



RESEARCH ARTICLE

Ramsey properties of algebraic graphs and hypergraphs

Benny Sudakov¹ and István Tomon²

¹ETH Zurich, Department of Mathematics, Rämistrasse 101, CH-8092 Zurich, Switzerland;

E-mail: benny.sudakov@math.ethz.ch.

²Umeå University, Department of Mathematics and Mathematical Statistics, Universitetstorget 4, SE-90187 Umeå, Sweden;

E-mail: istvan.tomon@umu.se.

Received: 16 March 2021; Accepted: 22 September 2022

2020 Mathematics Subject Classification: Primary – 05D10; Secondary – 05C65, 05C50

Abstract

One of the central questions in Ramsey theory asks how small the largest clique and independent set in a graph on N vertices can be. By the celebrated result of Erdős from 1947, a random graph on N vertices with edge probability $1/2$ contains no clique or independent set larger than $2 \log_2 N$, with high probability. Finding explicit constructions of graphs with similar Ramsey-type properties is a famous open problem. A natural approach is to construct such graphs using algebraic tools.

Say that an r -uniform hypergraph \mathcal{H} is *algebraic of complexity* (n, d, m) if the vertices of \mathcal{H} are elements of \mathbb{F}^n for some field \mathbb{F} , and there exist m polynomials $f_1, \dots, f_m : (\mathbb{F}^n)^r \rightarrow \mathbb{F}$ of degree at most d such that the edges of \mathcal{H} are determined by the zero-patterns of f_1, \dots, f_m . The aim of this paper is to show that if an algebraic graph (or hypergraph) of complexity (n, d, m) has good Ramsey properties, then at least one of the parameters n, d, m must be large.

In 2001, Rónyai, Babai and Ganapathy considered the bipartite variant of the Ramsey problem and proved that if G is an algebraic graph of complexity (n, d, m) on N vertices, then either G or its complement contains a complete balanced bipartite graph of size $\Omega_{n,d,m}(N^{1/(n+1)})$. We extend this result by showing that such G contains either a clique or an independent set of size $N^{\Omega(1/ndm)}$ and prove similar results for algebraic hypergraphs of constant complexity. We also obtain a polynomial regularity lemma for r -uniform algebraic hypergraphs that are defined by a single polynomial that might be of independent interest. Our proofs combine algebraic, geometric and combinatorial tools.

1. Introduction

The quantitative version of Ramsey's theorem, proved by Erdős and Szekeres [23], tells us that every graph on N vertices contains a clique or an independent set of size at least $\frac{1}{2} \log_2 N$. In 1947, Erdős [17] proved that this bound is best possible up to the constant factor, as the random graph on N vertices with edge probability $1/2$ contains no clique or independent set of size larger than $2 \log_2 N$, with high probability. Since then, it has become a central problem in graph theory to find explicit constructions of graphs having only logarithmic-sized cliques and independent sets. One natural approach to constructing such graphs is to use algebraic tools.

Let r, n, d, m be positive integers. Say that an r -uniform hypergraph \mathcal{H} is *algebraic of complexity* (n, d, m) if the following holds. The vertex set of \mathcal{H} is a subset of \mathbb{F}^n , where \mathbb{F} is some field, and there exist m polynomials $f_1, \dots, f_m : (\mathbb{F}^n)^r \rightarrow \mathbb{F}$ of degree at most d and a Boolean formula $\phi : \{\text{false}, \text{true}\}^m \rightarrow \{\text{false}, \text{true}\}$ such that $\{\mathbf{v}_1, \dots, \mathbf{v}_r\} \in V(\mathcal{H})^{(r)}$ is an edge if and only if

$$\phi([f_1(\mathbf{v}_1, \dots, \mathbf{v}_r) = 0], \dots, [f_m(\mathbf{v}_1, \dots, \mathbf{v}_r) = 0]) = \text{true}. \quad (1.1)$$

That is, the edges of \mathcal{H} are determined by zero patterns of polynomials f_1, \dots, f_m . We assume that for $\{\mathbf{v}_1, \dots, \mathbf{v}_r\} \in V(\mathcal{H})^{(r)}$, the left-hand side of (1.1) is invariant under the permutation of $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$, so the edges of \mathcal{H} are well defined. Also, say that an r -uniform hypergraph \mathcal{H} is *strongly algebraic of complexity* (n, d) if there exists a single polynomial $f : (\mathbb{F}^n)^r \rightarrow \mathbb{F}$ of degree at most d such that $\{\mathbf{v}_1, \dots, \mathbf{v}_r\} \in V(\mathcal{H})^{(r)}$ is an edge if and only if $f(\mathbf{v}_1, \dots, \mathbf{v}_r) \neq 0$. We assume that the statement $f(\mathbf{v}_1, \dots, \mathbf{v}_r) = 0$ is invariant under the permutation of $\mathbf{v}_1, \dots, \mathbf{v}_r$. (We could have also defined edges by $f(\mathbf{v}_1, \dots, \mathbf{v}_r) = 0$, but later it will be more convenient to work with this definition.)

1.1. Ramsey properties of algebraic graphs

One of the best-known explicit constructions of graphs having small cliques and independent sets is due to Frankl and Wilson [24]. For a prime p , they consider the graph G whose vertex set is the $p^2 - 1$ element subsets of $[n]$, and the sets A and B are joined by an edge if $|A \cap B| \equiv -1 \pmod p$. This graph is strongly algebraic of complexity $(n, 2)$, as we can identify each set with its characteristic vector over \mathbb{F}_p^n , and two such vectors \mathbf{v} and \mathbf{w} are joined by an edge if $\langle \mathbf{v}, \mathbf{w} \rangle \neq -1$. Note that G is also algebraic of complexity $(p^2 - 1, 1, (p^2 - 1)^2)$ over \mathbb{R} , as we can identify each set of size $p^2 - 1$ with a vector listing its elements, and whether the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{p^2-1}$ are joined by an edge can be decided by the $(p^2 - 1)^2$ linear equations $\mathbf{u}(i) = \mathbf{v}(j)$. The number of vertices of G is $N = \binom{n}{p^2-1}$, and the celebrated Frankl-Wilson theorem on restricted intersections implies that G has no clique or independent set of size larger than $\binom{n}{p-1} = O_p(N^{1/(p+1)})$. Choosing $n = p^3$, the largest clique and independent set in G has size $2^{O(\sqrt{\log N \log \log N})}$. Another candidate for a graph with good Ramsey properties is the *Paley-graph*. If $p \equiv 1 \pmod 4$ is a prime, the Paley-graph of order p is the graph whose vertex set is \mathbb{F}_p , and x and y are joined by an edge if $x + y$ is a quadratic residue: that is, $(x + y)^{(p-1)/2} \neq -1$. Such graphs are strongly algebraic of complexity $(1, (p - 1)/2)$, and it is widely believed that Paley-graphs have only polylogarithmic-sized cliques or independent sets. For the best-known explicit constructions of Ramsey graphs, see the recent works of Chattopadhyay, Zuckerman [7] and Cohen [11].

The above two constructions of algebraic graphs have at least one large parameter in their complexity. In this paper, we show that this is not a coincidence, and if an algebraic graph of complexity (n, d, m) has good Ramsey properties, then at least one of the parameters n, d, m must be large. In 2001, Rónyai, Babai and Ganapathy [35] considered a bipartite version of this problem. A *bi-clique* in a graph G is two disjoint sets $A, B \subset V(G)$ such that $|A| = |B|$ and every vertex in A is joined to every vertex in B by an edge. Note that if G is the random N -vertex graph with edge probability $1/2$, then the size of the largest bi-clique in both G and its complement is $O(\log N)$, with high probability. In contrast, Rónyai, Babai and Ganapathy [35] proved that if G is an algebraic graph on N vertices of complexity (n, d, m) , then either G or its complement contains a bi-clique of size at least $cN^{1/(n+1)}$, where $c = c(n, d, m) > 0$. However, the existence of large bi-cliques or their complements does not imply the existence of large cliques or independent sets. Our first theorem extends the result of Rónyai, Babai and Ganapathy as follows.

Theorem 1.1. *There exists a constant $c > 0$ such that the following holds. Let n, d, m, N be positive integers. Let G be an algebraic graph of complexity (n, d, m) on N vertices. Then G contains either a clique or an independent set of size at least $c'N^{1/\gamma}$, where $c' = c'(n, d, m) > 0$ and*

$$\gamma = cnm \min \left\{ d, \frac{n \log d}{\log n} \right\}.$$

That is, the growth of exponent γ as a function of n is linear, while as a function of d , it is at most logarithmic.

The more general *multicolour Ramsey problem* considers edge-colourings of the complete graph K_N with t colours and asks how small the size of the largest monochromatic clique can be. A colouring $c : E(K_N) \rightarrow [t]$ of the complete graph K_N with t colours is *algebraic of complexity* (n, d, m) , if there exist m polynomials $f_1, \dots, f_m : (\mathbb{F}^n)^2 \rightarrow \mathbb{F}$ of degree at most d and a function $\phi : \{\text{false}, \text{true}\}^m \rightarrow [t]$ such that $c(\mathbf{u}, \mathbf{v}) = \phi(\{f_i(\mathbf{u}, \mathbf{v}) = 0\}_{i \in [m]})$.

In case $t = p + 1$, where p is prime, the best-known construction (not necessarily explicit) of a colouring of K_N with t colours having small monochromatic cliques is a recent result of Conlon and Ferber [12], which was further improved for $t > 3$ by Wigderson [41]. In [12], it is proved that if n is a positive integer and $N = 2^{n/2} p^{3n/8 + o(n)}$, then K_N has a colouring with t colours containing no monochromatic clique of size n . Surprisingly, the colouring they provide is almost algebraic. More precisely, first they consider an algebraic $(t - 1)$ -colouring of complexity $(n, 2, t - 1)$, and then they recolour the last colour class with two new colours randomly.

We prove the extension of Theorem 1.1, which shows that in an algebraic colouring of complexity (n, d, m) , at least one of the parameters n, d, m must be large if we want to avoid large monochromatic cliques.

Theorem 1.2. *There exists a constant $c > 0$ such that the following holds. Let n, d, m, t, N be positive integers. Every algebraic colouring of K_N of complexity (n, d, m) with t colours contains a monochromatic clique of size at least $c'N^{1/\gamma}$, where $c' = c'(n, d, m)$ and*

$$\gamma = cnm \min \left\{ d, \frac{n \log d}{\log n} \right\}.$$

1.2. Ramsey properties of algebraic hypergraphs

The Ramsey problem for hypergraphs is also the subject of extensive study. For positive integers r and t , let $R_r(t)$ denote the smallest N such that any r -uniform hypergraph on N vertices contains either a clique or an independent set of size t . Erdős, Hajnal and Rado [20] and Erdős and Rado [21] proved that there exist constants $c, C > 0$ such that

$$tw_{r-1}(ct^2) < R_r(t) < tw_r(Ct),$$

where the tower function $tw_k(x)$ is defined as $tw_1(x) = x$ and $tw_k(x) = 2^{tw_{k-1}(x)}$. For recent developments on the topic, see also the works of Conlon, Fox and Sudakov [14, 15].

Semi-algebraic graphs and hypergraphs are closely related to algebraic ones and were first studied by Alon, Pach, Pinchasi, Radoičić and Sharir [2]. An r -uniform hypergraph \mathcal{H} is semi-algebraic of complexity (n, d, m) if $V(\mathcal{H}) \subset \mathbb{R}^n$, and there exist m polynomials $f_1, \dots, f_m : (\mathbb{R}^n)^r \rightarrow \mathbb{R}$ and a Boolean formula ϕ such that $\{\mathbf{v}_1, \dots, \mathbf{v}_r\} \in V(\mathcal{H})^{(r)}$ is an edge if and only if $\phi(\{f_i(\mathbf{v}_1, \dots, \mathbf{v}_r) \geq 0\}_{i \in [m]})$ is true.

Let $R_r^{n,d,m}(t)$ denote the smallest N such that any r -uniform semi-algebraic hypergraph of complexity (n, d, m) contains either a clique or an independent set of size t . Conlon et al. [13] studied the Ramsey problem for semi-algebraic hypergraphs and proved that there exist $c = c(r, n, m, d) > 0$ and $C = C(r, n, m, d) > 0$ such that

$$tw_{r-1}(ct) \leq R_r^{n,d,m}(t) < tw_{r-1}(t^C).$$

In the special case of $n = 1$, Bukh and Matoušek [6] proved that if \mathcal{H} is an r -uniform semi-algebraic hypergraph of complexity (n, d, m) on N vertices, then one can always find a clique or independent set of size at least $c' \log \log N$, where c' may depend not only on complexity but also on the defining polynomials f_1, \dots, f_m .

Quite surprisingly, algebraic hypergraphs behave very differently. We show that such hypergraphs of constant complexity contain polynomial-sized cliques or independent sets.

Theorem 1.3. *Let r, n, d, m be positive integers. Then there exist $\gamma = \gamma(r, n, d, m) > 0$ and $c = c(r, n, d, m) > 0$ such that the following holds. If \mathcal{H} is an r -uniform algebraic hypergraph of complexity*

(n, d, m) on N vertices, then \mathcal{H} contains either a clique or an independent set of size at least $cN^{1/\gamma}$. One can choose

$$\gamma = 2r^2m \left(\binom{n+d}{d} + 1 \right).$$

Let us remark that the existence of γ (whose dependence on n, d, m is not quite explicit) also follows from a model theoretic argument of Malliaris and Shelah, Theorem 3.5 in [33] (see also [9] for a shorter proof). They show that if \mathcal{H} is a p -stable r -uniform hypergraph on N vertices (see [9] for definitions), then \mathcal{H} contains either a clique or an independent set of size N^γ , where $\gamma = \gamma(r, p) > 0$. They also remark that r -uniform algebraic hypergraphs of complexity (n, d, m) are p -stable for some $p = p(r, n, d, m)$.

