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RAMSEY-TYPE PROBLEM FOR AN ALMOST MONOCHROMATIC K_4*

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Abstract. In this short note we prove that there is a constant c such that every k-edge-coloring of the complete graph K_n with $n \geq 2^{ck}$ contains a K_4 whose edges receive at most two colors. This improves on a result of Kostochka and Mubayi, and is the first exponential bound for this problem.

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1. Introduction.

The *Ramsey number* $R(t;k)$ is the least positive integer n such that every k-coloring of the edges of the complete graph K_n contains a monochromatic K_t. In 1916 Schur showed that $R(3;k)$ is at least exponential in k and at most a constant times $k!$. Despite various efforts over the past century to determine the asymptotics of $R(t;k)$, there were improvements only in the exponential constant in the lower bound and the constant factor in the upper bound. It is a major open problem to determine whether there is a constant c such that $R(3;k) \leq c^k$ for all k (see, e.g., the monograph [9]).

In 1981, Erdős [6] proposed studying the following generalization of the classical Ramsey problem. Let p, q be positive integers with $2 \leq q \leq \binom{p}{2}$. A (p,q)-coloring of K_n is an edge-coloring such that every copy of K_p receives at least q distinct colors. Let $f(n,p,q)$ be the minimum number of colors in a (p,q)-coloring of K_n. Determining the numbers $f(n,p,2)$ is equivalent to determining the multicolor Ramsey numbers $R(p;k)$, as an edge-coloring is a $(p,2)$-coloring if and only if it does not contain a monochromatic K_p. Over the last two decades, the study of $f(n,p,q)$ drew a lot of attention. Erdős and Gyárfás [7] proved several results on $f(n,p,q)$; e.g., they determined for which fixed p and q we have where $f(n,p,q)$ is at least linear in n, quadratic in n, or $\binom{n}{2}$ minus a constant. For fixed p, they also gave bounds on the smallest q for which $f(n,p,q)$ is asymptotically $\binom{n}{2}$. These bounds were significantly tightened by Sárközy and Selkow [15] using Szemerédi's regularity lemma. In a different paper, Sárközy and Selkow [14] show that $f(n,p,q)$ is linear in n for at most $\log p$ values of q. (Here, and throughout the paper, all logarithms are base 2.) There are also results on the behavior of $f(n,p,q)$ for particular values of p and q. Mubayi [13] gave an explicit construction of an edge-coloring which together with the already known lower bound shows that $f(n,4,4) = n^{1/2+o(1)}$. Using Behrend's construction of a dense set with no arithmetic progressions of length three, Axenovich [2] showed

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that $\frac{1+\sqrt{5}}{2}n - 3 \le f(n,5,9) \le 2n^{1+c/\sqrt{\log n}}$. These examples demonstrate that special cases of $f(n,p,q)$ lead to many interesting problems.

As was pointed out by Erdős and Gyárfás [7], one of the most intriguing problems among the small cases is the behavior of $f(n,4,3)$. This problem can be rephrased in terms of another more convenient function. Let $g(k)$ be the largest positive integer n for which there is a k-edge-coloring of K_n, in which every K_4 receives at least three colors, i.e., for which $f(n,4,3) \le k$. Restated, $g(k)+1$ is the smallest positive integer n for which every k-edge-coloring of the edges of K_n contains a K_4 that receives at most two colors. In 1981, Erdős [6] showed that $g(k)$ is superlinear in k by an easy application of the probabilistic method. Later, Erdős and Gyárfás used the Lovász local lemma to show that $g(k)$ is at least quadratic in k. Mubayi [12] improved these bounds substantially, showing that $g(k) \ge 2^{c(\log k)^2}$ for some absolute positive constant c. On the other hand, the progress on the upper bound was much slower. Until very recently, the best result was of the form $g(k) < k^{ck}$ for some constant c, which follows trivially from the multicolor k-color Ramsey number for K_4. This bound was improved by Kostochka and Mubayi [10], who showed that $g(k) < (\log k)^{ck}$ for some constant c. Here we further extend their neat approach and obtain the first exponential upper bound for this problem.

