Addenda: Cycle lengths in sparse graphs

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Abstract

In this note, we correct an error in Lemma 2.4 from our paper [4]. We defined a θ-graph to consist of three internally disjoint paths between two vertices. Lemma 2.4 claims that if H is a θ-graph whose vertices are colored red and blue, and C is a longest cycle in the θ-graph, then unless this is a proper coloring of H, there are paths of all possible lengths up to $|C| - 1$ starting with a red vertex and ending with a blue vertex. This is false in general, since we may come up with periodic colorings of a θ-graph if the lengths of the paths have a common factor. For instance, if all the paths have length zero modulo d, and we color any vertex at distance zero $\mod d$ from the ends of the paths red, and all other vertices blue, then in this coloring there is no path of length zero modulo d which starts at a red vertex and ends at a blue vertex. This gap is not difficult to repair, if we replace Lemma 2.4 in [4] with Lemma 1 in [5]:

Lemma 1.1 *If G is a θ-graph consisting of a cycle C with a chord and the vertices of C are colored red and blue, then unless C is bipartite and the coloring is proper there exist paths of all lengths up to $|C| - 1$ whose endpoints have different colors.*

The aim of this note is to point out how to replace each instance of the use of Lemma 2.4 in [4] with Lemma 1.1 in other words we point out how to obtain a θ-graph consisting of a cycle with a chord in each case, instead of a general θ-graph.

2 Finding long cycles with chords

The main use of the Lemma 2.4 in [4] was in finding a θ-graph containing a cycle of length $d^{[(g-1)/2]}$ in a graph of average degree at least $12(d + 1)$ and girth g. We now have to show that we can guarantee a long cycle with a chord instead of this θ-graph. This is done in the following lemma.

Lemma 2.1 *Let G be a graph of average degree at least $12(d + 1)$ and girth g. Then G contains cycle containing at least $t = \frac{1}{3}d^{[(g-1)/2]}$ vertices of degree at least $6(d + 1)$, each of which has no neighbors in $G - V(C)$.*

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The key tool to prove this is a variant of celebrated Pósa's Lemma [2]. First need some notation. If $P = v_1 v_2 \dots v_m$ is a longest path in a graph G and (v_1, v_i) is an edge of G, consider a path $Q = v_{i-1} v_{i-2} \dots v_1 v_i v_{i+1} \dots v_m$ of length $|P|$ is obtained from P by adding edge (v_1, v_i) and deleting edge (v_{i-1}, v_i). We say that Q was obtain from P via an elementary rotation which keeps endpoint v_m fixed. The set of all vertices $v_{i-1} \in V(P)$ which are endpoints of paths obtained by repeated elementary rotations from P with fixed endpoint v_m is denoted by $S(P)$. A variant of Pósa's lemma, proved in [2], says that G has a cycle containing $S(P) \cup N(S(P))$. Note that vertices in $S(P)$ have all their neighbors on this cycle, giving us plenty of chords.

Proof of Lemma 2.1. Pass to a subgraph H of G of minimum degree at least $6(d+1)$. By Lemma 2.1 in [4], for every $X \subset V(H)$ of size at most t, we have $|\partial X| > 2|X|$, where $\partial X = \{y \in V(H) \backslash X : \{x, y\} \in E(H)\}$. By Posas Lemma [2] (see also Lemma 6.3 in [1]), H contains a path P with $|S(P)| \geq t$. Let $S = S(P)$. Lemma 2.7 in [2] states that in this case H contains a cycle containing $S \cup N(S)$. Since every vertex of H has degree at least $6(d+1)$, at least t vertices of C have all their neighbors in C namely the vertices of S. □