Theorem 1.3 might give the feeling that semi-algebraic graphs and hypergraphs always have stronger Ramsey-type properties than algebraic ones. In the concluding remarks, we will show that this is not always true.

1.3. Erdős-Hajnal conjecture and graphs of bounded VC-dimension

Our results are also closely related to the celebrated Erdős-Hajnal conjecture. It was proved by Erdős and Hajnal [18] that if G is a graph on N vertices that contains no induced copy of some fixed graph H , then G contains a clique or an independent set of size at least $e^{c\sqrt{\log N}}$, where $c = c(H) > 0$ only depends on H . They also proposed the conjecture that G contains a clique or an independent set of size at least N^c for some $c = c(H) > 0$. This conjecture, referred to as the Erdős-Hajnal conjecture, is one of the central open problems in graph theory. Say that a family of graphs \mathcal{G} has the *Erdős-Hajnal property* if there exists a constant $c = c(\mathcal{G}) > 0$ such that every $G \in \mathcal{G}$ contains a clique or an independent set of size at least $|V(G)|^c$. The Erdős-Hajnal conjecture is equivalent to the statement that every hereditary family of graphs has the Erdős-Hajnal property unless it is the family of all graphs. Here, we say that a family \mathcal{F} of r -uniform hypergraphs is *hereditary* if \mathcal{F} is closed under taking induced subhypergraphs.

Although the Erdős-Hajnal conjecture is still wide open, its bipartite analogue was solved twenty years ago by Erdős, Hajnal and Pach [19]. They proved that if an N -vertex graph G contains no induced copy of H , then either G or its complement has a bi-clique of size at least N^c for some $c = c(H) > 0$. Fox and Sudakov [28] improved this by showing that either G contains a bi-clique of size N^c or it contains an independent set of size at least N^c . This suggests that finding polynomial-sized cliques or independent sets is considerably harder than finding bi-cliques or their complements.

The aforementioned bound of Erdős and Hajnal was recently improved for graphs of bounded VC-dimension. The concept of graphs of bounded VC-dimension extends both algebraic and semi-algebraic graphs of bounded complexity (as we shall see later). Let \mathcal{F} be a family of subsets of a base set X . If $U \subset X$, then $\mathcal{F}|_U = \{A \cap U : A \in \mathcal{F}\}$ is the *projection* of \mathcal{F} to U . Also, for every positive integer z , the *shatter function* of \mathcal{F} is defined as

$$\pi_{\mathcal{F}}(z) = \max_{U \in \mathcal{X}^{(z)}} |\mathcal{F}|_U|.$$

The *VC-dimension* (Vapnik-Chervonenkis dimension) of the family \mathcal{F} is the largest integer n such that $\pi_{\mathcal{F}}(n) = 2^n$: that is, there exists a set $U \subset X$ of size n such that $\mathcal{F}|_U = 2^U$. In this case, we say that U is *shattered* by \mathcal{F} . The VC-dimension, introduced by Vapnik and Chervonenkis [40], is one of the most widely used measures of complexity of set systems in computer science and computational geometry. We say that a graph G has *VC-dimension* n if the family $\{N(v) : v \in V(G)\}$ has VC-dimension n . Ramsey properties of graphs of bounded VC-dimension were recently studied by Fox, Pach and Suk [27]. They proved that if G is a graph on N vertices of VC-dimension n , then G contains either a clique or an independent set of size $e^{(\log N)^{1-o(1)}}$, where $o(1) \rightarrow 0$ as $N \rightarrow \infty$ while n is fixed. It is open whether this bound can be improved to N^c , where $c = c(n) > 0$: that is, whether the family of graphs of VC-dimension at most n has the Erdős-Hajnal property. Theorem 1.1 shows that this holds for the family of algebraic graphs of constant complexity.

Corollary 1.4. *Let n, d, m be positive integers. The family of algebraic graphs of complexity (n, d, m) has the Erdős-Hajnal property.*

1.4. Regularity lemma for algebraic hypergraphs

Graph and hypergraph regularity lemmas are among the most powerful tools in combinatorics. Let \mathcal{H} be an r -uniform hypergraph, and let V_1, \dots, V_r be disjoint subsets of $V(\mathcal{H})$. The *density* of (V_1, \dots, V_r) is $d(V_1, \dots, V_r) = \frac{|E(V_1, \dots, V_r)|}{|V_1| \dots |V_r|}$, where $E(V_1, \dots, V_r)$ is the set of edges containing exactly one vertex from each of V_1, \dots, V_r . A partition of $V(\mathcal{H})$ into parts V_1, \dots, V_K is *equitable* if the sizes of V_1, \dots, V_K differ by at most 1.

If G is a graph and (U, V) is a pair of disjoint subsets of the vertex set, then (U, V) is ϵ -regular if $|d(U, V) - d(U', V')| < \epsilon$ for every $U' \subset U, V' \subset V$ satisfying $|U'| \geq \epsilon|U|$ and $|V'| \geq \epsilon|V|$. The regularity lemma of Szemerédi [38] states that if G is a graph and $\epsilon > 0$, then G has an equitable partition into K parts with $1/\epsilon < K < m(\epsilon)$, where $m(\epsilon)$ depends only on ϵ , such that all but ϵ fraction of the pairs of parts are ϵ -regular. Unfortunately, the dependence of $m(\epsilon)$ on $1/\epsilon$ is Ackermann-type, so often, this regularity lemma is quite inefficient.

For certain special families of hypergraphs, one can obtain stronger results. Say that the r -tuple of sets (V_1, \dots, V_r) in the r -uniform hypergraph \mathcal{H} is ϵ -homogeneous if $d(V_1, \dots, V_r) \leq \epsilon$ or $d(V_1, \dots, V_r) \geq 1 - \epsilon$. Also, say that (V_1, \dots, V_r) is *homogeneous* if it is 0-homogeneous. It was proved in the aforementioned paper of Fox, Pach and Suk [27] that if \mathcal{H} has VC-dimension n , then \mathcal{H} has an equitable partition into K parts with $1/\epsilon < K < O_{r,n}((1/\epsilon)^{2n+1})$ such that all but at most ϵ fraction of the r -tuples of parts are ϵ -homogeneous. The authors refer to this as the ‘ultra-strong regularity lemma’, and their bound on the number of parts has an exponent that is optimal up to an absolute constant factor. This result improves on a sequence of earlier regularity lemmas for this class of hypergraphs [1, 8, 31]. One can do even better for semi-algebraic graphs. Fox, Pach and Suk [26] also proved that if \mathcal{H} is semi-algebraic of constant complexity, then \mathcal{H} has an equitable partition into K parts, where K is bounded by a polynomial of ϵ depending only on the complexity, such that all but at most ϵ fraction of the r -tuples of parts are homogeneous.

But what can we say about algebraic hypergraphs? It turns out that it is too much to ask for a partition in which all but a small fraction of r -tuples of parts are homogeneous. Indeed, if this were true, it would imply that if G is an algebraic graph of complexity (n, d, m) , then either G or its complement contains a bi-clique of linear size. In the concluding remarks, we present an example showing that this does not hold in general. On the other hand, Chernikov and Starchenko [8] showed (see Theorem 4.13) that if a hypergraph \mathcal{H} is p -stable, then \mathcal{H} has an equitable partition into K parts with $1/\epsilon < K < O_p((1/\epsilon)^{p+1})$ such that all r -tuples of parts are ϵ -homogeneous. Here, we will prove the following interesting regularity type lemma for strongly algebraic hypergraphs, which shows that strongly algebraic hypergraphs are halfway between semi-algebraic hypergraphs and hypergraphs of bounded VC-dimension.

Theorem 1.5. *Let r, n, d be positive integers. Then there exists $c = c(r, n, d)$ such that the following holds. Let \mathcal{H} be a strongly algebraic r -uniform hypergraph of complexity (n, d) . Then $V(\mathcal{H})$ has an equitable partition V_1, \dots, V_K with $8/\epsilon < K < c(1/\epsilon)^{r(2n+1)}$ parts such that all but at most ϵ -fraction of the r -tuples of parts are either empty or have density at least $1 - \epsilon$.*

Another regularity lemma in an algebraic setting was proved by Tao [39]. This lemma applies to graphs and hypergraphs whose vertices and edges are both defined by algebraic varieties of bounded complexity over some field \mathbb{F} and, more generally, are definable sets of bounded complexity (see [39] for details). For example, the above-mentioned Paley-graph falls into this category. Tao’s lemma shows that, remarkably, one can find an ϵ -regular partition of the vertices of the graph or hypergraph in question into a constant number of (definable) parts, where $\epsilon \approx |\mathbb{F}|^{-1/4}$. Interestingly, this seems quite different from our regularity lemma for strongly algebraic hypergraphs.

Let us present an application of Theorem 1.5 to the Ramsey problem for strongly algebraic hypergraphs. The following theorem roughly tells us that finding an independent set of size s for some constant

$s = s(r, n, d)$ in a strongly algebraic r -uniform hypergraph of complexity (n, d) is almost as ‘costly’ as finding a very large independent set. This suggests that the exponent in Theorem 1.3 can be improved, and in order to do so, it is enough to show that every such hypergraph contains either an independent set of size s or a large clique.

Theorem 1.6. *Let r, n, d be positive integers, $s = (r - 1)\binom{n+d}{d} + 1$ and $c, \alpha, \beta > 0$. Let \mathcal{F} be a hereditary family of strongly algebraic r -uniform hypergraphs of complexity (n, d) . Suppose that each $\mathcal{H} \in \mathcal{F}$ on N vertices contains either a clique of size at least cN^α or an independent set of size s . Then every $\mathcal{H} \in \mathcal{F}$ on N vertices contains either a clique of size $c_1N^{d\alpha\beta/n}$ or an independent set of size at least $c_2N^{1-\beta}$, where $c_1, c_2 > 0$ depend only on the parameters $r, n, d, c, \alpha, \beta$ and $d = d(r) > 0$ depends only on r .*

Our paper is organised as follows. In the next section, we introduce our notation and prepare several tools to prove our theorems. We also present the proof of Theorem 1.3 in this section. Then, in Section 3, we present the proof of Theorems 1.1 and 1.2. We continue with the proof of Theorem 1.5 in Section 4 and present the proof of Theorem 1.6 in Section 5. Finally, we provide some discussion in the concluding remarks.

2. Properties of algebraic hypergraphs

In this section, we prepare some tools for the proofs of our main results, and we prove Theorem 1.3. First, let us introduce the notation we use throughout the paper, which is mostly conventional.

2.1. Notation and preliminaries

If V is some set and s is a positive integer, $V^{(s)}$ denotes the family of s element subsets of V . Let \mathcal{H} be an r -uniform hypergraph. The density of \mathcal{H} is $d(\mathcal{H}) = |E(\mathcal{H})|/\binom{|V(\mathcal{H})|}{r}$. If $X \in V(\mathcal{H})^{(s)}$ for some $1 \leq s \leq r - 1$, then

$$N_{\mathcal{H}}(X) = N(X) = \{Y \in V(\mathcal{H})^{(r-s)} : X \cup Y \in E(\mathcal{H})\}$$

is the neighbourhood of X .

The following is a well-known result that tells us hypergraphs of density very close to 1 contain large cliques.

Lemma 2.1. *Let N be a positive integer, $\frac{1}{N^{r-1}} < \alpha < \frac{1}{2}$, and let \mathcal{H} be an r -uniform hypergraph on N vertices of density at least $1 - \alpha$. Then \mathcal{H} contains a clique of size at least $\frac{1}{4}(1/\alpha)^{1/(r-1)}$.*

Proof. Select each vertex of \mathcal{H} with probability $p = (2N\alpha^{1/(r-1)})^{-1} < 1$, and let U be the set of selected vertices. Let X be the number of non-edges spanned by U . Then $\mathbb{E}(X) \leq \alpha p^r \binom{N}{r}$. Delete a vertex of each non-edge in $\mathcal{H}[U]$, and let V be the resulting set; then V is a clique. We have

$$\mathbb{E}(|V|) \geq \mathbb{E}(|U| - X) = pN - \alpha p^r \binom{N}{r} \geq \frac{pN}{2} \geq \frac{1}{4} \left(\frac{1}{\alpha}\right)^{1/(r-1)},$$

where the last two inequalities hold by the choice of p . Hence, there exists a choice for U such that $|V| \geq \frac{1}{4}(1/\alpha)^{1/(r-1)}$. □

To describe hypergraphs defined by a single polynomial, let us introduce directed hypergraphs. A directed r -uniform hypergraph (or r -uniform dihypergraph) is a pair $\mathcal{H} = (V, E)$, where V is the set of vertices and E is a set of (ordered) r -tuples of distinct elements of V , called edges. Given an r -element subset f of V , an orientation of f is an r -tuple containing the elements of f . In a directed r -uniform hypergraph, we allow multiple orientations of the same r -element set. Say that an r -element set X of a dihypergraph \mathcal{H} is complete if all $r!$ orientations of X are edges. Let $[\mathcal{H}]$ be the hypergraph formed by

the complete edges of \mathcal{H} . A *clique* in an r -uniform directed hypergraph \mathcal{H} is a clique in $[\mathcal{H}]$. Also, an *independent set* of \mathcal{H} is a set of vertices in which no r -tuple forms an edge.