THEOREM 1.1. *For $k > 2^{100}$, we have $g(k) < 2^{2000k}$.*

While it is a longstanding open problem to determine whether $R(t;k)$ grows faster than exponential in k, it is not difficult to prove an exponential upper bound if we restrict the colorings to those that do not contain a rainbow K_s for fixed s. Let $M(k,t,s)$ be the minimum n such that every k-edge-coloring of K_n has a monochromatic K_t or a rainbow K_s. Axenovich and Iverson [4] showed that $M(k,t,3) \le 2^{kt^2}$. We improve on their bound by showing that $M(k,t,s) \le s^{4kt}$ for all k,t,s. In the other direction, we prove that for all positive integers k and t with k even and $t \ge 3$, $M(k,t,3) \ge 2^{kt/4}$, thus determining $M(k,t,3)$ up to a constant factor in the exponent.

The rest of this paper is organized as follows. In the next section, we prove our main result, Theorem 1.1. In section 3, we study the Ramsey problem for colorings without rainbow K_s. The last section of this note contains some concluding remarks. Throughout the paper, we systematically omit floor and ceiling signs whenever they are not crucial for the sake of clarity. We also do not make any serious attempt to optimize absolute constants in our statements and proofs.

2. Proof of Theorem 1.1. Our proof develops further on ideas in [10]. Like the Kostochka–Mubayi proof, we show that the K_4 we find is monochromatic or is a C_4 in one color and a matching in the other color. Call a coloring of K_t *rainbow* if all $\binom{t}{2}$ edges have different colors. Let $g(k,t)$ be the largest positive integer n such that there is a k-edge-coloring of K_n with no rainbow K_t, and in which the edges of every K_4 have at least three colors. We will study $g(k)$ by investigating the behavior of $g(k,t)$.

Before jumping into the details of the proof of Theorem 1.1, we first outline the proof idea. Note that $g(k) = g(k,k)$ for $k > 2$ as a rainbow K_k would use $\binom{k}{2} > k$ colors. We give a recursive upper bound on $g(k,t)$ which implies Theorem 1.1. We first prove a couple of lemmas which show that in any k-edge-coloring without a rainbow K_t, there are many vertices that have large degree in some color i. We then apply a simple probabilistic lemma to find a large subset V_2 of vertices such that every vertex subset of size d (with $d \ll t$) has many common neighbors in color i. We use this to get an upper bound on $g(k,t)$ as follows. Consider a k-edge-coloring of K_n with $n = g(k,t)$ without a rainbow K_t and with every K_4 containing at least three colors. There are two possible cases. If there is no rainbow K_d in the set V_2, then we obtain an upper bound on $g(k,t)$ using the fact that $|V_2|$ has size at most $g(k,d)$. If there is a set $R \subset V_2$ of d vertices which forms a rainbow K_d, then the $\binom{d}{2}$ colors that appear in this rainbow K_d cannot appear in the edges inside the set $N_i(R)$ of vertices that are adjacent to every vertex in R in color i, for otherwise we would obtain a K_4 having at most two colors (the color i and the color that appears in both R and in $N_i(R)$). In this case we obtain an upper bound on $g(k,t)$ using the fact that $|N_i(R)| \le g(k-\binom{d}{2}, t)$. Finally, if the coloring has no rainbow K_d with d constant, it is easy to show an exponential upper bound.

For an edge-coloring of K_n, a vertex x, and a color i, let $d_i(x)$ denote the degree of vertex x in color i. Our first lemma shows that if, for every vertex x and color i, $d_i(x)$ is not too large, then the coloring contains many rainbow cliques.

LEMMA 2.1. *If an edge-coloring of the complete graph K_n satisfies $d_i(x) \le \delta n$ for each $x \in V(K_n)$ and each color i, then this coloring has at most $\frac{5}{8}\delta t^4\binom{n}{t}$ nonrainbow copies of K_t.*

Proof. If a K_t is not rainbow, then it has two adjacent edges of the same color or two nonadjacent edges of the same color. We will use this fact to give an upper bound on the number of K_t's that are not rainbow.