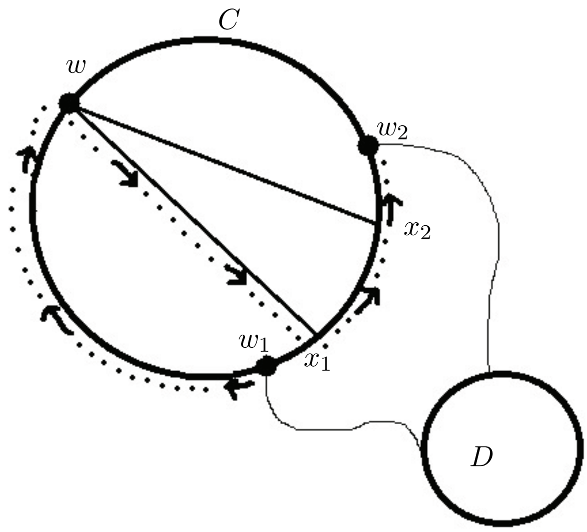
Lemma 2.1 can be now used to fix the proof of both Lemma 2.3 and Lemma 3.1 in [4], which before used the wrong definition of θ-graph.

3 Chromatic number

The same repair is made for graphs of large chromatic number and girth. In that case, Lemma 2.4 in [4] was used in the following way: in any red-blue coloring of the vertices of a non-bipartite θ-graph containing a cycle C, there exist paths of all lengths up to $|C|-1$ such that the endpoints of each path have different colors. As we stated before, Lemma 2.4 is false, but becomes true if we insist that the θ-graph consists of a cycle with a chord, as stated in Lemma 1.1. So in this case, we require (instead of a non-bipartite θ-graph) a long cycle with a chord such that the cycle plus the chord forms a non-bipartite graph. The following lemma achieves this:

Lemma 3.1 *Let G be a graph of chromatic number at least $6(d+1)+1$ and girth $g \geq 3$ where $d \geq 0$. Then G contains a non-bipartite θ-graph containing a cycle of length at least $\frac{t}{2} = \frac{1}{6} d^{\lfloor (g-1)/2 \rfloor}$.*

Proof. We may assume G is minimal $6(d+1)+1$-chromatic. By Lemma 1.1, and since G has minimum degree at least $6(d+1)$, G contains a cycle C containing at least t vertices with no neighbors in $G - V(C)$ – we call these special vertices. These vertices are each incident with at least $6(d+1) - 2 > 2$ chords of C. If the subgraph induced by $V(C)$ is not bipartite, then the required θ-graph is the subgraph induced by $V(C)$. If the subgraph induced by $V(C)$ is bipartite, then since G has chromatic number at least seven, $G - V(C)$ has an odd cycle D. Since G is minimal $6(d+1)+1 > 3$ chromatic, it has not cut vertex. By 2-connectivity of G, there exist two vertex disjoint paths from $V(D)$ to $V(C)$ whose internal vertices are in $G - V(C) - V(D)$. In particular, there exist $w_1, w_2 \in V(C)$ such that there are both even and odd length paths Q_1 and Q_2 and $V(Q_1) \cap V(C) = \{w_1, w_2\} = V(Q_2) \cap V(C)$. Let P_1 and P_2 denote the two $w_1 w_2$-subpaths of C, and suppose P_1 contains at least $t/2$ special vertices. If P_1 has a chord, then $P_1 \cup Q_1$ or $P_1 \cup Q_2$ is the required non-bipartite θ-graph. If P_1 has no chord, then at least $t/2$ special vertices of P_1 each have at least $6(d+1)-2>2$ neighbors in $V(P_2)\backslash\{w_1,w_2\}$. Pick a special vertex $w \in V(P_1)$ and neighbors $x_1, x_2 \in \Gamma(w)\backslash V(P_2)$. We assume that the order of appearance of vertices on C is w_1, w_2, w, x_2, x_1 clockwise. For $a, b \in V(C)$, let $C[a,b]$ denote the subpath of C from a to b in the clockwise order where a precedes b. Consider the paths $R_1 = C - C[x_1, w_1] - C[w, w_2]$ and $R_2 = C - C[w_2, x_2] - C[w_1, w]$. The path R_1 is shown by dotted lines and arrows in the figure below. Then $R_1 \cup R_2 = C$ and $\{w, x_2\}$ is a chord of R_1 and $\{w, x_1\}$ is a chord of R_2. One of these paths has length at least $|C|/2$, and together with Q_1 or Q_2 forms the required non-bipartite θ-graph. ∎



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