.

Proposition 6.3. *Let H_0 be a fixed graph whose every vertex is contained in a triangle. Then for all $\varepsilon > 0$, there exists p_ε such that for all $0 < p \leq p_\varepsilon$, $G = G(n, p)$ a.a.s. has a spanning subgraph G' with $\delta(G') > (1 - \varepsilon)np$ such that at least $\varepsilon p^{-2}/3$ vertices of G' are not contained in a copy of H_0 .*

Proof. Let X be a set of size $|X| = \varepsilon p^{-2}/3$ and delete all the edges of G inside X . Since X is a set of constant size, the effect of these edges is asymptotically negligible. For a vertex $v \in V \setminus X$, we expect that it has $|X|p = \varepsilon p^{-1}/3$ neighbors in X , and by Lemma 6.1(iv) with $\alpha = 1$, a.a.s. there are at most $e^{-\Omega(\varepsilon p^{-1})}n$ vertices in $V \setminus X$ which have degree greater than $2|X|p = 2\varepsilon p^{-1}/3$ into X . Let W be the collection of all such vertices and remove all the edges between X and W . Note that if $p \leq p_\varepsilon := c\varepsilon/\log(\varepsilon^{-1})$ for sufficiently small constant c , then we have $e^{-\Omega(\varepsilon p^{-1})}n \leq \varepsilon np/2$. Thus we will not remove too many edges from any of the vertices in X (and also from vertices in W since X is a set of constant size).

Then for all $y \in V \setminus (X \cup W)$ delete edges according to the following rule. For every triangle xyz in G with $x \in X$, $y, z \in V \setminus (X \cup W)$, remove the edge yz . By Lemma 6.1(ii), a.a.s. x and y have at most $(9/8)np^2$ common neighbors. Moreover $\deg(y, X) \leq 2\varepsilon p^{-1}/3$ because $y \notin W$, and therefore we have deleted at most $(2\varepsilon p^{-1}/3) \cdot (9/8)np^2 \leq 3\varepsilon np/4$ edges from y . Also note that there are no further edges removed from y since the process is symmetric and xyz forms a triangle if and only if xzy forms a triangle. Let G' be the new graph. Then $\delta(G') \geq (1 - \varepsilon)np$ and the deleting process guarantees that every vertex $x \in X$ is not contained in a triangle. However, since every point of H_0 is contained in a triangle, there cannot exist a copy of H_0 in G' which contains a vertex from X . Thus we have found a required G' . \square

Remark. It is easy to see that in this lemma, the constant p must be sufficiently small. Indeed, if p is close to 1, then every subgraph of $G(n, p)$ with minimum degree $(1 - 1/r + \gamma)np$ in fact has minimum degree greater than $(1 - 1/r + \gamma/2)n$ and thus Komlós, Sárközy, and Szemerédi's theorem [24] shows that a perfect packing does exist.

The consequence of this proposition is quite interesting. Given H_0 as in the proposition, if we take H to be the graph consisting of n' vertex disjoint copies of H_0 and let $n = hn'$, then Proposition 6.3 is equivalent to saying that, for any $\gamma < 1/r$ and sufficiently small p , H a.a.s. cannot be embedded into some $G' \subset G(n, p)$ with $\delta(G') \geq (1 - 1/r + \gamma)np$. Note that such graph H satisfies all the assumptions of Theorem 1.1 except the one requiring H to have enough vertices with independent neighborhood. Therefore, this proposition indicates the necessity of this condition for the theorem.

The proof of Proposition 6.3 also shows that, even for arbitrary graph H which is not necessarily a disjoint union of copies of a fixed graph, if every vertex of H is contained in a triangle, then H cannot be embedded into G' . On the other hand, as we have seen from Corollary 5.3, if H is allowed to be slightly smaller than G (constant difference is enough), then we can embed H into the subgraph G' . However, even though the required gap between the sizes of G and H is only of constant size, this constant might be rather huge because it comes from the regularity lemma. This suggests a very natural question of determining the correct order of magnitude of this gap. In the remaining part of this section, we will investigate this question in the case when H is the union of vertex disjoint copies of H_0 .

Let K_{t_1, \dots, t_r} be the complete r -partite graph with parts having size t_1, \dots, t_r respectively. Next lemma shows that for certain graphs, the assertion of Proposition 6.3 is essentially best possible.