Let $\nu(i,t,n)$ be the number of copies of K_t in K_n in which there are at least two adjacent edges of color i. To bound the number of such K_t we can first choose the vertex, then the two edges with color i incident to this vertex and then the remaining $t-3$ vertices. Hence, the number of K_t's for which there is a vertex with degree at least two in some color is at most

$$\begin{aligned}\sum_i \nu(i,t,n) &\le \sum_i \sum_{x\in V} \binom{d_i(x)}{2}\binom{n-3}{t-3} \le n\delta^{-1}\binom{\delta n}{2}\binom{n-3}{t-3}\\ &\le \frac{\delta n^3}{2}\left(\frac{t}{n}\right)^3\binom{n}{t} = \frac{1}{2}\delta t^3\binom{n}{t}.\end{aligned}$$

Here we used the fact that $\sum_i \binom{d_i(x)}{2} \le \delta^{-1}\binom{\delta n}{2}$, since $d_i(x) \le \delta n$, $\sum_i d_i(x) = n-1$, and the function $f(y) = \binom{y}{2}$ is convex.

Let $\psi(i,t,n)$ be the number of copies of K_t in K_n in which there is a matching of size at least two in color i. Let e_i denote the number of edges of color i. Since

$$e_i \le \frac{n}{2}\max_{x\in V} d_i(x) \le \frac{\delta}{2}n^2,$$

then the number of K_t's in which there is a matching of size at least two in some color is at most

$$\sum_i \psi(i,t,n) \le \sum_i \binom{e_i}{2}\binom{n-4}{t-4} \le \delta^{-1}\binom{\delta n^2/2}{2}\binom{n-4}{t-4} \le \frac{\delta t^4}{8}\binom{n}{t},$$

where again we used the convexity of the function $f(y) = \binom{y}{2}$ together with $e_i \le \delta n^2/2$ and $\sum_i e_i \le n^2/2$. Hence, the number of K_t's which are not rainbow is at most $\frac{1}{2}\delta t^3\binom{n}{t} + \frac{1}{8}\delta t^4\binom{n}{t} \le \frac{5}{8}\delta t^4\binom{n}{t}$, completing the proof. ∎

For the proof of Theorem 1.1, we do not need the full strength of this lemma since we will use only the existence of at least one rainbow K_t. We also would like to mention the following stronger result. Call an edge-coloring *m-good* if each color appears at most m times at each vertex. Let $h(m,t)$ denote the minimum n such that every m-good edge-coloring of K_n contains a rainbow K_t. The above lemma demonstrates that $h(m,t)$ is at most mt^4. It is shown by Alon et al. [1] that there are constant positive constants c_1 and c_2 such that

$$c_1 mt^3/\log t \leq h(m,t) \leq c_2 mt^3/\log t.$$

The following easy corollary of Lemma 2.1 demonstrates that in every k-edge-coloring without a rainbow K_t, there is a color and a large set of vertices which have large degree in that color.

COROLLARY 2.2. *In every k-edge-coloring of K_n without a rainbow K_t, there is a subset $V_1 \subset V(K_n)$ with $|V_1| \geq \frac{n}{2k}$ and a color i such that $d_i(x) \geq \frac{n}{2t^4}$ for each vertex $x \in V_1$.*

Proof. Let $V' \subset V(K_n)$ be those vertices x for which there is a color i such that $d_i(x) \geq \frac{n}{2t^4}$.

Case 1: $|V'| < n/2$. In this case, letting $V'' = V(K_n) \setminus V'$, $|V''| \geq n/2$ and no vertex in V'' has degree at least $\frac{n}{2t^4} \leq |V''|/t^4$ in any given color. By Lemma 2.1 applied to the coloring of K_n restricted to V'' with $\delta = t^{-4}$, there are at least $\frac{3}{8}\binom{|V''|}{t}$ rainbow K_t's, contradicting the assumption that the coloring is free of rainbow K_t's.

Case 2: $|V'| \geq n/2$. In this case, by the pigeonhole principle, there is a color i and at least $\frac{n}{2k}$ vertices x for which $d_i(x) \geq \frac{n}{2t^4}$, completing the proof. ∎

The following lemma is essentially the same as results in [11] and [16]. Its proof uses a probabilistic argument commonly referred to as dependent random choice, which appears to be a powerful tool in proving various results in Ramsey theory (see, e.g., [8] and its references). In a graph G, the *neighborhood* $N(v)$ of a vertex v is the set of vertices adjacent to v. For a vertex subset U of a graph G, the *common neighborhood* $N(U)$ is the set of vertices adjacent to all vertices in U.