Given $I \subset [r]$ and an $|I|$ -tuple of vertices $X = (v_i)_{i \in I}$, the I -neighbourhood of X is $N_I(X) = N_{\mathcal{H}, I}(X) = \{(v_j)_{j \in [r] \setminus I} \in V^{r-|I|} : (v_1, \dots, v_r) \in E\}$. Note that the vertices of X appear in the corresponding directed edges in the order given by I . In the case of directed graphs, we write simply $N^+(v)$ and $N^-(v)$ for the out- and in-neighbourhood of v , respectively. If X is an r -tuple and $k \in [r]$, then $X(k)$ is the k th element of X and $\hat{X}(k) = (X(1), \dots, X(k-1), X(k+1), \dots, X(r))$.

2.2. Linear algebra

In this section, we collect some facts from linear algebra and introduce the flattening rank of tensors. Let $T : A_1 \times \dots \times A_r \rightarrow \mathbb{F}$ be an r -dimensional tensor, where A_1, \dots, A_r are finite sets and \mathbb{F} is a field. For $i \in [r]$, the i -flattening rank of T , denoted by $\text{frank}_i(T)$, is defined as follows. Let $B_i = A_1 \times \dots \times A_{i-1} \times A_{i+1} \times \dots \times A_r$; then T can be viewed as a matrix M with rows indexed by A_i and columns indexed by B_i . Then $\text{frank}_i(T) := \text{rank}(M)$. Note that $\text{frank}_i(T) = 1$ if and only if $T \neq 0$, and there exist two functions $f : A_i \rightarrow \mathbb{F}$ and $g : B_i \rightarrow \mathbb{F}$ such that $T(a_1, \dots, a_r) = f(a_i)g(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_r)$. Also, the i -flattening rank of T is the minimum t such that T is the sum of t tensors of i -flattening rank 1. Equivalently, $\text{frank}_i(T)$ is the dimension of the vector space generated by the rows of T in the i th dimension.

It is easy to see that the i -flattening rank satisfies the usual properties of rank. It is subadditive, and if T' is a subtensor of T , then $\text{frank}_i(T') \leq \text{frank}_i(T)$. Here, $T' : A'_1 \times \dots \times A'_r \rightarrow \mathbb{F}$ is a *subtensor* of $T : A_1 \times \dots \times A_r \rightarrow \mathbb{F}$ if $A'_i \subset A_i$ for $i \in [r]$ and $T'(a_1, \dots, a_r) = T(a_1, \dots, a_r)$ for $(a_1, \dots, a_r) \in A'_1 \times \dots \times A'_r$. A detailed discussion of i -flattening rank and its combinatorial applications can be found in [34]. In this paper, we use the following property of the i -flattening rank of tensors defined by polynomials.

Lemma 2.2. *Let $f : (\mathbb{F}^n)^r \rightarrow \mathbb{F}$ be a polynomial of degree at most d , and let $V \subset \mathbb{F}^n$. Define the r -dimensional tensor $T : V^r \rightarrow \mathbb{F}$ such that $T(X) := f(X)$ for $X \in V^r$. Then $\text{frank}_i(T) \leq \binom{n+d}{d}$ for $i \in [r]$.*

Proof. Let $\Lambda = \{\alpha \in \mathbb{N}^n : \sum_{i=1}^n \alpha(i) \leq d\}$. For $\mathbf{x} \in \mathbb{F}^n$ and $\alpha \in \mathbb{N}^n$, let $h_\alpha(\mathbf{x}) = \mathbf{x}(1)^{\alpha(1)} \dots \mathbf{x}(n)^{\alpha(n)}$. Then the polynomial f can be written as

$$f(X) = \sum_{\alpha \in \Lambda} h_\alpha(X(i))g_\alpha(\hat{X}(i)),$$

where $X \in V^r$ is considered as an r -tuple of elements of V , and $g_\alpha : \mathbb{F}^{(r-1)n} \rightarrow \mathbb{F}$ is some polynomial for $\alpha \in \Lambda$. But then T is the sum of $|\Lambda| \leq \binom{n+d}{d}$ tensors of i -flattening rank 1, so $\text{frank}_i(T) \leq \binom{n+d}{d}$. \square

We use this lemma in the next section to define a family of forbidden directed subhypergraphs in dihypergraphs defined by a single polynomial. Working with the flattening rank (versus matrix rank) is not necessary for this application, but later (see Section 5) we will exploit more of its properties.

2.3. Forbidden subhypergraphs

In this section, we show that algebraic dihypergraphs defined by a single polynomial avoid the following simple family of dihypergraphs.

Definition 1. For positive integers r, s, k with $k \in [r]$, let $\mathcal{M}(r, s, k)$ be the family of r -uniform dihypergraphs \mathcal{M} having the following form. There are given, not all necessarily distinct, vertices $u_{i,j}$ for $(i, j) \in [s] \times [r]$ forming the vertex set of \mathcal{M} . For $(i, i') \in [s] \times [s]$, let $X_{i,i'}$ be the r -tuple (of distinct) vertices satisfying $X_{i,i'}(j) = u_{i,j}$ for $j \in [r] \setminus \{k\}$ and $X_{i,i'}(k) = u_{i',k}$. Then $X_{i,i}$ is an edge of \mathcal{M} for $i \in [s]$, and $X_{i,i'}$ is not an edge for $1 \leq i < i' \leq s$. The rest of the r -tuples can be either edges or non-edges. Also, let $\mathcal{M}(r, s)$ be the union of the families $\mathcal{M}(r, s, k)$ for $k \in [r]$.

Lemma 2.3. *Let \mathcal{H} be a strongly algebraic r -uniform dihypergraph of complexity (n, d) . Then \mathcal{H} contains no member of $\mathcal{M}(r, s)$ for $s > \binom{n+d}{d}$.*

Proof. Let $V = V(\mathcal{H})$, and let f be the polynomial defining \mathcal{H} . Define the tensor $T : V^r \rightarrow \mathbb{F}$ such that for $X \in V^r$, $T(X) := f(X)$. For $k \in [r]$, let $M_k : V \times V^{r-1} \rightarrow \mathbb{F}$ be the matrix defined as $M_k(X(k), \hat{X}(k)) := T(X)$ for $X \in V^r$. Then by Lemma 2.2, we have $\text{rank}(M_k) = \text{frank}_k(T) \leq \binom{n+d}{d}$.

Let $s > \binom{n+d}{n}$, and suppose that \mathcal{H} contains a copy of a member of $\mathcal{M}(r, s, k)$ for some $k \in [r]$. Then there exist $\mathbf{u}_{i,j} \in V$ for $(i, j) \in [s] \times [r]$ such that the following holds. For $(i, i') \in [s] \times [s]$, define $U_{i,i'} \in V^r$ such that $U_{i,i'}(j) = \mathbf{u}_{i,j}$ for $j \neq k$, and $U_{i,i'}(k) = \mathbf{u}_{i',k}$. Then $f(U_{i,i}) \neq 0$ for $i \in [s]$ and $f(U_{i,i'}) = 0$ if $1 \leq i < i' \leq s$. But then the submatrix of M_k induced by the rows $\mathbf{u}_{1,k}, \dots, \mathbf{u}_{s,k}$ and columns $\hat{U}_{1,1}(k), \dots, \hat{U}_{s,s}(k)$ is an upper-triangular matrix, which has full rank. Therefore, $\text{rank}(A) \geq s$, which is a contradiction. \square

Lemma 2.4. *Let $\mathcal{H}_1, \dots, \mathcal{H}_m$ be r -uniform dihypergraphs on an N element vertex set V . If \mathcal{H}_i contains no member of $\mathcal{M}(r, s)$ for $i \in [m]$, then there exists $U \subset V$ such that $\mathcal{H}_i[U]$ is either a clique or an independent set for $i \in [m]$, and $|U| \geq cN^{1/2r^2ms}$, where $c = c(r, s, m) > 0$ depends only on r, s, m .*

Proof. We prepare the proof with the following claims.

Claim 2.5. *Let \mathcal{G} be a nonempty r -uniform hypergraph on $M \geq 100r$ vertices, and suppose that the density of \mathcal{G} is at most $1 - \alpha$. Then there exists $X \in V(\mathcal{G})^{(r-1)}$ such that $1 \leq |N(X)| \leq (1 - \alpha/2r)M$.*

Proof. Let α' be the unique real number such that $|E(\mathcal{G})| = \binom{(1-\alpha')M}{r} \leq (1 - \alpha) \binom{M}{r}$. Then one can check that $\alpha' \geq \alpha/2r$. We prove by induction on r that \mathcal{G} contains an $(r - 1)$ -element set X such that $1 \leq |N(X)| \leq (1 - \alpha')M$. This is certainly true if $r = 1$, so suppose that $r \geq 2$.

Let $U \subset V(\mathcal{G})$ be the set of vertices with at least 1 neighbour. Then there exists $u \in U$ such that

$$|N(u)| \leq \frac{r|E(\mathcal{G})|}{|U|}.$$

Note that $\binom{|U|}{r} \geq |E(\mathcal{G})|$, as every edge is contained in U , so $|U| \geq (1 - \alpha')M$. But then

$$|N(u)| \leq \frac{r \binom{(1-\alpha')M}{r}}{(1 - \alpha')M} = \binom{(1 - \alpha')M - 1}{r - 1} < \binom{(1 - \alpha')M}{r - 1}.$$

Let \mathcal{G}' be the link graph of u . Then \mathcal{G}' is a nonempty $(r - 1)$ -uniform hypergraph such that $|E(\mathcal{G}')| \leq \binom{(1-\alpha')M}{r-1}$. Therefore, by our induction hypothesis, there exists an $(r - 2)$ -element set X' such that $1 \leq |N_{\mathcal{G}'}(X')| \leq (1 - \alpha')M$. But then $X = X' \cup \{u\}$ satisfies $1 \leq |N_{\mathcal{G}}(X)| \leq (1 - \alpha')M$ as well, finishing the proof. \square

Claim 2.6. *Let \mathcal{G} be a nonempty r -uniform dihypergraph on $M \geq 100r$ vertices, and suppose that the density of \mathcal{G} is at most $1 - \alpha$. Then there exist $\ell \in [r]$ and $Y \in V(\mathcal{G})^{r-1}$ such that*

$$1 \leq |N_{[r] \setminus \{\ell\}}(Y)| \leq \left(1 - \frac{\alpha}{2r \cdot r!}\right)M.$$

Proof. First, consider the case that $[\mathcal{G}]$ is nonempty. Then by the previous claim, there exists $X \in V(\mathcal{G})^{(r-1)}$ such that $1 \leq |N_{[\mathcal{G}]}(X)| \leq (1 - \alpha/2r)M$. Let $U = V(\mathcal{G}) \setminus N_{[\mathcal{G}]}(X)$, and note that $|U| \leq \frac{\alpha}{2r}M$. For every $u \in U$, the set $X \cup \{u\}$ has an orientation not present in \mathcal{G} . Hence, there exist $U' \subset U$, $\ell \in [r]$ and an orientation Y of X such that $|U'| \geq |U|/r!$ and U' is disjoint from $N_{[r] \setminus \{\ell\}}(Y)$. But then $|N_{[r] \setminus \{\ell\}}(Y)| \leq (1 - \frac{\alpha}{2r \cdot r!})M$. Also, $N_{[\mathcal{G}]}(Y) \subset N_{[r] \setminus \{\ell\}}(Y)$, so $|N_{[r] \setminus \{\ell\}}(Y)| \geq 1$ as well.

Now consider the case when $[\mathcal{G}]$ is empty. Let $\beta = 1/r!$. In this case, we show that there exist $\ell \in [r]$ and $Y \in V(\mathcal{G})^{r-1}$ such that $1 \leq |N_{[r] \setminus \{\ell\}}(Y)| \leq (1 - \beta)M$. Suppose this is not the case: that is, for each $W \in E(\mathcal{G})$ and $\ell \in [r]$, we have $|N_{[r] \setminus \{\ell\}}(\hat{W}(\ell))| > (1 - \beta)M$. We define the sets of edges F_0, \dots, F_r of \mathcal{G} as follows. Let F_0 contain a single edge W_0 of \mathcal{G} . For $i \in [r]$, F_i will have the following

form. There exist $x_1, \dots, x_i \in V(\mathcal{G})$ such that F_i contains all r -tuples W for which W agrees with W_0 in exactly $(r - i)$ coordinates, and the rest of the coordinates are a permutation of x_1, \dots, x_i . Note that then $|F_i| = r(r - 1) \dots (r - i + 1)$. If F_i is already defined satisfying this property, we define F_{i+1} as follows. For each $W \in F_i$, let I_W be the set of coordinates $\ell \in [r]$ for which $W(\ell) = W_0(\ell)$. Consider the set

$$U = \bigcap_{W \in F_i} \bigcap_{\ell \in I_W} N_{[r] \setminus \{\ell\}}(\hat{W}(\ell)).$$

Then U is the intersection of $(r - i)|F_i| \leq r!$ subsets of $V(\mathcal{G})$ of size at least $(1 - \beta)M$; hence, U is nonempty. Therefore, there exists $x_{i+1} \in U$. Let F_{i+1} be the set of all edges W' for which $\hat{W}'(\ell) = \hat{W}(\ell)$ and $W'(\ell) = x_{i+1}$ for some $W \in F_i$ and $\ell \in I_W$. Then F_{i+1} has the desired properties. In particular, F_r contains every orientation of $\{x_1, \dots, x_r\}$ contradicting that $[\mathcal{G}]$ is nonempty. This finishes the proof.