Lemma 6.4. Let H_0 be the complete r -partite graph $K_{1,m,\dots,m}$. Then there exists a constant $C = C(r)$ such that for all $0 < p \leq 1$, there exists $\varepsilon = \varepsilon(r, p)$ such that $G = G(n, p)$ a.s. has the following property. For every spanning subgraph $G' \subset G$ with minimum degree $\delta(G') \geq (1 - 1/r)np$, and every set $T \subset V(G')$ of size $|T| \leq \varepsilon n$, all but at most Cp^{-2} vertices of $V \setminus T$ are contained in a copy of H_0 in G' which does not intersect T .

Proof. Let $V = V(G)$ and $\varepsilon = \varepsilon(r, p)$, $C = C(\varepsilon)$ are constants which we choose later. Given G' and T as above, let $X \subset V \setminus T$ be an arbitrary set of size Cp^{-2} , and let $Y = V \setminus (X \cup T)$. By assuming that the events of Lemma 6.1 hold, we will show that there exists a copy of H_0 in G' which intersects X but not T .

For a vertex $x \in X$, let N_x be the set of neighbors of x in Y in the graph G , that is $N_x := N_G(x) \cap Y$, and note that the size of N_x is at least $(1 - 3\varepsilon p^{-1})np$ by Lemma 6.1(i) and the fact $|X \cup T| \leq 2\varepsilon n$. Then in the graph G' , since the degree of x is at least $(1 - 1/r)np$, we can arbitrarily fix a set $N'_x \subset N_{G'}(x) \cap Y$ of size $|N'_x| = (1 - 1/r - 2\varepsilon p^{-1})np$. We claim that there exists a vertex $x \in X$ such that $e_{G'}(N'_x) \geq (1 - 1/(r - 2) + \gamma)|N'_x|^2 p/2$ for some constant $\gamma > 0$. Then, by Corollary 2.6, N'_x contains the complete $(r - 1)$ -partite graph with parts of size m , which together with x will form a copy of $K_{1,m,\dots,m}$ that intersects X but not T .

Thus it remains to verify the claim. To prove this claim we count the number of triangles xy_1y_2 in G' such that $x \in X, y_1, y_2 \in N'_x$. Let this number be M . To lower bound M , first bound the number of triangles xy_1y_2 in G such that $x \in X, y_1, y_2 \in N_x$, and y_1y_2 is an edge of the graph G' (we will later subtract the triangles whose y_1 or y_2 is not in N'_x). Let this number be M_0 . Since $|X \cup T| \leq 2\varepsilon n$, by Lemma 6.1(iii),

$$\begin{aligned} e_{G'}(Y) &\geq e_{G'}(V) - e_{G'}(V, X \cup T) \geq \left(1 - \frac{1}{r}\right) \frac{n^2 p}{2} - e_G(V, X \cup T) \\ &\geq \left(1 - \frac{1}{r} - O(\varepsilon p^{-1})\right) \frac{n^2 p}{2}. \end{aligned}$$

Let $\varepsilon' = \varepsilon'(r)$ be a small constant. If $C = C(r)$ is large enough, by Lemma 6.1(v), there are at most $e^{-\Omega_{\varepsilon'}(|X|p^2)} n^2 p = e^{-\Omega_{\varepsilon'}(C)} n^2 p = O(\varepsilon' n^2 p)$ edges $\{v, w\}$ in $G[Y]$ which form a triangle with fewer than $(1 - \varepsilon')C$ vertices $x \in X$. These two facts provide the following bound on M_0 :

$$M_0 \geq (e_{G'}(Y) - O(\varepsilon' n^2 p))(1 - \varepsilon')C \geq \left(1 - \frac{1}{r} - O(\varepsilon p^{-1}) - O(\varepsilon')\right) \frac{Cn^2 p}{2}.$$

To obtain a bound on M from M_0 , we can subtract the number of triangles xy_1y_2 as above such that either y_1 or y_2 is not in N'_x . Since $|N_x| = (1 - O(\varepsilon p^{-1}))np$,

$$|N_x \setminus N'_x| = |N_x| - |N'_x| = (1 - O(\varepsilon p^{-1}))np - (1 - 1/r - 2\varepsilon p^{-1})np = (1/r - O(\varepsilon p^{-1}))np.$$