LEMMA 2.3. *Let $G = (V,E)$ be a graph with n vertices and let $V_1 \subset V$ be a subset with $|V_1| = m$ in which each vertex has degree at least αn. If $\beta \leq m^{-d/h}$, then there is a subset $V_2 \subset V_1$ with $|V_2| \geq \alpha^h m - 1$ such that every d-tuple in V_2 has at least βn common neighbors.*

Proof. Let $U = \{x_1, \ldots, x_h\}$ be a subset of h random vertices from V chosen uniformly with repetitions, and let $V_1' = N(U) \cap V_1$. We have

$$\mathbb{E}[|V_1'|] = \sum_{v \in V_1} \mathbf{Pr}(v \in N(U)) = \sum_{v \in V_1} \left(\frac{|N(v)|}{n}\right)^h \geq \alpha^h m.$$

The probability that a given set $W \subset V_1$ of vertices is contained in V_1' is $(\frac{|N(W)|}{n})^h$. Let Z denote the number of d-tuples in V_1' with less than βn common neighbors. So

$$\mathbb{E}[Z] = \sum_{W \subset V_1, |W|=d, |N(W)|<\beta n} \mathbf{Pr}(W \subset V_1') \leq \binom{m}{d}\beta^h \leq m^d \beta^h \leq 1.$$

Hence, the expectation of $|V_1'| - Z$ is at least $\alpha^h m - 1$ and thus, there is a choice U_0 for U such that the corresponding value of $|V_1'| - Z$ is at least $\alpha^h m - 1$. For every d-tuple D of vertices of V_1' with less than βn common neighbors, delete a vertex $v_D \in D$ from V_1'. Letting V_2 be the resulting set, it is clear that V_2 has the desired properties, completing the proof. ∎

The proof of the next lemma uses the standard pigeonhole argument together with Lemma 2.1.

LEMMA 2.4. *Let d, k be integers with $d, k \geq 2$. Then every k-edge-coloring of K_n with $n \geq d^{12k}$ and without a rainbow K_d has a monochromatic K_4. In particular, we have $g(k, d) < d^{12k}$.*

Proof. Suppose for contradiction that there is a k-edge-coloring of K_n with $n \geq d^{12k}$ and without a rainbow K_d and without a monochromatic K_4. By Lemma 2.1 with $t = d$ and $\delta = d^{-4}$, this graph contains a vertex x_1 with degree at least $\frac{n}{d^4}$ in some color c_1. Pick this vertex x_1 and let N_1 be the set of vertices adjacent to x_1 by color c_1. We will define a sequence $x_1, \ldots, x_{2k+1}$ of vertices, a sequence $c_1, \ldots, c_{2k+1}$ of colors, and a sequence $V(K_n) \supset N_1 \supset \cdots \supset N_{2k+1}$ of vertex subsets. Once x_j, c_j, and N_j have been defined, pick a vertex x_{j+1} in N_j such that there are at least $\frac{|N_j|}{d^4}$ vertices in N_j connected to x_{j+1} by edges of the same color c_{j+1}. Let N_{j+1} be the set of vertices in N_j that are adjacent to x_{j+1} by edges of color c_{j+1}. Note that $|N_{j+1}| \geq d^{-4}|N_j|$ so

$$|N_{2k+1}| \geq (d^{-4})^{2k+1} n \geq 1.$$

Therefore, there is a color c that is represented at least three times in the list $c_1, \ldots, c_{2k+1}$ and the three vertices $x_{j_1}, x_{j_2}, x_{j_3}$ together with a vertex from N_{2k+1} form a monochromatic K_4 in color c, where $c_{j_1} = c_{j_1} = c_{j_3} = c$ with $j_1 < j_2 < j_3$. $\square$

LEMMA 2.5. *Let d, k, t be positive integers with $3 \leq d \leq t$ and $d \geq 40 \log t$. If $k \geq \binom{d}{2}$, then*

$$g(k,t) \leq \max\left(4kg(k,t)^{\frac{20 \log t}{d}} g(k,d), 2^{\binom{d}{2}} g\left(k - \binom{d}{2}, t\right)\right). \tag{2.1}$$

Otherwise, we have $g(k, t) = g(k, d)$.