Let us illustrate the previous argument in the case $r = 3$. Here, F_0 contains some edge abc . Then we can find $x = x_1$ such that abx, axc, xbc are all edges: they form F_1 . Then we can find some $y = x_2$ such that $ayx, ybx, axy, yxc, xby, xyc$ are all edges: they form F_2 . Finally, we can find some $z = x_3$ such that $zyx, yzx, xzy, yxz, xzy, xyz$ are all edges, forming F_3 . □

Let $\alpha = 2r \cdot r!N^{-1/2rms}$. Let $U_0 = V(\mathcal{H})$, $s_{0,i,k} = s$ for $(i, k) \in [m] \times [r]$, and define the nested sequence of vertex sets $U_0 \supset U_1 \supset \dots$ and sequence of integers $s_{\ell,i,k}$ for $\ell = 1, \dots$ and $(i, k) \in [m] \times [r]$ as follows. Suppose that U_ℓ and $s_{\ell,i,k}$ are already defined for some $\ell \geq 0$ such that $\mathcal{H}_i[U_\ell]$ contains no member of $\mathcal{M}(r, s_{\ell,i,k}, k)$ for $(i, k) \in [m] \times [r]$. Let $I \subset [m]$ be the set of indices i such that $\mathcal{H}_i[U_\ell]$ is nonempty. For such indices i , we must have $s_{\ell,i} > 1$, as the family $\mathcal{M}(r, 1)$ is composed of a single dihypergraph containing a single edge. Consider three cases.

Case 1. $I = \emptyset$.

In this case, we stop and set $U = U_\ell$.

Case 2. For every $i \in I$, the density of $[\mathcal{H}_i][U_\ell]$ is at least $1 - \alpha$.

Let $\mathcal{H}_I = \bigcap_{i \in I} \mathcal{H}_i$; then $d(\mathcal{H}_I[U_\ell]) \geq 1 - \alpha|I| \geq 1 - \alpha m$. Therefore, by Lemma 2.1, $\mathcal{H}_I[U_\ell]$ contains a clique U of size at least $\frac{1}{4}(1/\alpha m)^{1/(r-1)}$. Note that $\mathcal{H}_i[U]$ is a clique for every $i \in I$, and $\mathcal{H}_i[U]$ is an independent set for every $i \in [m] \setminus I$.

Case 3. There exists $i \in I$ such that the density of $[\mathcal{H}_i][U_\ell]$ is less than $1 - \alpha$.

As $\mathcal{H}_i[U_\ell]$ is not empty, we get by Claim 2.6 that there exist $X \subset U_\ell^{r-1}$ and $k \in [r]$ such that

$$1 \leq |N_{\mathcal{H}_i[U_\ell], [r] \setminus \{k\}}(X)| \leq \left(1 - \frac{\alpha}{2r \cdot r!}\right)|U_\ell|.$$

Let $U_{\ell+1} = U_\ell \setminus N_{\mathcal{H}_i[U_\ell], [r] \setminus \{k\}}(X)$. Also set $s_{\ell,i',k'} = s_{\ell+1,i',k'}$ if $(i', k') \neq (i, k)$, and let $s_{\ell+1,i,k} = s_{\ell,i,k} - 1$. Note that $|U_{\ell+1}| \geq \frac{\alpha}{2r \cdot r!}|U_\ell|$, and $\mathcal{H}_j[U_\ell]$ contains no member of $\mathcal{M}(r, s_{\ell+1,i',k'}, k)$ for $(i', k') \in [m] \times [r]$. The latter is clear if $(i', k') \neq (i, k)$. If $(i', k') = (i, k)$, let f be any edge in $\mathcal{H}_i[U_\ell]$ with $\hat{f}(k) = X$. If $\mathcal{H}_i[U_{\ell+1}]$ contains a member of $\mathcal{M}(r, s_{\ell+1,i,k}, k)$, then together with f , this forms a member of $\mathcal{M}(r, s_{\ell,i,k}, k)$ in $\mathcal{H}_i[U_\ell]$, a contradiction.

Let L be the index ℓ for which we stop. For $\ell = 1, \dots, L$, let $s_\ell = \sum_{i=1}^m \sum_{k=1}^r s_{\ell,i,k}$. Then $s_0 = rms$, $s_L \geq rm$, and $s_{\ell+1} = s_\ell - 1$, so we get $L \leq rm(s - 1)$. Also, $|U_{\ell+1}| \geq \frac{\alpha}{2r \cdot r!}|U_\ell|$, so $|U_L| \geq (\alpha/2r \cdot r!)^L N \geq (\alpha/2r \cdot r!)^{rms} N \geq N^{1/2}$.

If we stop in Case 1, then $U = U_\ell$, so $|U| \geq N^{1/2}$. If we stop in Case 2, then $|U| \geq \frac{1}{4}(1/\alpha m)^{1/(r-1)} \geq cN^{1/2msr(r-1)}$, where $c = c(r, s, m) > 0$ depends only on r, s, m . This finishes the proof. □

From this, we can immediately deduce Theorem 1.3.

Proof of Theorem 1.3. Let $f_1, \dots, f_m : (\mathbb{F}^n)^r$ be polynomials of degree at most d and ϕ be a Boolean formula defining \mathcal{H} , and let \mathcal{H}_i be the algebraic dihypergraph defined by f_i on $V(\mathcal{H})$ for $i \in [m]$. By Lemma 2.3, \mathcal{H}_i contains no member of $\mathcal{M}(r, s)$ for $s = \binom{n+d}{n} + 1$. But then by Lemma 2.4, there exists $U \subset V(\mathcal{H})$ such that $|U| \geq cN^{1/2r^2ms}$ and $\mathcal{H}_i[U]$ is either a clique or an independent set for $i \in [m]$.

If $X \in V(\mathcal{H})^{(r)}$ and Y is an orientation of X , then X is an edge of \mathcal{H} if and only if

$$\phi([Y \notin E(\mathcal{H}_i)]_{i \in [m]}) = \text{true}.$$

By the definition of U , the left-hand side is the same for every $Y \in U^r$, so $\mathcal{H}[U]$ is either a clique or an independent set. □

Let us remark that the bound in Lemma 2.4 is optimal in the following sense. Let G be a graph, which we view as a digraph in which each edge is oriented both ways. Note that if G contains no two sets of size $\lfloor s/2 \rfloor$ with no edges between them, then G contains no member of $\mathcal{M}(2, s)$ as well. But standard probabilistic arguments show that there are graphs G on N vertices containing no two such sets, and every clique and independent set in G is of size $N^{O(1/s)}$. In the next section, we show that this bound can be significantly improved if we exploit some other properties of algebraic graphs as well.

2.4. Zero-patterns

One of the key results in our proof is the following lemma of Rónyai, Babai and Ganapathy [35] on the number of zero-patterns of polynomials.

Given a sequence of m polynomials $\mathbf{f} = (f_1, \dots, f_m)$ in n variables over the field \mathbb{F} , a *zero-pattern* of \mathbf{f} is a sequence $\epsilon \in \{0, *\}^m$ for which there exists $x \in \mathbb{F}^n$ such that $f_i(x) = 0$ if and only if $\epsilon(i) = 0$. Let $Z(\mathbf{f})$ denote the number of zero-patterns of \mathbf{f} . Using elegant algebraic tools, Rónyai, Babai and Ganapathy [35] proved the following.

Lemma 2.7. *Let $f_1, \dots, f_m : \mathbb{F}^n \rightarrow \mathbb{F}$ be a sequence of polynomials of degree at most d . Then the number of zero-patterns of $\mathbf{f} = (f_1, \dots, f_m)$ is at most $\binom{md+n}{n}$. In particular, $Z(\mathbf{f}) \leq cm^n$, where $c = c(n, d)$ depends only on n and d .*

2.5. Weak-VC-dimension

In this section, we show that algebraic graphs of complexity (n, d, m) behave similarly to graphs of VC-dimension n . By the celebrated Sauer-Shelah lemma [36, 37], if a family of sets \mathcal{F} has VC-dimension n , then

$$\pi_{\mathcal{F}}(z) \leq \sum_{i=0}^n \binom{z}{i} \leq cz^n,$$

where $c = c(n)$ depends only on n .

Let us relax the notion of a VC-dimension as follows, and note that we now work with undirected hypergraphs. Say that the family \mathcal{F} has *weak-VC-dimension* (c, n) if $\pi_{\mathcal{F}}(z) \leq cz^n$ for every positive integer z . Also, an r -uniform hypergraph \mathcal{H} has *weak-VC-dimension* (c, n) if the family $\{N(v) : v \in V(\mathcal{H})\} \subset 2^{V(\mathcal{H})^{(r-1)}}$ has weak-VC-dimension (c, n) (that is, $\{N(v) : v \in V(\mathcal{H})\}$ is viewed as a family of subsets of the base set $\mathcal{V} = V(\mathcal{H})^{(r-1)}$). Also, say that a family of hypergraphs \mathcal{F} has *weak-VC-dimension* n if there exists a constant $c = c(\mathcal{F})$ such that every $\mathcal{H} \in \mathcal{F}$ has weak-VC-dimension (c, n) .

Lemma 2.8. *Let \mathcal{H} be an r -uniform algebraic hypergraph of complexity (n, d, m) , and let $\mathcal{F} = \{N(v) : v \in V(\mathcal{H})\}$. Then*

$$\pi_{\mathcal{F}}(z) \leq \binom{zmd+n}{n} \leq cz^n,$$

where $c = c(n, d, m)$ depends only on n, d, m .

Proof. Let $f_1, \dots, f_m : (\mathbb{F}^n)^r \rightarrow \mathbb{F}$ be polynomials of degree at most d whose zero-patterns determine the edges of \mathcal{H} .

Let $\mathcal{V} = V(\mathcal{H})^{(r-1)}$: that is, the base set of \mathcal{F} . For $U \in \mathcal{V}$ and $i \in [m]$, define the polynomial $f_{i,U} : \mathbb{F}^n \rightarrow \mathbb{F}$ as $f_{i,U}(\mathbf{x}) = f_i(U', \mathbf{x})$, where $U' \in (\mathbb{F}^n)^{r-1}$ is a fixed orientation of U . If $\mathcal{U} \subset \mathcal{V}$ such

that $|\mathcal{U}| = z$, then each element of the family $\{N(v) \cap \mathcal{U} : v \in V(\mathcal{H})\}$ is determined by a zero-pattern of the zm polynomials $f_{i,U}$ for $i \in [m]$ and $U \in \mathcal{U}$. Therefore, by Lemma 2.7, we have

$$\pi_{\mathcal{F}}(z) \leq \binom{zmd+n}{n}. \quad \square$$

Solving the inequality $\binom{zmd+n}{n} < 2^z$, Lemma 2.8 shows that an algebraic graph of complexity (n, d, m) has VC-dimension at most $n \log(ndm)$. Also, the VC-dimension of such graphs cannot be bounded by a function of n alone: every graph is an algebraic graph of complexity $(1, d, 1)$ and $(1, 1, m)$ for some positive integers d and m . However, Lemma 2.8 also tells us that the family of algebraic graphs of complexity (n, d, m) has weak-VC-dimension at most n .

Let $\delta > 0$. Say that the family \mathcal{F} of subsets of the base set X is δ -separated if $|A \Delta B| \geq \delta|X|$ holds for any two distinct $A, B \in \mathcal{F}$. The following packing lemma is due to Haussler [29]. While it is originally stated for families of bounded VC-dimension, the same proof implies the following result as well.

Lemma 2.9. *Let n be a positive integer and $c > 0$; then there exists $c' = c'(c, n)$ such that the following holds. Let \mathcal{F} be a δ -separated family of weak-VC-dimension (c, n) . Then $|\mathcal{F}| \leq c'(1/\delta)^n$.*

Next, we show that Lemma 2.8 and Lemma 2.9 can be combined to prove that if G is a graph of bounded weak-VC-dimension whose density is not too close to 1, then G contains two large subsets with few edges between them. This lemma is inspired by Theorem 1.3 in the paper of Fox, Pach and Suk [27], which is a regularity-type result for hypergraphs of bounded VC-dimension. Later, we will also use this theorem; see Lemma 4.1 for the precise statement.