Thus, if $\varepsilon = \varepsilon(p)$ is small enough, by Lemma 6.1(iii) we have,

$$\begin{aligned} M &\geq M_0 - \sum_{x \in X} (e_{G'}(N_x \setminus N'_x, N'_x) + e_{G'}(N_x \setminus N'_x)) \\ &\geq M_0 - \sum_{x \in X} (1 + O(\varepsilon p^{-1})) \left(\left(\frac{1}{r} - O(\varepsilon p^{-1}) \right) \left(1 - \frac{1}{r} + O(\varepsilon p^{-1}) \right) n^2 p^3 \right. \\ &\quad \left. + \left(\frac{1}{r} - O(\varepsilon p^{-1}) \right)^2 \frac{n^2 p^3}{2} \right) \\ &\geq \left(1 - \frac{1}{r} - O(\varepsilon p^{-1}) - O(\varepsilon') \right) \frac{Cn^2 p}{2} \\ &\quad - \sum_{x \in X} \left(\frac{1}{r} \left(1 - \frac{1}{r} \right) n^2 p^3 + \frac{1}{r^2} \frac{n^2 p^3}{2} + O(\varepsilon p^{-1}) n^2 p^3 \right) \\ &= \left(1 - \frac{3}{r} + \frac{1}{r^2} - O(\varepsilon p^{-1}) - O(\varepsilon') \right) \frac{Cn^2 p}{2}. \end{aligned}$$

On the other hand we have, $M = \sum_{x \in X} e_{G'}(N'_x)$. Thus combining these two equations and using the fact $|X| = Cp^{-2}$, we can find a vertex $x_0 \in X$ such that

$$e_{G'}(N'_{x_0}) \geq \frac{M}{|X|} \geq \left(1 - \frac{3}{r} + \frac{1}{r^2} - O(\varepsilon p^{-1}) - O(\varepsilon')\right) \frac{n^2 p^3}{2}$$

$$\geq \left(1 - \frac{1}{r-2} + \gamma\right) \left(1 - \frac{1}{r}\right)^2 \frac{n^2 p^3}{2} \geq \left(1 - \frac{1}{r-2} + \gamma\right) \frac{|N'_{x_0}|^2 p}{2},$$

for some constant $\gamma > 0$ depending on r , small enough ε' depending on r , and ε depending on r and p . This concludes the proof. \square

Next, we extend Lemma 6.4 to all graphs H_0 .

Lemma 6.5. *Let H_0 be a fixed r -chromatic graph. Then there exists a constant $C = C(r)$ such that for every $0 < p \leq 1$, there exists $\varepsilon = \varepsilon(r, p)$ such that $G = G(n, p)$ a.s. has the following property. For every spanning subgraph $G' \subset G$ with minimum degree $\delta(G') \geq (1 - 1/r)np$, and every set $T \subset V(G')$ of size $|T| \leq \varepsilon n$, all but at most Cp^{-2} vertices of $V \setminus T$ are contained in a copy of H_0 in G' which does not intersect T .*

Proof. Let $V = V(G)$, and $C = C_{6.4}(r)$. Let $\varepsilon \leq \varepsilon_{6.4}(r, p)$ and $D = D(r, p, \varepsilon)$ be constants to be chosen later. We may assume that H_0 is a complete r -partite graph with equal parts of size s . Throughout the proof we condition on the event that the statements of Lemma 6.1 hold.

Given G' and T as above, let $X \subset V \setminus T$ be an arbitrary set of size Cp^{-2} . We will show that there exists a copy of H_0 in G' which intersects X but not T . By Lemma 6.4 we can find a complete r -partite graph with parts $\{x\} \cup Z_1 \cup \dots \cup Z_{r-1}$ such that $x \in X$ and $|Z_i| = Dsp^{-1}$ (note that in Lemma 6.4, the part size m can be an arbitrary constant). Let $Z = Z_1 \cup \dots \cup Z_{r-1}$ and $Y = V \setminus (X \cup Z \cup T)$. Note that $|Y| \geq (1 - 2\varepsilon)n$ for large enough n . We construct a set $A \subset Y$ of size $s - 1$ and sets $Z'_i \subset Z_i$ of size s for $1 \leq i \leq r - 1$ such that $A \cup Z'_1 \cup \dots \cup Z'_{r-1}$ forms a complete r -partite graph.