Proof. Note that if $k < \binom{d}{2}$, then a k-edge-coloring cannot have a rainbow K_d. Therefore, $g(k, t) = g(k, d)$ in this case. So we assume $k \geq \binom{d}{2}$. By the definition of $g(k, t)$, there is a k-edge-coloring of K_n with $n = g(k, t)$ with no rainbow K_t and in which every K_4 receives at least three colors. Consider such a coloring. By Corollary 2.2, there is a color i and a subset $V_1 \subset V(K_n)$ with $|V_1| \geq \frac{n}{2k}$ and $d_i(x) \geq \frac{n}{2t^4}$ for every vertex $x \in V_1$. Apply Lemma 2.3 to the graph of color i with $\alpha = \frac{1}{2t^4}$, $\beta = 2^{-\binom{d}{2}}$, $m = |V_1| \geq \frac{n}{2k}$, and $h = 4d^{-1} \log n$. We can apply Lemma 2.3 since $\beta < 2^{-d^2/4} = n^{-d/h} \leq |V_1|^{-d/h}$. So there is a subset $V_2 \subset V_1$ such that

$$|V_2| \geq \alpha^h m - 1 \geq \alpha^h m/2 \geq (2t^4)^{-4d^{-1} \log n} \cdot \frac{n}{4k} \geq n^{1 - \frac{20 \log t}{d}}/(4k)$$

and every subset of V_2 of size d has at least $\beta n = 2^{-\binom{d}{2}} n$ common neighbors in color i.

There are two possibilities: Either every K_d in V_2 is not rainbow, or there is a K_d in V_2 that is rainbow. In the first case, the k-edge-coloring restricted to V_2 is free of rainbow K_d, so

$$g(k, d) \geq |V_2| \geq n^{1 - \frac{20 \log t}{d}}/(4k).$$

Since $n = g(k, t)$, we can restate this inequality as

$$g(k, t) \leq 4kg(k, t)^{\frac{20 \log t}{d}} g(k, d).$$

In the second case, there is a rainbow d-tuple $R \subset V_2$ such that $N_i(R)$, the common neighborhood of R in color i, has cardinality at least βn. The $\binom{d}{2}$ colors present in R

cannot be present in $N_i(R)$ since otherwise we would have a K_4 using only two colors (the color i and the color that appears in both R and in $N_i(R)$). In this case we have

$$g\left(k-\binom{d}{2},t\right) \geq |N_i(R)| \geq \beta n = 2^{-\binom{d}{2}} g(k,t).$$

In either case we have

$$g(k,t) \leq \max\left(4kg(k,t)^{\frac{20\log t}{d}} g(k,d), 2^{\binom{d}{2}} g\left(k-\binom{d}{2},t\right)\right),$$

which completes the proof. $\square$

Having finished all the necessary preparation, we are now ready to prove Theorem 1.1, which says that $g(k) \leq 2^{2000k}$ for $k > 2^{100}$. The iterated logarithm $\log^* n$ is defined by $\log^* n = 0$ if $n \leq 1$ and otherwise $\log^* n = 1 + \log^* \log n$. It is straightforward to verify that $\log^* n < \log n$ holds for $n > 8$.

Proof of Theorem 1.1. Note that $g(k) = g(k,k)$ since no k-edge-coloring contains a rainbow K_k. Assume $k > 2^{100}$ and suppose for contradiction that there is a k-edge-coloring of K_n with $n = g(k) \geq 2^{2000k}$ such that every K_4 has at least three colors.