Lemma 2.10. *Let n be a positive integer and $c > 0$; then there exists $c_0 = c_0(c, n) > 0$ such that the following holds. Let $\alpha, \beta > 0$, and let G be a graph of weak-VC-dimension (c, n) on N vertices such that $d(G) \leq 1 - \alpha$. Then G contains two disjoint sets A and B such that $|A| \geq \frac{\alpha}{4}N$, $|B| \geq c_0\alpha(\alpha\beta)^nN$, and every vertex in B has at most $\beta|A|$ neighbors in A .*

Proof. Let $U \subset V(G)$ be the set of vertices v such that $|N(v)| \leq (1 - \alpha/2)N$. Then

$$(1 - \alpha) \binom{N}{2} \geq |E(G)| \geq \frac{(N - |U|)}{2} \left(1 - \frac{\alpha}{2}\right)N,$$

from which we get

$$|U| > \frac{\alpha}{2}N.$$

Let $\gamma = \alpha\beta/4$, and let $Z \subset U$ be maximal such that the family $\{N(v) : v \in Z\}$ is γ separated. Then by Lemma 2.9, we have $|Z| \leq c'(1/\gamma)^n$, where $c' = c'(c, n)$ depends only on the parameters c and n . By the maximality of Z , for every $v \in U$, there exists $z \in Z$ such that $|N(v) \Delta N(z)| \leq \gamma N$. Hence, we can find $z_0 \in Z$ such that the set B is formed by the vertices v such that $|N(v) \Delta N(z_0)| \leq \gamma N$ satisfies

$$|B| \geq \frac{|U|}{|Z|} \geq \frac{\alpha\gamma^n}{2c'}N.$$

By deleting some elements of B , we can assume that $|B| \leq \frac{\alpha}{4}$ and that B still satisfies the previous inequality. Set $A = V(G) \setminus (B \cup N(z_0))$; then we show that A and B suffice.

Indeed,

$$|A| \geq N - |B| - |N(z_0)| \geq N - \frac{\alpha}{4}N - \left(1 - \frac{\alpha}{2}\right)N \geq \frac{\alpha}{4}N.$$

Also, every $v \in B$ satisfies $|N(v) \Delta N(z_0)| \leq \gamma N$, and z_0 has no neighbor in A , so $|N(v) \cap A| \leq \gamma N \leq \beta|A|$. □

3. A Ramsey-type result for algebraic graphs

In this section, we prove Theorems 1.1 and 1.2. If G is a directed graph and (A, B) is a pair of disjoint subsets of $V(G)$, say that (A, B) is *well-directed* if either every edge of G between A and B goes from A to B , or every edge between A and B goes from B to A . We show that if G is an algebraic digraph defined by a single polynomial and A and B are two sets of vertices such that $[G]$ has few edges between A and B , then we can find a large subset $A' \subset A$ and $B' \subset B$ such that (A', B') is well-directed. This is the consequence of the fact that G contains no member of $\mathcal{M}(2, s)$ for some constant s . For simplicity, write $\mathcal{M}^-(s)$ and $\mathcal{M}^+(s)$ instead of $\mathcal{M}(2, s, 1)$ and $\mathcal{M}(2, s, 2)$, respectively. Note that each member of $\mathcal{M}^-(s)$ (and $\mathcal{M}^+(s)$) is composed of s directed edges $(u_1, v_1), \dots, (u_s, v_s)$ such that (u_i, v_j) is not an edge for $i > j$ ($j > i$, respectively).

Lemma 3.1. *Let n, d be positive integers. Then there exist $c_1 = c_1(n, d) > 0$ and $c_2 = c_2(n, d) > 0$ such that the following holds. Let G be a strongly algebraic digraph of complexity (n, d) on N vertices such that $d([G]) \leq 1 - \alpha$. Then $V(G)$ contains two disjoint subsets A and B such that $|A| \geq c_1\alpha N$, $|B| \geq c_2\alpha^{n+1}N$, and (A, B) is well-directed.*

Proof. Let $s = \binom{n+2d}{n} + 1$ and $\beta = \frac{1}{6 \cdot 3^{2s}}$. Note that $[G]$ is a strongly algebraic graph of complexity $(n, 2d)$. Indeed, if G is defined by the polynomial $f : (\mathbb{F}^n)^2 \rightarrow \mathbb{F}$ of degree at most d , then the polynomial $f'(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}, \mathbf{y}) \cdot f(\mathbf{y}, \mathbf{x})$ defines $[G]$ and has degree at most $2d$. Therefore, by Lemma 2.8, $[G]$ has weak-VC-dimension (c, n) for some $c = c(n, d)$. Hence, by Lemma 2.10, there exist two subsets A_0 and B_0 of $V(G)$ such that $|A_0| \geq \frac{\alpha}{4}N$, $|B_0| \geq c_0\alpha(\alpha\beta)^nN = c'_2\alpha^{n+1}N$ (where $c'_2 = c'_2(n, d)$ depends only on n and d), and in $[G]$, every vertex in B_0 has at most $\beta|A_0|$ neighbours in A_0 . We show that $c_1 = \frac{1}{4 \cdot 3^{2s}}$ and $c_2 = c'_2/2$ suffices.

We define the sequence of sets $A_0 \supset A_1 \supset \dots$ and sequence of edges $f_1, f_2, \dots \in E(G)$ as follows. For $i = 1, 2, \dots$,

- $|A_i| \geq |A_0|3^{-i}$.
- $f_i = (u_i, v_i)$ has one vertex in A_{i-1} and one vertex in B .
- If $v_i \in B_0$, then v_i has no in-neighbor in A_i , and if $u_i \in B_0$, then u_i has no out-neighbor in A_i .

Suppose that A_{i-1} is already defined for some $1 \leq i < 2s$. Consider two cases. First, consider the case that for every $b \in B_0$, either $N^+(b) \cap A_{i-1} = \emptyset$ or $N^-(b) \cap A_{i-1} = \emptyset$. Then there exists $B \subset B_0$ such that $|B| \geq |B_0|/2$ and either every $b \in B$ satisfies $N^+(b) \cap A_{i-1} = \emptyset$, or every $b \in B$ satisfies $N^-(b) \cap A_{i-1} = \emptyset$. Set $A = A_{i-1}$; then (A, B) is well-directed. As $|A_{i-1}| > |A_0|3^{-2s} \geq c_1\alpha N$ and $|B| \geq c_2N$, the sets A and B satisfy our conditions. In this case, we stop.

Now consider the case when there exists $b \in B_0$ such that both $N^+(b) \cap A_{i-1}$ and $N^-(b) \cap A_{i-1}$ are nonempty. If $a \in N^+(b) \cap N^-(b) \cap A_{i-1}$, then $\{a, b\} \in E([G])$, so $|N^+(b) \cap N^-(b) \cap A_{i-1}| \leq \beta|A_0| < \frac{1}{6}|A_{i-1}|$. But then, either $|N^+(b) \cap A_{i-1}| \leq \frac{2}{3}|A_{i-1}|$ or $|N^-(b) \cap A_{i-1}| \leq \frac{2}{3}|A_{i-1}|$. In the first case, set $A_i = A_{i-1} \setminus N^+(b)$ and $f_i = (b, a)$ for some $a \in N^+(b)$; in the second case, $A_i = A_{i-1} \setminus N^-(b)$ and $f_i = (a, b)$ for some $a \in N^-(b)$. Then A_i and f_i satisfy the desired properties.

Note that we must have stopped for some $i \leq 2s$. Indeed, either at least half of the edges f_1, \dots, f_i go from A_0 to B_0 , or at least half of the edges go from B_0 to A_0 . In the first case, G contains a member of $\mathcal{M}^-(\lceil i/2 \rceil)$; in the second case, G contains a member of $\mathcal{M}^+(\lceil i/2 \rceil)$. □

The following theorem will immediately imply Theorem 1.2, which in turn implies Theorem 1.1.

Theorem 3.2. *There exists $c > 0$ such that the following holds. Let n, d, m, N be positive integers, \mathbb{F} be a field and $0 < \beta < 1$. Let $f_1, \dots, f_m : (\mathbb{F}^n)^2 \rightarrow \mathbb{F}$ be polynomials of degree at most d , and let $V \subset \mathbb{F}^n$ such that $|V| = N$. For $I \subset [m]$, let G_I be the digraph defined on V such that (\mathbf{x}, \mathbf{y}) is an edge if $f_i(\mathbf{x}, \mathbf{y}) \neq 0$ for $i \in I$ and $f_i(\mathbf{x}, \mathbf{y}) = 0$ for $i \notin I$. If N is sufficiently large, then either G_\emptyset contains a clique of size at least $N^{1-\beta}$ or there exists $I \subset [m]$, $I \neq \emptyset$ such that G_I contains a clique of size at least $\frac{1}{4m}N^{\beta/\gamma}$, where*

$$\gamma = cnm \min \left\{ d, \frac{n \log d}{\log n} \right\}.$$

Proof. For $i = 1, \dots, m$, let G_i be the algebraic digraph on V defined by f_i . Then $G_I = (\bigcap_{i \in I} G_i) \cap (\bigcap_{i \in [m] \setminus I} \overline{G}_i)$. Here, \overline{G}_i is the digraph in which every directed edge is included that is not present in G_i .

Let $0 < \alpha < 1$, $s_{0,1} = \dots = s_{0,m} = \binom{n+d}{d} + 1$ and $U_0 = V$. In what comes, we define the nested sequence of sets $U_0 \supset U_1 \supset \dots$ and sequence of positive integers $s_{\ell,i}$ for $\ell = 0, 1, \dots$ and $i \in [m]$ such that $G_i[U_\ell]$ contains no member of $\mathcal{M}^+(s_{\ell,i})$. By Lemma 2.3, this is satisfied for $\ell = 0$. Suppose that U_ℓ and $s_{\ell,i}$ has been already defined for some $\ell \geq 0$.

Let $I \subset [m]$ be the set of indices i such that $s_{\ell,i} > 1$. Consider three cases.

Case 1. $I = \emptyset$.

In this case, we stop and set $U = U_\ell$. Note that $G_i[U]$ is empty for $i = 1, \dots, m$, so $G_\emptyset[U]$ is a clique.

Case 2. For every $i \in I$, we have $d([G_i][U_\ell]) \geq 1 - \alpha$.

Let $G_I^* = \bigcap_{i \in I} [G_i]$; then $d(G_I^*[U_\ell]) \geq 1 - \alpha|I| \geq 1 - \alpha m$. Therefore, by Lemma 2.1, $G_I^*[U_\ell]$ contains a clique U of size at least $\frac{1}{4\alpha m}$. Note that $G_i[U]$ is a clique for every $i \in I$, and $G_i[U]$ is an independent set for every $i \in [m] \setminus I$, so $G_I[U]$ is a clique. In this case, we stop as well.

Case 3. There exists $i \in I$ such that $d([G_i][U_\ell]) < 1 - \alpha$.

Then by Lemma 3.1, there exist two disjoint sets A and B in U_ℓ such that $|A| \geq c_1\alpha|U_\ell|$, $|B| \geq c_2\alpha^{n+1}|U_\ell|$ and (A, B) is well-directed in G_i . Let $t = \max\{1, \lfloor s_{\ell,i}/n^{1/2} \rfloor\}$ and $u = s_{\ell,i} - t$. Note that if $G_i[A]$ contains a member of $\mathcal{M}^+(u)$ and $G_i[B]$ contains a member of $\mathcal{M}^+(t)$, then $G_i[U_\ell]$ contains a member of $\mathcal{M}^+(s_{\ell,i})$. Indeed, suppose that the directed edges $(x_1, y_1), \dots, (x_u, y_u)$ form a copy of a member of $\mathcal{M}^+(u)$ in $G_i[A]$ and $(x'_1, y'_1), \dots, (x'_t, y'_t)$ form a copy of a member of $\mathcal{M}^+(t)$ in $G_i[B]$. If every edge between A and B goes from A to B , then setting $x'_{i+t} = x_i$ and $y'_{i+t} = y_i$ for $i \in [u]$, the edges $(x'_1, y'_1), \dots, (x'_{u+t}, y'_{u+t})$ form a copy of a member of $\mathcal{M}^+(s_{\ell,i})$. On the other hand, if every edge between A and B goes from B to A , then setting $x_{i+u} = x'_i$ and $y_{i+u} = y'_i$ for $i \in [t]$, the edges $(x_1, y_1), \dots, (x_{u+t}, y_{u+t})$ form a copy of a member of $\mathcal{M}^+(s_{\ell,i})$.

Therefore, either $G_i[A]$ contains no member of $\mathcal{M}^+(u)$, in which case set $U_{\ell+1} = A$ and $s_{\ell+1,i} = u$ and say that ℓ is a *small-jump*, or $G_i[B]$ contains no member of $\mathcal{M}(t)$, in which case set $U_{\ell+1} = B$, $s_{\ell+1,i} = t$ and say that ℓ is a *big-jump*. For $j \in [m] \setminus \{i\}$, set $s_{\ell+1,j} = s_{\ell,j}$.