By Lemma 6.1(iv), there are at most $e^{-\Omega_\varepsilon(Ds)}n$ vertices in $V \setminus X$ such that $\deg_{G'}(y, Z_i) > (1 + \varepsilon)|Z_i|p = (1 + \varepsilon)Ds$ for any fixed $1 \leq i \leq r - 1$. Hence if $D = D(\varepsilon, p)$ is large enough, there are at most $re^{-\Omega_\varepsilon(Ds)}n = O(\varepsilon np)$ vertices $y \in Y$ which have $\deg_{G'}(y, Z_i) > (1 + \varepsilon)Ds$ for at least one $1 \leq i \leq r - 1$. Let Y_0 be these vertices. Then we have the crude bound $e_{G'}(Y_0, Z) \leq O(\varepsilon np)|Z|$. Let Y_1 be the collection of vertices in $Y \setminus Y_0$ which have at least $\varepsilon|Z_i|p = \frac{\varepsilon|Z|p}{r-1}$ neighbors in Z_i in the graph G' for all $1 \leq i \leq r - 1$, and $Y_2 := Y \setminus (Y_0 \cup Y_1)$. Then since $Y_1 \subset Y \setminus Y_0$,

$$e_{G'}(Y_1, Z) \leq \sum_{i=1}^{r-1} e_{G'}(Y_1, Z_i) \leq \sum_{i=1}^{r-1} (1 + \varepsilon)|Y_1||Z_i|p = |Y_1| \cdot (1 + \varepsilon)|Z|p,$$

and since $Y_2 = Y \setminus (Y_0 \cup Y_1)$,

$$e_{G'}(Y_2, Z) \leq |Y_2| \cdot \left((1 + \varepsilon) \frac{|Z|p}{r-1} (r-2) + \varepsilon \frac{|Z|p}{r-1} \right).$$

Thus we have,

$$e_{G'}(Y, Z) \leq e_{G'}(Y_0, Z) + e_{G'}(Y_1, Z) + e_{G'}(Y_2, Z)$$

$$\leq O(\varepsilon np)|Z| + |Y_1| \cdot (1 + \varepsilon)|Z|p + n \cdot \left((1 + \varepsilon) \frac{|Z|p}{r-1} (r-2) + \varepsilon \frac{|Z|p}{r-1} \right)$$

$$= \left(\frac{|Y_1|}{n} + \frac{r-2}{r-1} + O(\varepsilon) \right) |Z|np.$$

On the other hand, by the minimum degree condition of G' ,

$$e_{G'}(Y, Z) = \sum_{z \in Z} (\deg_{G'}(z, V) - \deg_{G'}(z, V \setminus Y)) \geq \left(\left(1 - \frac{1}{r} \right) np - (n - |Y|) \right) |Z| \geq \left(1 - \frac{1}{r} - 2 \frac{\varepsilon}{p} \right) |Z| np.$$

By combining the previous inequalities and dividing each side by $|Z|np$ we have,

$$\frac{|Y_1|}{n} \geq 1 - \frac{1}{r} - 2 \frac{\varepsilon}{p} - \frac{r-2}{r-1} - O(\varepsilon) \geq \frac{1}{2r(r-1)}.$$

The last inequality holds if we pick $\varepsilon = \varepsilon(r, p)$ small enough. Thus there are at least $\frac{1}{2r(r-1)}n$ vertices which have at least $\varepsilon|Z_i|p = \varepsilon D_s$ neighbors in Z_i for all $1 \leq i \leq r-1$. Let $D \geq \varepsilon^{-1}$, and for each such vertex fix s points in each Z_i which are adjacent to that vertex. Since there are only $\binom{|Z_i|}{s}$ possible subsets of size s in each Z_i , and these numbers are constants, if n is large enough then by pigeonhole principle we can find $s-1$ vertices y_1, y_2, \dots, y_{s-1} which are adjacent to the same s -tuple of vertices in every Z_i . Let $A = \{y_1, y_2, \dots, y_{s-1}\}$, and for each i , let Z'_i be the s -tuple which is adjacent to these vertices. Recall that $x \in X$ was a vertex chosen at the beginning, which forms a complete r -partite graph together with the sets Z_1, \dots, Z_{r-1} . Since Z'_i are subsets of Z_i , $Z'_1 \cup \dots \cup Z'_{r-1} \cup (A \cup \{x\})$ forms a complete r -partite graph with s vertices in each part which intersects X but not T . \square

We are now ready to prove the following theorem which when combined with Proposition 6.3 establishes Theorem 1.2.