Let $t_1 = k$, and if $t_i > 2^{100}$, let $t_{i+1} = (\log t_i)^2$. We first exhibit several inequalities which we will use. We have $t_{i+1} > 100 \log t_i$ and $20\frac{\log t_i}{t_{i+1}} = 20/\log t_i \leq \frac{1}{5}$. Let ℓ be the largest positive integer for which t_ℓ is defined, so $100^2 < t_\ell \leq 2^{100}$. Note that $\ell < 2\log^* k$, as one can easily check that $t_{j+1} = (\log t_j)^2 = (2\log\log t_{j-1})^2 < \log t_{j-1}$. Since $\ell < 2\log^* k \leq 2\log k$ and $n \geq 2^{2000k}$, then $(4k)^\ell < n^{1/12}$. For $1 \leq i \leq \ell - 1$, we have $20/\log t_{\ell-i} < 5^{-i}$. Indeed, for $i = 1$, since $t_{\ell-1} > 2^{100}$, we have $20/\log t_{\ell-1} < 1/5$. Suppose by induction on i that we already have $20/\log t_{\ell-i} < 5^{-i}$. Then $t_{\ell-i} > 2^{20\cdot 5^i}$ and therefore we have $20/\log t_{\ell-i-1} = 20/\sqrt{t_{\ell-i}} \leq 20 \cdot 2^{-10\cdot 5^i} < 5^{-i-1}$. Therefore, $\sum_{i=1}^{\ell-1} 20/\log t_i < \sum_{i=1}^{\infty} 5^{-i} \leq 1/4$. Putting this together, we have

$$(4k)^{\ell-1} n^{\sum_{i=1}^{\ell-1} 20/\log t_i} < n^{1/3}.$$

To get an upper bound on $g(k,k)$ we repeatedly apply Lemma 2.5. Given $k' \leq k$ and $t = t_i$, to bound $g(k',t)$, we use this lemma with $d = t_{i+1}$. Note that we have $d = t_{i+1} > 100\log t_i$, so indeed the condition of the lemma holds. If $k' < \binom{t_{i+1}}{2}$, then $g(k',t_i) = g(k',t_{i+1})$. Otherwise, we have one of two possible upper bounds given by (2.1). If the maximum of the two terms in (2.1) is the left bound, then

$$g(k',t) \leq 4k' g(k',t)^{\frac{20\log t}{d}} g(k',d) \leq 4kn^{\frac{20\log t}{d}} g(k',d) = 4kn^{20/\log t_i} g(k',d);$$

otherwise we have $g(k',t) \leq 2^j g(k'-j,t)$ with $j = \binom{d}{2}$. Since $\frac{g(k',t)}{g(k',d)} \leq 4kn^{20/\log t_i}$ if the left bound holds, we can accumulate only up to a total upper bound factor of

$$\prod_{i=1}^{l-1} 4kn^{20/\log t_i} = (4k)^{\ell-1} n^{\sum_{i=1}^{\ell-1} 20/\log t_i} < n^{1/3}$$

in all of the applications of the left bound. When we use the right bound, we pick up a factor of $\frac{g(k',t)}{g(k'-j,t)} \leq 2^j$ with $j = \binom{d}{2}$ and also decrease k' by j. Since in the end of the process $k' \geq 3$, this can give only another multiplicative factor of at most 2^k in all of the applications of the right bound.

As we already mentioned above, if $k' < \binom{t_{i+1}}{2}$, then $g(k',t_i) = g(k',t_{i+1})$. Therefore when we finish repeatedly applying Lemma 2.5 we end up with a term of the form $g(k_0,t_\ell)$ with $k_0 \leq k$. In that case, we use that $t_\ell \leq 2^{100}$ together with Lemma 2.4 to bound it by $g(k,t_\ell) \leq t_\ell^{12k} \leq 2^{1200k}$. Putting this all together, we obtain the upper bound

$$n = g(k) = g(k,k) < n^{1/3}2^k g(k,t_\ell) < 2^{1201k}n^{1/3},$$

which implies that $n < 2^{2000k}$. This completes the proof. $\square$

3. Monochromatic or rainbow cliques. In this section, we prove bounds on the smallest n, denoted by $M(k,t,s)$, such that every k-edge-coloring of K_n contains a monochromatic K_t or a rainbow K_s. The following proposition is a straightforward generalization of Lemma 2.4.

PROPOSITION 3.1. *We have $M(k,t,s) \leq s^{4kt}$.*

Let $M_s(t_1,\ldots,t_k)$ be the maximum n such that there is a k-edge-coloring of K_n with colors $\{1,\ldots,k\}$ without a rainbow K_s and without a monochromatic K_{t_i} in color i for $1 \leq i \leq k$. The above proposition follows from repeated application of the following recursive bound.