Let $s_\ell = s_{\ell,1} \dots s_{\ell,m} \cdot n^{-z/2}$, where z is the number of 1s among $s_{\ell,1}, \dots, s_{\ell,m}$. Let L be the index ℓ at which we stop, let a be the number of big jumps, and let b be the number of small jumps. Note that if ℓ is a small jump, then $s_{\ell+1} \leq s_\ell(1 - n^{-1/2}/2)$ and $|U_{\ell+1}| \geq c_1\alpha|U_\ell|$. Also, if ℓ is a big jump, then $s_{\ell+1} \leq s_\ell n^{-1/2}$ and $|U_{\ell+1}| \geq c_2\alpha^{n+1}|U_\ell|$. Therefore,

$$s_L \leq s_0 n^{-a/2} (1 - n^{-1/2}/2)^b.$$

As $s_L \geq n^{-m/2}$ and $s_0 = \binom{n+d}{d} + 1)^m \leq \min\{n^{dm}, d^{nm}\}$, we get

$$a \leq \frac{2 \log(s_0/s_L)}{\log n} \leq 4m \min\left\{d, \frac{n \log d}{\log n}\right\}$$

and

$$b \leq 4n^{1/2} \log(s_0/s_L) \leq 8n^{1/2} m \min\{d \log n, n \log d\}.$$

Also, we get

$$|U_L| \geq N(c_1\alpha)^b (c_2\alpha^{n+1})^a > c_3\alpha^{(n+1)a+b} N,$$

where $c_3 = c_3(n, m, d) > 0$. Choosing $\alpha = N^{-\beta/\gamma'}$, where $\gamma' = 32mn \min\{d, n \log d / \log n\}$, the left-hand side is at least $N^{1-\beta}$ for sufficiently large N . Therefore, if we stopped in Case 1, then we found a set U such that $G_\emptyset[U]$ is a clique of size $|U_L| \geq N^{1-\beta}$. On the other hand, if we stopped at Case 2, then we found $I \subset [m]$, $I \neq [m]$ and a set U such that $G_I[U]$ is a clique of size at least $\frac{1}{4m\alpha} \geq \frac{1}{4m} N^{\beta/\gamma'}$. This finishes the proof. □

Proof of Theorem 1.2. Consider an algebraic colouring of K_N of complexity (n, d, m) with t colours. Let the polynomials f_1, \dots, f_m define the colouring of K_N . Apply Theorem 3.2 with $\beta = 1/2$. Then there exists $I \subset [m]$ such that G_I contains a clique of size at least $\frac{1}{4m}N^{1/2\gamma}$, where

$$\gamma = cnm \min\left\{d, \frac{n \log d}{\log n}\right\}.$$

Note that for every $I \subset [m]$, the graph G_I is monochromatic; hence, K_N contains a monochromatic clique of size at least $\frac{1}{4m}N^{1/2\gamma}$. □

4. Regularity lemma for algebraic hypergraphs

In this section, we prove Theorem 1.5. We remark that in the rest of the paper, we shall only work with undirected hypergraphs. We prepare the proof of Theorem 1.5 with several lemmas. The following regularity lemma is proved in [27] for hypergraphs of bounded VC-dimension; however, the same proof works almost word for word for hypergraphs of bounded weak-VC-dimension as well.

Lemma 4.1. *Let r be a positive integer and $c, n > 0$. Then there exists $c' = c'(r, c, n) > 0$ such that the following holds. Let $0 < \epsilon < 1/4$. Let \mathcal{H} be an r -uniform hypergraph of weak-VC-dimension (c, n) . Then $V(\mathcal{H})$ has an equitable partition V_1, \dots, V_K with $8/\epsilon < K < c'(1/\epsilon)^{2n+1}$ parts such that all but at most ϵ -fraction of the r -tuples of parts are ϵ -homogeneous.*

Let us restate the definition of $\mathcal{M}(r, s)$ for undirected hypergraphs. The hypergraph \mathcal{M} is a member of $\mathcal{M}(r, s)$ if the vertex set of \mathcal{M} is the union of (not necessarily disjoint) $(r - 1)$ -element sets Z_1, \dots, Z_s and s vertices z_1, \dots, z_s , $Z_i \cup \{z_i\}$ is an edge of \mathcal{M} for $i \in [s]$, and $Z_i \cup \{z_j\}$ is not an edge for $1 \leq i < j \leq s$. The rest of the r -tuples can be either edges or non-edges. If Z_1, \dots, Z_s and z_1, \dots, z_s satisfy these properties, say that they define a member of $\mathcal{M}(r, s)$.

In what follows, we show that if we also assume that \mathcal{H} contains no member of $\mathcal{M}(r, s)$, then those r -tuples of parts that have density at most ϵ can be made empty by removing a few vertices. We first need the following technical lemma, which can be viewed as a variant of Claim 2.5 for sparse hypergraphs.

Lemma 4.2. *Let \mathcal{G} be a nonempty r -partite r -uniform hypergraph with vertex classes W_1, \dots, W_r such that $d(W_1, \dots, W_r) \leq \epsilon$. Then there exist $\ell \in [r]$ and $X \in \bigcup_{i \in [r] \setminus \{\ell\}} W_i$ such that $|X| = r - 1$ and $1 \leq |N(X)| \leq \epsilon^{1/r}|W_\ell|$.*

Proof. We proceed by induction on r . In case $r = 1$, the statement is trivial, so assume $r \geq 2$.

Let $U \subset V(\mathcal{G})$ be the set of vertices with positive degree, and let $U_i = W_i \cap U$ for $i \in [r]$. Then $|U_1| \cdots |U_r| \geq \epsilon|W_1| \cdots |W_r|$, so there exists $j \in [r]$ such that $|U_j| \geq \epsilon^{1/r}|W_j|$. Without loss of generality, assume that $j = r$. Then by simple averaging, there exists $x \in U_r$ such that

$$|N(x)| \leq \frac{|E(\mathcal{G})|}{|U_r|} \leq \epsilon^{(r-1)/r}|W_1| \cdots |W_{r-1}|.$$

Let \mathcal{G}' be the $(r - 1)$ -partite $(r - 1)$ -uniform hypergraph with vertex classes W_1, \dots, W_{r-1} formed by the neighbourhood of x . By our induction hypothesis, there exist $\ell \in [r - 1]$ and $X' \in \bigcup_{i \in [r-1] \setminus \{\ell\}} W_i$ such that $|X'| = r - 2$ and $1 \leq |N(X')| \leq \epsilon^{1/r}|W_\ell|$. But then ℓ and $X = X' \cup \{x\}$ satisfy our desired properties. □

The next lemma will be the heart of the proof of Theorem 1.5. Before we state it, let us introduce a definition. Let \mathcal{H} be an r -uniform hypergraph, and let (V_1, \dots, V_K) be disjoint subsets of its vertex set. Suppose that $Z_1, \dots, Z_s \in V(\mathcal{H})^{(r-1)}$ and $z_1, \dots, z_s \in V(\mathcal{H})$ define a member of $\mathcal{M}(r, s)$ in \mathcal{H} . If z_1, \dots, z_s belong to the same part V_j , and no two vertices of $Z_i \cup \{z_i\}$ belong to the same part for $i \in [s]$, then say that $\bigcup_{i=1}^s Z_i \cup \{z_1, \dots, z_s\}$ induces a *focused* copy of a member of $\mathcal{M}(r, s)$ and Z_1, \dots, Z_s and z_1, \dots, z_s defines a *focused* member of $\mathcal{M}(r, s)$.

Lemma 4.3. *Let r, s be positive integers. Then there exists $c = c(r, s) > 0$ such that the following holds. Let $\epsilon_0 > 0$, let \mathcal{H} be an r -uniform hypergraph, and let U_1, \dots, U_L be disjoint subsets of $V(\mathcal{H})$. Suppose that \mathcal{H} contains no focused copy of a member of $\mathcal{M}(r, s)$. Let \mathcal{G} be an r -uniform hypergraph on $[L]$ such that if $\{i_1, \dots, i_r\}$ is an edge of \mathcal{G} ; then $d_{\mathcal{H}}(U_{i_1}, \dots, U_{i_r}) \leq \epsilon_0$. Then for $i \in [L]$, there exists $V_i \subset U_i$ such that $|V_i| \geq (1 - c\epsilon_0^{1/r})|U_i|$, and $(V_{i_1}, \dots, V_{i_r})$ is empty for every $\{i_1, \dots, i_r\} \in E(\mathcal{G})$.*

Proof. We prove this by induction on r . We proceed with the base case $r = 2$ and the general case $r \geq 3$ simultaneously. Let $T := \emptyset$, and let $\epsilon_1 = \epsilon_0^{1/r}$. Let $X \in [L]^{(r-1)}$ and $k \in [L] \setminus X$. If $X \cup \{k\} \notin E(\mathcal{G})$, set $F_{X,k} = \emptyset$. Otherwise, let $F_{X,k}$ be the $(r - 1)$ -partite $(r - 1)$ -uniform hypergraph on $\bigcup_{x \in X} U_x$ whose edges are the $(r - 1)$ -tuples $Z \in V(\mathcal{H})^{(r-1)}$ such that Z has one vertex in U_x for $x \in X$ and $|N_{\mathcal{H}}(Z) \cap U_k| \geq \epsilon_1|U_k|$. As $d_{\mathcal{H}}(\{U_x\}_{x \in X}, U_k) \leq \epsilon_0$, we have

$$d_{F_{X,k}}(\{U_x\}_{x \in X}) \leq \frac{\epsilon_0}{\epsilon_1} = \epsilon_1^{r-1}. \tag{4.1}$$

Also, define $T_k \subset U_k$ as the union of the sets $U_k \cap N_{\mathcal{H}}(Z')$, where $Z' \in (V(\mathcal{H}) \setminus U_k)^{(r-1)}$ has all its vertices in different parts and $|U_k \cap N_{\mathcal{H}}(Z')| \leq \epsilon_1|U_k|$.

Claim 4.4.

$$|T_k| \leq s\epsilon_1|U_k|.$$

Proof. Suppose that $|T_k| > s\epsilon_1|U_k|$. Then we can find greedily $Z_1, \dots, Z_s \in (V(\mathcal{H}) \setminus U_k)^{(r-1)}$ and $z_1, \dots, z_s \in T_k$ such that $z_i \in N(Z_i)$ for $i \in [s]$ and $z_i \notin N(Z_j)$ for $i > j$. But then Z_1, \dots, Z_s and z_1, \dots, z_s induces a focused copy of a member of $\mathcal{M}(r, s)$, a contradiction. \square

For $k \in [L]$, let $U'_k = U_k \setminus T_k$. Note that if $X \cup \{k\} \in E(\mathcal{G})$ and $Z \cup \{z\}$ is an edge of \mathcal{H} for some Z having one vertex in every $U'_x, x \in X$ and $z \in U'_k$, then $Z \in E(F_{X,k})$. Remove every edge Z from $F_{X,k}$ that has no neighbor in U'_k . Let $F_X = \bigcup_{k \in [L] \setminus X} F_{X,k}$.

Now let us consider the base case $r = 2$. Then every $F_{X,k}$ is a just a subset of U_X , which by (4.1) has size at most $\epsilon_1|U_X|$. This implies that $|F_X| \leq s\epsilon_1|U_X|$ for $X \in [L]$. Indeed, otherwise, by a simple greedy argument, we can find $z_1, \dots, z_s \in F_X$ and k_1, \dots, k_s such that $z_i \in F_{X,k_i}$ and $z_i \notin F_{X,k_j}$ for $i > j$. Choose an arbitrary vertex $w_i \in N(z_i) \cap U'_{k_i}$. Then w_1, \dots, w_s and z_1, \dots, z_s define a focused copy of a member of $\mathcal{M}(2, s)$, a contradiction. For $i \in [L]$, set $V_i = U'_i \setminus F_i$. Then $|V_i| \geq |U_i|(1 - 2s\epsilon_1)$, and there are no edges between V_i and V_j for $\{i, j\} \in E(\mathcal{G})$. Therefore, choosing $c(2, s) = 2s$ proves this case. Now consider the general case $r \geq 3$.

Claim 4.5.

$$d_{F_X}(\{U_x\}_{x \in X}) < 3rs\epsilon_1.$$

Proof. Let $F := F_X$, let $W_x := U'_x$ for $x \in X$. In what follows, step by step, we define $|X|$ sequences of pairs $(Z_{i,x}, z_{i,x}) \in (\bigcup_{y \in X \setminus \{x\}} U'_y)^{(r-1)} \times U'_x$ for $x \in X$ and $i = 0, 1, \dots$, while removing elements from W_x and edges from F . Suppose that j_x is the largest index i for which $(Z_{i,x}, z_{i,x})$ is already defined. If $(\{W_x\}_{x \in X})$ spans no edge in F , then stop. Otherwise, there exists $k \in [L]$ such that $F_{X,k}$ is not empty on $(\{W_x\}_{x \in X})$. By applying Lemma 4.2 to the $(r - 1)$ -tuple $(\{U_x\}_{x \in X})$ and hypergraph $F_{X,k} \cap F$ (which by (4.1) has density at most ϵ_1^{r-1}), there exist $x \in X$ and $Z \in \bigcup_{y \in X \setminus \{x\}} W_y$ such that $1 \leq |N_{F_{X,k} \cap F}(Z) \cap W_x| \leq \epsilon_1|U_x|$. Let $z_{j_x+1,x}$ be an arbitrary element of $N_{F_{X,k} \cap F}(Z) \cap W_x$. Furthermore, remove the elements of $N_{F_{X,k} \cap F}(Z)$ from W_x , and remove the edges of $F_{X,k}$ from F . Also, let $Z_{j_x+1,x} := Z$ and define $k_{j_x+1,x} := k$.

After we stop, let s_x be the largest index i such that $(Z_{i,x}, z_{i,x})$ is defined. Note that for every $x \in X$ and $i \leq s_x, Z_{1,x}, \dots, Z_{i,x}$ and $z_{1,x}, \dots, z_{i,x}$ define a focused copy of a member of $\mathcal{M}(r - 1, i)$. Let $u_{i,x}$ be an arbitrary element of $U'_{k_{i,x}} \cap N_{\mathcal{H}}(Z_{i,x} \cup \{z_{i,x}\})$. Then $Z_{1,x} \cup \{u_{1,x}\}, \dots, Z_{i,x} \cup \{u_{i,x}\}$ and $z_{1,x}, \dots, z_{i,x}$

define a focused copy of a member of $\mathcal{M}(r, i)$ in \mathcal{H} . Therefore, we have $s_x < s$. Also, at each step, when we modify W_x , we remove at most $\epsilon_1|U_x|$ elements; so after we stop, we have $|W_x| \geq |U'_x| - s_x\epsilon_1|U_x|$.