Theorem 6.6. *Let H_0 be a fixed r -chromatic graph. There exists a constant $C = C(r)$ such that for every fixed $0 < p \leq 1$ and $\gamma > 0$, if a spanning subgraph $G' \subset G$ satisfies $\delta(G') \geq (1 - 1/r + \gamma)np$, then G' contains vertex disjoint copies of H_0 covering all but at most Cp^{-2} vertices.*

Proof. Let $\Delta = \Delta(H_0)$ and $h = |V(H_0)|$. Let $C = \max\{2C_{6.5}, 2rh\}$, $d = d_{3.1}(r, p, \gamma)$, and $b_0 = b_{3.1}$. Then let

$$\varepsilon = \frac{1}{2} \min \left\{ \varepsilon_{2.7} \left(\frac{d}{2}, \Delta, c, r \right), \varepsilon_{3.1}(r, p, \gamma), \varepsilon_{6.5}(r, p), \frac{d}{2} \right\},$$

and $\xi = \xi_{3.1}(r, p, \gamma, \varepsilon)$.

Assume that $G' \subset G(n, p)$ is given as above. Lemma 3.1 applied to G' provides us a subgraph $G'' \subset G'$, a graph R over the vertex set $[k] \times [r]$, a set B with $|B| = b \leq b_0$, sets $(V_{i,j}^*)$, and an r -equitable integer partition $(m_{i,j})_{1 \leq i \leq k, 1 \leq j \leq r}$ satisfying (i)–(iv) of Lemma 3.1. Let $n' := n - |B| = \sum_{i,j} m_{i,j}$ be the number of vertices not in B . Since copies of H_0 in G'' are also copies in G' , by abusing notation, we denote G' for the graph G'' . Note that by doing this, we can only guarantee $\delta(G') \geq (1 - 1/r + 4\gamma/5)np$.

We first find copies of H_0 containing vertices of B and only using vertices from $B \cup (\bigcup_{i,j} V_{i,j}^*)$. Let $T = V \setminus (B \cup (\bigcup_{1 \leq i \leq k, 1 \leq j \leq r} V_{i,j}^*))$ and note that

$$|T| \leq n - (1 - \varepsilon) \sum_{i,j} m_{i,j} - |B| \leq n - (1 - \varepsilon)(n - |B|) - |B| \leq \varepsilon n \leq \varepsilon_{6.5} n.$$

By Lemma 6.5 if $|B| \geq (C/2)p^{-2}$ then we can find a copy of H_0 in G' which intersects B but does not intersect T . Move the vertices of this copy to T . Repeat this process, as long as $|B| \geq (C/2)p^{-2}$, one can find a copy of H_0 intersecting B but not T (note that $|T| \leq \varepsilon n + |B|h \leq \varepsilon_{6.5} n$ at any point of this process). In the end we will have vertex disjoint copies of H_0 and at most $(C/2)p^{-2}$ vertices left in B . The leftover vertices of B will remain uncovered. Our next task is to find an H_0 -packing in the remaining part. Let S be the vertices belonging to the copies of H_0 found so far.

Let $\delta_{i,j} = |V_{i,j}^* \cap S|$ and construct $(n_{i,j})_{1 \leq i \leq k, 1 \leq j \leq r}$ as following. For $i \in [k-1]$, let t_i be the largest integer smaller than $\min_s (m_{i,s} - \delta_{i,s})$ which is divisible by h , and let $n_{i,j} = t_i$ for all $j \in [r]$. Then pick

t_k so that $\sum_{i=1}^k \sum_{j=1}^r (t_i + \delta_{i,j}) \in (n' - rh, n']$ is divisible by rh . Recall that $\sum_{i=1}^k \sum_{j=1}^r m_{i,j} = n'$. Since $|m_{i,j} - m_{i,j'}| \leq 1$ for all i, j, j' , we are modifying each $m_{i,j}$ by at most $(k - 1) \cdot \max_{s,t} (\delta_{s,t} + 1) + rh$ to construct $n_{i,j}$ for all i, j . Since $\delta_{i,j} \leq |S|$ for all i, j and S has constant size, it shows that $|m_{i,j} - n_{i,j}|$ is at most some constant. Thus the integer partition $(n_{i,j})$ satisfies the following properties:

- (i) $n' - \sum_{i,j} \delta_{i,j} \geq \sum_{i,j} n_{i,j} \geq n' - \sum_{i,j} \delta_{i,j} - rh$,
- (ii) $n_{i,j} \in [m_{i,j} - (\xi/2)n, m_{i,j} + (\xi/2)n]$,
- (iii) $n_{i,j} = n_{i,j'}$ for all $1 \leq i \leq k, 1 \leq j, j' \leq r$, and
- (iv) h divides $n_{i,j}$ for all $1 \leq i \leq k, 1 \leq j \leq r$.

It then follows that $n_{i,j} + \delta_{i,j} \in [m_{i,j} - \xi n, m_{i,j} + \xi n]$. So by Lemma 3.1 we can find sets $V_{i,j}$ such that $|V_{i,j}| \geq n_{i,j} + \delta_{i,j}$ which are (d, ε) -super-regular on K_k^r . Let $V'_{i,j} = V_{i,j} \setminus S$, and we have $|V'_{i,j}| \geq |V_{i,j}| - \delta_{i,j} \geq n_{i,j}$. Remove some vertices so that $|V'_{i,j}| = n_{i,j}$. The number of removed vertices is at most $n' - \sum_{i,j} (n_{i,j} + \delta_{i,j}) \leq rh$. These vertices together with the remaining vertices of B will form the $(C/2)p^{-2} + rh \leq Cp^{-2}$ uncovered vertices. Further note that we removed only at most some constant number of vertices from each $V_{i,j}$ to obtain $V'_{i,j}$.

Since $(V_{i,j})_{1 \leq i \leq k, 1 \leq j \leq r}$ is (d, ε) -super-regular on K_k^r and we removed only at most constant number of vertices from each part to obtain $V'_{i,j}$, we can conclude that $(V'_{i,j})_{i,j}$ is $(d - \varepsilon, 2\varepsilon)$ -super-regular on K_k^r (Lemma 4.3). Thus we may apply the blow-up lemma to the super-regular partitions $(V'_{i,j})_{1 \leq j \leq r}$ for each fixed $i \in [k]$ to find a perfect H_0 -packing in each of them. By (iii) and (iv) of the previous paragraph, it suffices to show that the complete r -partite graph with h vertices in each class contains a perfect H_0 -packing, or equivalently, have an r -coloring in which every color class has size h . Assume that H_0 has an r -coloring with color classes of size h_1, \dots, h_r . Then by renaming the colors, we can color the i -th copy of H_0 so that the j -th color class of it has h_{i+j-1} vertices (addition of indices are modulo r). In this way, we will end up with a coloring of r vertex disjoint copies of H_0 in which every color class has size $\sum_{i=1}^r h_i = h$. \square

7. Concluding remarks

- In this paper, we proved that for all integers r and $p \in (0, 1]$, there exists β such that if H is an r -chromatic graph on n vertices with bounded degree, bandwidth at most βn , and has enough vertices whose neighbors form an independent set, then $G(n, p)$ a.a.s. has the following property. Every spanning subgraph $G' \subset G(n, p)$ with minimum degree at least $(1 - 1/r + \gamma)np$ contains a copy of H . It would be interesting to know whether this theorem holds for $p \ll 1$ or not. As mentioned in the introduction, Böttcher, Kohayakawa, and Taraz [9] proved that for fixed $\eta, \gamma > 0, \Delta > 1$ there exist positive constants β and c such that if $p \geq c(\log n/n)^{1/\Delta}$ then a.a.s. every subgraph of $G(n, p)$ with minimum degree at least $(1/2 + \gamma)np$ contains a copy of any bipartite graph H with $(1 - \eta)n$ vertices, maximum degree Δ and bandwidth at most βn . However, it is plausible that one can even embed a spanning bipartite graph H under the same conditions. The technique we used in this paper cannot be applied mainly because of the lack of the corresponding blow-up lemma in the range $p \ll 1$. It is hopeful that a sparse version of the blow-up lemma (if one exists) will allow us to extend the same proof.