LEMMA 3.2. *We have*

$$M_s(t_1,\ldots,t_k) \leq s^4 \max_{1\leq i\leq k} M_s(t_1,\ldots,t_i-1,\ldots,t_k).$$

Proof. By Lemma 2.1, for every edge-coloring of K_n without a rainbow K_s, there is a vertex v with degree at least n/s^4 in some color i. If the coloring of K_n does not contain a monochromatic K_{t_i} in color i, then the neighborhood of v in color i has at least n/s^4 vertices and does not contain K_{t_i-1} in color i, completing the proof. $\square$

Using a slightly better estimate by Alon et al. [1] (which we mentioned earlier) instead of Lemma 2.1, one can improve the constant in the exponent of the above proposition from 4 to 3. Together with the next lemma, Proposition 3.1 determines $M(k,t,3)$ up to a constant factor in the exponent.

LEMMA 3.3. *For all positive integers k and t with k even and $t \geq 3$, we have $M(k,t,3) > 2^{kt/4}$.*

Proof. To prove the lemma, it suffices by induction to prove $M(k,t,3) - 1 \geq 2^{t/2}(M(k-2,t,3)-1)$ for all $k \geq 2$ and $t \geq 3$. Consider a 2-edge-coloring C_1 of K_m with $m = 2^{t/2}$ and without a monochromatic K_t. Such a 2-edge-coloring exists by the well-known lower bound of Erdős [5] on the 2-color Ramsey number $R(t;2)$. Consider also a $(k-2)$-edge-coloring C_2 of K_r with $r = M(k-2,t,3)-1$ without a rainbow triangle and without a monochromatic K_t. We use these two colorings to make a new edge-coloring C_3 of K_{mr} with k colors: We first partition the vertices of K_{mr} into m vertex subsets $V_1,\ldots,V_m$ each of size r, and color any edge $e=(v,w)$ with $v \in V_i, w \in V_j$, and $i \neq j$ by the color of (i,j) in the 2-edge-coloring C_1 of K_m, and color within each V_i identical to the coloring C_2 of K_r. First we show that coloring C_3 has no rainbow triangle. Indeed, consider three vertices of K_{mr}. If all three vertices lie in the same vertex subset V_i, then the triangle between them is not rainbow by the assumption on coloring C_2. If exactly two of the three vertices lie in the same vertex subset, then the two edges from these vertices to the third vertex will receive the same color. Finally, if they lie in three different vertex subsets, then the triangle between them receives only colors from C_1 and is not rainbow since C_1 is a 2-coloring. Similarly, one can see that coloring C_3 has no monochromatic K_t, which completes the proof. $\square$

4. Concluding remarks. In this paper we proved that there exists a constant c such that every k-edge-coloring of K_n with $n \geq 2^{ck}$ contains a K_4 whose edges receive at most two colors. On the other hand, for $n \leq 2^{c(\log k)^2}$, Mubayi constructed a k-edge-coloring of K_n in which every K_4 receives at least three colors. There is still a large gap between these results. We believe that the lower bound is closer to the truth, and the correct growth is likely to be subexponential in k.

Our upper bound is equivalent to $f(n,4,3) \geq (\log n)/2000$ for n sufficiently large. Kostochka and Mubayi showed that $f(n,2a,a+1) \geq c_a \frac{\log n}{\log\log\log n}$, where c_a is a positive constant for each integer $a \geq 2$. Like the Kostochka–Mubayi proof, our proof can be generalized to demonstrate that for every integer $a \geq 2$ there is $c_a > 0$ such that $f(n,2a,a+1) \geq c_a \log n$ for every positive integer n. For brevity, we do not include the details.

We do not yet have a good understanding of how $M(k,t,s)$, which is the smallest positive integer n such that every k-edge-coloring of K_n has a monochromatic K_t or a rainbow K_s, depends on s. From the definition, it is an increasing function in s. For constant s, we showed that $M(k,t,s)$ grows only exponentially in k. On the other hand, for $\binom{s}{2} > k$, we have $M(k,t,s) = R(t;k)$, so understanding the behavior of $M(k,t,s)$ for large s is equivalent to understanding the classical Ramsey numbers $R(t;k)$.

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