Let us bound $d_{F_X}(\{U_x\}_{x \in X})$. Recall that $d_{F_{X, k_{i,x}}}(\{U_x\}_{x \in X}) \leq \epsilon_1^{r-1}$ by (4.1). Let $F^* = \bigcup_{x \in X} \bigcup_{i \in [s_x]} F_{X, k_{i,x}}$. Then

$$d_{F^*}(\{U_x\}_{x \in X}) < \sum_{x \in X} \sum_{i \in [s_x]} d_{F_{X, k_{i,x}}}(\{U_x\}_{x \in X}) < s(r-1)\epsilon_1^{r-1}.$$

If $k \neq k_{i,x}$ for every $x \in X$ and $i \in [s_x]$, then every edge of $F_{k,X}$ touches $U_x \setminus W_x$ for some $x \in X$. Note that $|U_x \setminus W_x| = |U_x \setminus U'_x| + |U'_x \setminus W_x| \leq 2s\epsilon_1|U_x|$. Hence $\frac{|U_x \setminus W_x|}{|U_x|} < 2s\epsilon_1$. Therefore, we have

$$d_{F_X \setminus F^*}(\{U_x\}_{x \in X}) < 2(r-1)s\epsilon_1.$$

Finally, by our construction, $(\{W_x\}_{x \in X})$ spans no edges. Thus, $d_{F_X}(\{U_x\}_{x \in X}) < 3rs\epsilon_1$. □

Since $|U'_x| \geq (1 - s\epsilon_1)|U_x|$, we can write $\prod_{x \in X} |U'_x| \geq (1 - s\epsilon_1)^{r-1} \prod_{x \in X} |U_x| \geq \frac{1}{2} \prod_{x \in X} |U_x|$, assuming $s\epsilon_1 < 1/10r$. Therefore, from the last claim, we have that $d_{F_X}(\{U'_x\}_{x \in X}) \leq 6rs\epsilon_1$.

Let $\mathcal{H}' = \bigcup_{X \in [L]^{(r-1)}} F_X$. We show that \mathcal{H}' contains no focused copy of a member of $\mathcal{M}(r-1, s)$ with respect to the disjoint sets U'_1, \dots, U'_L . Otherwise, there exist $W_1, \dots, W_s \in V(\mathcal{H}')^{(r-2)}$ and $w_1, \dots, w_s \in U'_j$ for some $j \in [L]$ defining a focused member of $\mathcal{M}(r-1, s)$. Let $Z_i = W_i \cup \{w_i\}$. But then for $i \in [s]$, there exist $k_i \in [L]$ and $X_i \in [L]^{(r-1)}$ such that $Z_i \in F_{X_i, k_i}$. Picking $z_i \in N_{\mathcal{H}}(Z_i) \cap U'_i$ arbitrarily, $\{W_i \cup \{z_i\}\}_{i \in [s]}$ and $\{w_i\}_{i \in [s]}$ define a focused member of $\mathcal{M}(r, s)$ in \mathcal{H} , a contradiction.

But then we can apply our induction hypothesis with the following parameters: the $(r-1)$ -uniform hypergraph \mathcal{H}' instead of \mathcal{H} , the disjoint sets U'_1, \dots, U'_L instead of U_1, \dots, U_L , $6rs\epsilon_1$ instead of ϵ_0 and the complete $(r-1)$ -uniform hypergraph on $[L]$ instead of \mathcal{G} . Recalling that $|U'_i| \geq (1 - s\epsilon_1)|U_i|$ and $\epsilon_1 = \epsilon_0^{1/r}$, we conclude that for $i \in [L]$, there exists $V_i \subset U'_i$ such that

$$|V_i| \geq (1 - c(r-1, s)(6rs\epsilon_1)^{1/(r-1)!})|U'_i| \geq (1 - c(r, s)\epsilon_0^{1/r!})|U_i|$$

for some suitable $c(r, s) > 0$, and $(V_{i_1}, V_{i_2}, \dots, V_{i_{r-1}})$ is empty for $(i_1, \dots, i_{r-1}) \in [L]^{(r-1)}$. But note that then $(V_{i_1}, \dots, V_{i_r})$ is also empty in \mathcal{H} if $\{i_1, \dots, i_r\} \in E(\mathcal{G})$. This finishes the proof. □

Now everything is set to prove the main theorem of this section.

Proof of Theorem 1.5. Let $\epsilon_0 > 0$, which we shall specify later as a function of ϵ . By Lemma 2.8 and Lemma 4.1, there exists an equitable partition of $V(\mathcal{H})$ into L parts U_1, \dots, U_L such that $8/\epsilon_0 < L < c'(1/\epsilon_0)^{2n+1}$ and all but at most ϵ_0 -fraction of the r -tuples of parts are ϵ_0 -homogeneous.

Let $s = \binom{n+d}{d} + 1$; then \mathcal{H} contains no member of $\mathcal{M}(r, s)$ by Lemma 2.3. In particular, \mathcal{H} contains no focused copy of a member of $\mathcal{M}(r, s)$ with respect to the partition U_1, \dots, U_L . Let $c_0 = c(r, s)$ be the constant given by Lemma 4.3. Then for every $i \in [L]$, there exists $V'_i \subset U_i$ such that

$$|V'_i| \geq (1 - c_0\epsilon_0^{1/r!})|U_i| \geq (1 - c_0\epsilon_0^{1/r!}) \left\lfloor \frac{N}{L} \right\rfloor,$$

and $(V'_{i_1}, \dots, V'_{i_r})$ is empty if $d(U_{i_1}, \dots, U_{i_r}) < \epsilon_0$. Let K be a minimal positive integer such that $N/K < \min_{i \in [L]} |V'_i|$; then $K < L + 2c_0\epsilon_0^{1/r!}L < 2L$. In particular, when we choose ϵ_0 , it will satisfy $c_0\epsilon_0^{1/r!} < 1/2$. Let V_1, \dots, V_K be any equitable partition of $V(\mathcal{H})$ such that $V_i \subset V'_i$ for $i \in [L]$: this clearly exists by the definition of K .

Say that an r -tuple $\{i_1, \dots, i_r\} \in [K]^r$ is *good* if $d(V_{i_1}, \dots, V_{i_r}) > 1 - 2^r\epsilon_0$ or $(V_{i_1}, \dots, V_{i_r})$ is empty. Otherwise, say that $\{i_1, \dots, i_r\}$ is *bad*. Note that if $\{i_1, \dots, i_r\} \in [L]^r$, then $\{i_1, \dots, i_r\}$ is good

if $(U_{i_1}, \dots, U_{i_r})$ is ϵ -homogeneous. Indeed, if $d(U_{i_1}, \dots, U_{i_r}) > 1 - \epsilon_0$, then

$$d(V_{i_1}, \dots, V_{i_r}) > 1 - \frac{|U_{i_1}| \dots |U_{i_r}|}{|V_{i_1}| \dots |V_{i_r}|} \epsilon_0 \geq 1 - 2^r \epsilon_0,$$

and if $d(U_{i_1}, \dots, U_{i_r}) < \epsilon_0$, then $(V_{i_1}, \dots, V_{i_r})$ is empty by definition. The number of r -tuples $\{i_1, \dots, i_r\} \in [K]^{(r)}$ having a nonempty intersection with $\{L + 1, \dots, K\}$ is fewer than $(K - L)K^{r-1}$. Hence, the total number of bad r -tuples is at most

$$\epsilon_0 \binom{L}{r} + (K - L)K^{r-1} < \epsilon_0 \binom{K}{r} + 2c_0 \epsilon_0^{1/r!} K^r \leq c_1 \epsilon_0^{1/r!} \binom{K}{r},$$

where $c_1 = c_1(r, s) > c_0$ depends only on r and s . Choose ϵ_0 such that both $c_1 \epsilon_0^{1/r!}, 2^r \epsilon_0 \leq \epsilon < 1/4$. Then V_1, \dots, V_K is an equitable partition of \mathcal{H} such that $8/\epsilon < 8/\epsilon_0 < K \leq 2c'(1/\epsilon_0)^{2n+1} < c(1/\epsilon)^{r!(2n+1)}$ for some $c = c(r, s) > 0$, and all but at most an ϵ -fraction of the r -tuples of parts $(V_{i_1}, \dots, V_{i_r})$ satisfy that either $d(V_{i_1}, \dots, V_{i_r}) > 1 - 2^r \epsilon_0 > 1 - \epsilon$, or $(V_{i_1}, \dots, V_{i_r})$ is empty. \square

5. A Ramsey-type result for algebraic hypergraphs

In this section, we prove Theorem 1.6. First, let us extend the list of forbidden hypergraphs in algebraic hypergraphs of constant complexity. To do this, we use a result about the maximum flattening rank of tensors that are almost diagonal. Let $T : A_1 \times \dots \times A_r \rightarrow \mathbb{F}$ be an r -dimensional tensor. Let the *max-flattening rank* of T , denoted by $\text{mfrank}(T)$, be the maximum of $\text{frank}_i(T)$ for $i \in [r]$. Say that an r -dimensional tensor $T : A^r \rightarrow \mathbb{F}$ is *semi-diagonal* if the following holds. Let $a_1, \dots, a_r \in A$. Then $T(a_1, \dots, a_r) = 0$ if a_1, \dots, a_r are pairwise distinct, and $T(a_1, \dots, a_r) \neq 0$ if $a_1 = \dots = a_d$. If a_1, \dots, a_r are neither all equal or all distinct, there is no restriction on $T(a_1, \dots, a_r)$. We prove the following result in the companion note [34].

Theorem 5.1. *Let $T : A^r \rightarrow \mathbb{F}$ be an r -dimensional semi-diagonal tensor. Then*

$$\text{mfrank}(T) \geq \frac{|A|}{r - 1}.$$

Let $\mathcal{N}_{r,s}$ be the family of r -uniform hypergraphs \mathcal{N} having the following form. There are s disjoint edges $f_i = \{u_{i,1}, \dots, u_{i,r}\}$ for $i \in [s]$, whose union is the vertex set of \mathcal{N} , and the r element set $\{u_{i_1,1}, \dots, u_{i_r,r}\}$ is not an edge for any r -tuple $(i_1, \dots, i_r) \in [s]^r$, where i_1, \dots, i_r are pairwise distinct. The rest of the r element subsets of the vertex set can be either edges or non-edges.

Lemma 5.2. *Let \mathcal{H} be a strongly algebraic r -uniform hypergraph of complexity (n, d) . Then \mathcal{H} contains no member of $\mathcal{N}_{r,s}$ for $s > (r - 1) \binom{n+d}{d}$.*

Proof. Let $V = V(\mathcal{H}) \subset \mathbb{F}^n$, and let $f : (\mathbb{F}^n)^r \rightarrow \mathbb{F}$ be the polynomial defining \mathcal{H} . Define tensor $T : V^r \rightarrow \mathbb{F}$ to be $T(\mathbf{x}_1, \dots, \mathbf{x}_r) = f(\mathbf{x}_1, \dots, \mathbf{x}_r)$. Then by Lemma 2.2 $\text{mfrank}(T) \leq \binom{n+d}{d}$.

Suppose that \mathcal{H} contains a member of $\mathcal{N}_{r,s}$ for some $s > (r - 1) \binom{n+d}{d}$. Then there exists $\mathbf{u}_{i,j} \in \mathbb{F}^n$ for $(i, j) \in [s] \times [r]$ such that $f(\mathbf{u}_{i,1}, \dots, \mathbf{u}_{i,r}) \neq 0$ for $i \in [s]$ and $f(\mathbf{u}_{i_1,1}, \dots, \mathbf{u}_{i_r,r}) = 0$ for $(i_1, \dots, i_r) \in [s]^r$ for which i_1, \dots, i_r are pairwise distinct. Let T' be the subtensor of T induced on $\{\mathbf{u}_{1,1}, \dots, \mathbf{u}_{s,1}\} \times \dots \times \{\mathbf{u}_{1,r}, \dots, \mathbf{u}_{s,r}\}$. Identifying $\mathbf{u}_{i,j}$ by i , the tensor $T' : [s]^r \rightarrow \mathbb{F}$ is semi-diagonal, so

$$\text{mfrank}(T) \geq \text{mfrank}(T') \geq \frac{s}{r - 1}$$

by Theorem 5.1. This is a contradiction, finishing the proof. \square

Now everything is set to prove the main theorem of this section.

Proof of Theorem 1.6. Let $\mathcal{H} \in \mathcal{F}$, and let $N = |V(\mathcal{H})|$. Then \mathcal{H} is an r -uniform strongly algebraic hypergraph of complexity (n, d) . Let $\epsilon = N^{-\beta/r!(2n+1)}$. Then by Theorem 1.5, there is an equitable partition of $V(\mathcal{H})$ into parts V_1, \dots, V_K for some $8/\epsilon < K < c'(1/\epsilon)^{r!(2n+1)}$ such that all but at most ϵ -fraction of the r -tuples of parts either are empty or have density at least $1 - \epsilon$. Here, $c' = c'(r, n, d) > 0$.