- In view of the results of Komlós [20], Shokoufandeh and Zhao [31,32], and Kühn and Osthus [27], which establishes the best possible minimum degree condition for packing problems, it is likely that in Theorem 1.2, the minimum degree condition $(1 - 1/r + \gamma)np$ can be further relaxed. However, we did not further pursue towards this direction as our primary goal was to study the packing problem in connection to Theorem 1.1.

- For a graph G , let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ be the eigenvalues of its adjacency matrix. The quantity $\lambda(G) = \max\{\lambda_2, -\lambda_n\}$ is called the second eigenvalue of G . A graph $G = (V, E)$ is called an (n, d, λ) -graph if it is d -regular, has n vertices, and the second eigenvalue of G is at most λ . It is well known (see e.g., survey [25]) that if λ is much smaller than the degree d , then G has certain random-like properties. Thus λ could serve as some kind of “measure of randomness” in G . By using an almost

identical argument as in the proof of Theorem 1.1 we can prove the bandwidth theorem for pseudo-random graphs as well.

Theorem 7.1. *For all integers r, Δ , and reals $\gamma > 0$ and $0 < p \leq 1$, there exists a constant $\beta > 0$ such that, for an (n, d, λ) graph G with $d = np$ and $\lambda = o(n)$, if n is large enough, then any spanning subgraph $G' \subset G$ with minimum degree $\delta(G') \geq (1 - 1/r + \gamma)np$ contains a copy of every graph H on n vertices which satisfies the following properties.*

- (i) H is r -chromatic,
- (ii) has maximum degree at most Δ ,
- (iii) has bandwidth at most βn with respect to a labeling of vertices by $1, 2, \dots, n$, and
- (iv) for every interval $[a, a + \beta^2(n/\lambda)] \subset [1, n]$, there exists a vertex $v \in H$ such that $N_H(v)$ is an independent set.

The sketch of the proof will be given in the arXiv version of our paper.

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References

- [1] N. Alon, J. Spencer, *The Probabilistic Method*, third ed., John Wiley & Sons Inc., Hoboken, NJ, 2008.
- [2] N. Alon, R. Yuster, H -factors in dense graphs, *J. Combin. Theory Ser. B* 66 (1996) 269–282.
- [3] J. Balogh, B. Csaba, W. Samotij, Local resilience of almost spanning trees in random graphs, *Random Structures Algorithms* 38 (2011) 121–139.
- [4] S. Ben-Shimon, M. Krivelevich, B. Sudakov, Local resilience and Hamiltonicity Maker–Breaker games in random regular graph, *Combin. Probab. Comput.* 20 (2011) 173–211.
- [5] J. Böttcher, M. Schacht, A. Taraz, Spanning 3-colourable subgraphs of small bandwidth in dense graphs, *J. Combin. Theory Ser. B* 98 (2008) 752–777.
- [6] J. Böttcher, M. Schacht, A. Taraz, Proof of the bandwidth conjecture of Bollobás and Komlós, *Math. Ann.* 343 (2009) 175–205.
- [7] J. Böttcher, Y. Kohayakawa, A. Taraz, Problem session, in: *Combinatorics and Probability Workshop, Oberwolfach*, 2009.
- [8] J. Böttcher, K. Pruessmann, A. Taraz, A. Würfl, Bandwidth, expansion, treewidth, separators, and universality for bounded degree graphs, *European J. Combin.* 31 (2010) 1217–1227.
- [9] J. Böttcher, Y. Kohayakawa, A. Taraz, Almost spanning subgraphs of random graphs after adversarial edge removal, arXiv:1003.0890v1 [math.CO].
- [10] F. Chung, Labelings of graphs, in: *Selected Topics in Graph Theory*, vol. 3, Academic Press, San Diego, CA, 1988, pp. 151–168.
- [11] D. Conlon, T. Gowers, Combinatorial theorems in sparse random sets, arXiv:1011.4310v1 [math.CO].
- [12] D. Dellamonica, Y. Kohayakawa, M. Marcinişzyn, A. Steger, On the resilience of long cycles in random graphs, *Electron. J. Combin.* 15 (2008) R32.
- [13] R. Diestel, *Graph Theory*, third ed., Graduate Texts in Mathematics, vol. 173, Springer-Verlag, Berlin, 2005.
- [14] P. Erdős, A. Stone, On the structure of linear graphs, *Bull. Amer. Math. Soc.* 52 (1946) 1087–1091.
- [15] A. Frieze, M. Krivelevich, On two Hamilton cycle problems in random graphs, *Israel J. Math.* 166 (2008) 221–234.
- [16] S. Gerke, A. Steger, The sparse regularity lemma and its applications, in: *Surveys in Combinatorics 2005*, in: *London Math. Soc. Lecture Note Ser.*, vol. 327, Cambridge Univ. Press, Cambridge, 2005, pp. 227–258.
- [17] A. Hajnal, E. Szemerédi, Proof of a conjecture of P. Erdős, in: *Combinatorial Theory and Its Applications, II*, Proc. Colloq., Balatonfüred, 1969, North-Holland, Amsterdam, 1970, pp. 601–623.
- [18] J. Kim, B. Sudakov, V. Vu, On the asymmetry of random regular graphs and random graphs, *Random Structures Algorithms* 21 (2002) 216–224.
- [19] J. Komlós, The blow-up lemma, *Combin. Probab. Comput.* 8 (1999) 161–176.
- [20] J. Komlós, Tiling Turán theorems, *Combinatorica* 20 (2000) 203–218.
- [21] J. Komlós, M. Simonovits, Szemerédi’s regularity lemma and its applications in graph theory, in: *Combinatorics*, Paul Erdős is Eighty, vol. 2, Keszthely, 1993, in: *Bolyai Soc. Math. Stud.*, vol. 2, János Bolyai Math. Soc., Budapest, 1996, pp. 295–352.
- [22] J. Komlós, G. Sárközy, E. Szemerédi, Blow-up lemma, *Combinatorica* 17 (1997) 109–123.
- [23] J. Komlós, G. Sárközy, E. Szemerédi, Proof of the Seymour conjecture for large graphs, *Ann. Comb.* 2 (1998) 43–60.
- [24] J. Komlós, G. Sárközy, E. Szemerédi, Proof of the Alon–Yuster conjecture, *Discrete Math.* 235 (2001) 255–269.
- [25] M. Krivelevich, B. Sudakov, Pseudo-random graphs, in: *More Sets, Graphs and Numbers*, in: *Bolyai Soc. Math. Stud.*, vol. 15, Springer, Berlin, 2006, pp. 199–262.

- [26] M. Krivelevich, C. Lee, B. Sudakov, Resilient pancyclicity of random and pseudorandom graphs, *SIAM J. Discrete Math.* 24 (2010) 1–16.
- [27] D. Kühn, D. Osthus, Embedding large subgraphs into dense graphs, in: *Surveys in Combinatorics 2009*, in: *London Math. Soc. Lecture Notes*, vol. 365, Cambridge University Press, 2009, pp. 137–167.
- [28] D. Kühn, D. Osthus, The minimum degree threshold for perfect graph packings, *Combinatorica* 29 (2009) 65–107.
- [29] M. Schacht, Extremal results for random discrete structures, manuscript.
- [30] P. Seymour, Problem section, in: *Combinatorics: Proceedings of the British Combinatorial Conference, 1973*.
- [31] A. Shokoufandeh, Y. Zhao, Proof of a conjecture of Komlós, *Random Structures Algorithms* 23 (2003) 180–205.
- [32] A. Shokoufandeh, Y. Zhao, On a tiling conjecture for 3-chromatic graphs, *Discrete Math.* 277 (2004) 171–191.
- [33] B. Sudakov, V. Vu, Local resilience of graphs, *Random Structures Algorithms* 33 (2008) 409–433.
- [34] E. Szemerédi, Regular partitions of graphs, in: *Problèmes combinatoires et théorie des graphes*, *Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976*, in: *Colloq. Internat. CNRS*, vol. 260, CNRS, Paris, 1978, pp. 399–401.