For $i \in [K]$, pick a vertex $v_i \in V_i$ randomly with uniform distribution. Let \mathcal{H}' be the hypergraph induced on the vertex set $\{v_1, \dots, v_K\}$; then $\mathcal{H}' \in \mathcal{F}$ as well. Define the r -uniform hypergraph \mathcal{G} on vertex set $[K]$ as follows. The r -element set $\{i_1, \dots, i_r\} \in [K]^{(r)}$ is an edge of \mathcal{G} if and only if either $\{v_{i_1}, \dots, v_{i_r}\} \in E(\mathcal{H})$ and $d(V_{i_1}, \dots, V_{i_r}) \geq 1 - \epsilon$, or $\{v_{i_1}, \dots, v_{i_r}\} \notin E(\mathcal{H})$ and $(V_{i_1}, \dots, V_{i_r})$ is empty. Note that if $(V_{i_1}, \dots, V_{i_r})$ either is empty or has density at least $1 - \epsilon$, then $\{i_1, \dots, i_r\}$ is an edge with probability at least $1 - \epsilon$. Therefore, $\mathbb{E}(d(\mathcal{G})) \geq 1 - 2\epsilon$, so there is a choice for v_1, \dots, v_K such that $d(\mathcal{G}) \geq 1 - 2\epsilon$. Fix such a choice. By Lemma 2.1, \mathcal{G} contains a clique J of size at least $\frac{1}{8}\epsilon^{-1/(r-1)}$. Let \mathcal{H}^* be the subhypergraph of \mathcal{H} induced on the vertex set $\{v_j : j \in J\}$. As $\mathcal{H}^* \in \mathcal{F}$, \mathcal{H}^* contains either an independent set of size s or a clique of size $c|J|^\alpha = c_1 N^{\alpha\beta/r!(r-1)(2n+1)}$ for some $c_1 = c_1(r, n, d, c, \alpha, \beta) > 0$. In the latter case, we are done, so assume that \mathcal{H}^* contains an independent set $\{v_i : i \in I\}$ of size s .

We finish the proof by noting that at least one of $U_i, i \in I$ is an independent set in \mathcal{H} . Otherwise, if U_i contains an edge f_i for every $i \in I$, then $\bigcup_{i \in I} f_i$ spans a member of $\mathcal{N}_{r,s}$ in \mathcal{H} , contradicting Lemma 5.2. Therefore, \mathcal{H} contains an independent set of size at least $\frac{1}{2}|N/K| > c_2 N^{1-\beta}$, where $c_2 = c_2(r, n, d, c, \alpha, \beta) > 0$. □

6. Concluding remarks

Following Fox and Pach [25], we say that a family of graphs \mathcal{G} has the *strong-Erdős-Hajnal property* if there exists a constant $c = c(\mathcal{G}) > 0$ such that for every $G \in \mathcal{G}$, either G or its complement contains a bi-clique of size at least $c|V(G)|$. In [2], it is proved that the strong-Erdős-Hajnal property implies the Erdős-Hajnal property in hereditary graph families and that the family of semi-algebraic graphs of complexity (n, d, m) has the strong-Erdős-Hajnal property. On the other hand, it is known that this is not the case for algebraic graphs. For infinitely many N , one describes a strongly algebraic graph G of complexity $(3, 2)$ such that the size of the largest bi-clique in both G and its complement is $O(N^{3/4})$.

If q is a prime power, the Erdős-Rényi graph ER_q is defined as follows. The vertices of ER_q are the elements of the projective plane over \mathbb{F}_q , and (x_0, x_1, x_2) and (y_0, y_1, y_2) are joined by an edge if $x_0y_0 + x_1y_1 + x_2y_2 = 0$. The graph ER_q has $N = q^2 + q + 1$ vertices, is $(q + 1)$ -regular and has at most $2(q + 1)$ vertices with loops. This graph contains no copy of $K_{2,2}$ and has eigenvalues $\pm\sqrt{q}$ and $q + 1$, where the multiplicity of $q + 1$ is 1; see, for example, [30]. By the expander mixing lemma (see, for example, Theorem 2.11 in [30]), one can bound the number of the edges/non-edges of ER_q between two disjoint subsets of vertices using its eigenvalues. This lemma implies that both ER_q and its complement contains no bi-clique of size larger than

$$\frac{N\sqrt{q}}{q + 1} = O(N^{3/4}).$$

Note that the complement of ER_q is strongly algebraic of complexity $(3, 2)$. Indeed, we can view the vertices of ER_q as elements of \mathbb{F}_q^3 by replacing each $(x_0, x_1, x_2) \in \mathbb{P}\mathbb{F}_q^2$ with one element of the equivalence class $C_{(x_0, x_1, x_2)} = \{(\lambda x_0, \lambda x_1, \lambda x_2) : \lambda \in \mathbb{F}_q \setminus \{0\}\} \subset \mathbb{F}_q^3$. For a variant of this construction, see, for example, [10], Section 6.1.

Finally, let us mention that although algebraic graphs of bounded complexity do not have strong Ramsey properties (as we proved in this paper), they are one of the main sources of best examples for so-called Turán-type questions. Given an r -uniform hypergraph \mathcal{H} , the *extremal number (or Turán number) of \mathcal{H}* denoted by $ex(N, \mathcal{H})$ is the maximum number of edges in an r -uniform hypergraph on N vertices that contains no copy of \mathcal{H} as a subhypergraph. If H is a graph, the asymptotic value of the extremal number of H is known by the Erdős-Stone theorem, unless H is bipartite. The case of bipartite graphs

is notoriously hard. In many cases, when the order of $\text{ex}(N, H)$ is known for some bipartite graph H , the construction achieving the right order of magnitude is an algebraic graph of bounded complexity; see, for example, [3, 4, 5, 22]. Algebraic hypergraphs of bounded complexity also appear in connection to Turán type results; see, for example, [32, 16].

Acknowledgements. We would like to thank the anonymous referee for their useful comments and suggestions. Moreover, we would like to thank Jacob Fox, János Pach and Artem Chernikov for pointing out the related references and results.

Conflicts of Interest. The authors have no conflicts of interest to declare.

Financial support. Both authors were supported by the SNSF grant 200021_196965. István Tomon also acknowledges the support of the Russian Government in the framework of MegaGrant no 075-15-2019-1926.

References

- [1] N. Alon, E. Fischer and I. Newman, ‘Efficient testing of bipartite graphs for forbidden induced subgraphs’, *SIAM J. Comput.* **37**(3) (2007), 959–976.
- [2] N. Alon, J. Pach, R. Pinchasi, R. Radoičić and M. Sharir, ‘Crossing patterns of semi-algebraic sets’, *J. Combin. Theory Ser. A* **111** (2005), 310–326.
- [3] W. G. Brown, ‘On graphs that do not contain a Thomsen graph’, *Canad. Math. Bull.* **9** (1966), 281–285.
- [4] B. Bukh, ‘Random algebraic construction of extremal graphs’, *Bull. Lond. Math. Soc.* **47**(6) (2015), 939–945.
- [5] B. Bukh and D. Conlon, ‘Rational exponents in extremal graph theory’, *J. Eur. Math. Soc. (JEMS)* **20**(7) (2018), 1747–1757.
- [6] B. Bukh and J. Matoušek, ‘Erdős-Szekerés-type statements: Ramsey function and decidability in dimension 1’, *Duke Math. J.* **163**(12) (2012), 2243–2270.
- [7] E. Chattopadhyay and D. Zuckerman, ‘Explicit two-source extractors and resilient functions’, in *Proc. 48th Annu. ACM Sympos. Theor. Comput.* (ACM, New York, 2016), 670–683.
- [8] A. Chernikov and S. Starchenko, ‘Definable regularity lemmas for NIP hypergraphs’, *Q. J. Math.* **72**(4) (2021), 1401–1433.
- [9] A. Chernikov and S. Starchenko, ‘A note on the Erdős-Hajnal property for stable graphs’, *Proc. Amer. Math. Soc.* **146**(2) (2018), 785–790.
- [10] A. Chernikov and S. Starchenko, ‘Regularity lemma for distal structures’, *J. Eur. Math. Soc. (JEMS)* **20** (2018), 2437–2466.
- [11] G. Cohen, ‘Two-source dispersers for polylogarithmic entropy and improved Ramsey graphs’, *SIAM J. Comput.* **50**(3) (2021), 16–30.
- [12] D. Conlon and A. Ferber, ‘Lower bounds for multicolor Ramsey numbers’, *Adv. Math.* **378** (2021), 107528.
- [13] D. Conlon, J. Fox, J. Pach, B. Sudakov and A. Suk, ‘Ramsey-type results for semi-algebraic relations’, *Trans. Amer. Math. Soc.* **366** (2014), 5043–5065.
- [14] D. Conlon, J. Fox and B. Sudakov, ‘Hypergraph Ramsey numbers’, *J. Amer. Math. Soc.* **23**(1) (2010), 247–266.
- [15] D. Conlon, J. Fox and B. Sudakov, ‘An improved bound for the stepping-up lemma’, *Discrete Appl. Math.* **161**(9) (2013), 1191–1196.
- [16] D. Conlon, C. Pohoata and D. Zakharov, ‘Random multilinear maps and the Erdős box problem’, *Discrete Anal.* **17** (2021), 8.
- [17] P. Erdős, ‘Some remarks on the theory of graphs’, *Bull. Amer. Math. Soc.* **53** (1947), 292–294.
- [18] P. Erdős and A. Hajnal, ‘Ramsey-type theorems’, *Discrete Appl. Math.* **25** (1989), 37–52.
- [19] P. Erdős, A. Hajnal and J. Pach, ‘Ramsey-type theorem for bipartite graphs’, *Geombinatorics* **10** (2000), 64–68.
- [20] P. Erdős, A. Hajnal and R. Rado, ‘Partition relations for cardinal numbers’, *Acta Math. Acad. Sci. Hungar.* **16** (1965), 93–196.
- [21] P. Erdős and R. Rado, ‘Combinatorial theorems on classifications of subsets of a given set’, *Proc. Lond. Math. Soc.* **3** (1952), 417–439.
- [22] P. Erdős, A. Rényi and V. T. Sós, ‘On a problem of graph theory’, *Studia Sci. Math. Hungar.* **1** (1966), 215–235.
- [23] P. Erdős and G. Szekerés, ‘A combinatorial problem in geometry’, *Compos. Math.* **2** (1935), 463–470.
- [24] P. Frankl and R. M. Wilson, ‘Intersection theorems with geometric consequences’, *Combinatorica* **1** (1981), 357–368.
- [25] J. Fox and J. Pach, ‘Erdős-Hajnal-type results on intersection patterns of geometric objects’, in *Horizons of Combinatorics, Bolyai Society Mathematical Studies* vol. 17 (Springer, Berlin, 2008), 79–103.
- [26] J. Fox, J. Pach and A. Suk, ‘A polynomial regularity lemma for semi-algebraic hypergraphs and its applications in geometry and property testing’, *SIAM J. Comput.* **45** (2016), 2199–2223.
- [27] J. Fox, J. Pach and A. Suk, ‘Erdős-Hajnal conjecture for graphs with bounded VC-dimension’, *Discret. Comput. Geom.* **61**(4) (2019), 809–829.
- [28] J. Fox and B. Sudakov, ‘Density theorems for bipartite graphs and related Ramsey-type results’, *Combinatorica* **29** (2009), 153–196.
- [29] D. Haussler, ‘Sphere packing numbers for subsets of the Boolean n -cube with bounded Vapnik-Chervonenkis dimension’, *J. Combin. Theory Ser. A* **69** (1995), 217–232.
- [30] M. Krivelevich and B. Sudakov, ‘Pseudo-random graphs’, in *More Sets, Graphs and Numbers, Bolyai Society Mathematical Studies* vol. 15 (Springer, Berlin, 2006), 199–262.

- [31] L. Lovász and B. Szegedy, ‘Szemerédi’s lemma for the analyst’, *Geom. Funct. Anal.* **17**(1) (2007), 252–270.
- [32] J. Ma, X. Yuan and M. Zhang, ‘Some extremal results on complete degenerate hypergraphs’, *J. Combin. Theory Ser. A* **154** (2018), 598–609.
- [33] M. Malliaris and S. Shelah, ‘Regularity lemmas for stable graphs’, *Trans. Amer. Math. Soc.* **366**(3) (2014), 1551–1585.
- [34] D. Munhá Correia, B. Sudakov and I. Tomon, ‘Flattening rank and its combinatorial applications’, *Linear Algebra Appl.* **625** (2021), 113–125.
- [35] L. Rónyai, L. Babai and M. K. Ganapathy, ‘On the number of zero-patterns of a sequence of polynomials’, *J. Amer. Math. Soc.* **14**(3) (2001), 717–735.
- [36] N. Sauer, ‘On the density of families of sets’, *J. Combin. Theory Ser. A* **13** (1972), 145–147.
- [37] S. Shelah, ‘A combinatorial problem; stability and order for models and theories in infinitary languages’, *Pacific J. Math.* **41** (1972), 247–261.
- [38] E. Szemerédi, ‘Regular partitions of graphs’, in *Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay)*, vol. **260** (CNRS, Paris, 1978), 399–401.
- [39] T. Tao, ‘Expanding polynomials over finite fields of large characteristic, and a regularity lemma for definable sets’, *Contrib. Discrete Math.* **10** (2015), 22–98. See also his blog on this topic: terrytao.wordpress.com/2013/10/29/a-spectral-theory-proof-of-the-algebraic-regularity-lemma/
- [40] V. Vapnik and A. Chervonenkis, ‘On the uniform convergence of relative frequencies of events to their probabilities’, *Theory Probab. Appl.* **16** (1971), 264–280.
- [41] Y. Wigderson, ‘An improved lower bound on multicolor Ramsey numbers’, *Proc. Amer. Math. Soc.* **149**(6) (2021), 2371–2374.