

A Construction of Almost Steiner Systems

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Abstract: Let n , k , and t be integers satisfying $n > k > t \geq 2$. A Steiner system with parameters t , k , and n is a k -uniform hypergraph on n vertices in which every set of t distinct vertices is contained in exactly one edge. An outstanding problem in Design Theory is to determine whether a nontrivial Steiner system exists for $t \geq 6$. In this note we prove that for every $k > t \geq 2$ and sufficiently large n , there exists an almost Steiner system with parameters t , k , and n ; that is, there exists a k -uniform hypergraph on n vertices such that every set of t distinct vertices is covered by either one or two edges. © 2013 Wiley Periodicals, Inc. *J. Combin. Designs* 22: 488–494, 2014

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1. INTRODUCTION

Let n , k , t , and λ be positive integers satisfying $n > k > t \geq 2$. A t - (n, k, λ) -design is a k -uniform hypergraph $\mathcal{H} = (X, \mathcal{F})$ on n vertices with the following property: every t -set of vertices $A \subset X$ is contained in exactly λ edges $F \in \mathcal{F}$. The special case $\lambda = 1$ is known as a *Steiner system* with parameters t , k , and n , named after Jakob Steiner who pondered the existence of such systems in 1853. Steiner systems, t -designs¹ and other combinatorial designs turn out to be useful in a multitude of applications, for example,

¹That is, t - (n, k, λ) -designs for some parameters n, k , and λ .

in coding theory, storage systems design, and wireless communication. For a survey of the subject, the reader is referred to [2].

A counting argument shows that a Steiner triple system—that is, a $2-(n, 3, 1)$ -design—can only exist when $n \equiv 1$ or $n \equiv 3 \pmod{6}$. For every such n , this is achieved via constructions based on symmetric idempotent quasi-groups. Geometric constructions over finite fields give rise to some further infinite families of Steiner systems with $t = 2$ and $t = 3$. For instance, for a prime power q and an integer $m \geq 2$, affine geometries yield $2-(q^m, q, 1)$ -designs, projective geometries yield $2-(q^m + \dots + q^2 + q + 1, q + 1, 1)$ -designs and spherical geometries yield $3-(q^m + 1, q, 1)$ -designs.

For $t = 4$ and $t = 5$, only finitely many nontrivial constructions of Steiner systems are known; for $t \geq 6$, no constructions are known at all.

Before stating our result, let us extend the definition of t -designs as follows. Let n , k , and t be positive integers satisfying $n > k > t \geq 2$ and let Λ be a set of positive integers. A t - (n, k, Λ) -design is a k -uniform hypergraph $\mathcal{H} = (X, \mathcal{F})$ on n vertices with the following property: for every t -set of vertices $A \subset X$, the number of edges $F \in \mathcal{F}$ that contains A belongs to Λ . Clearly, when $\Lambda = \{\lambda\}$ is a singleton, a t - $(n, k, \{\lambda\})$ -design coincides with a t - (n, k, λ) -design as defined above.

Not able to construct Steiner systems for large t , Erdős and Hanani [3] aimed for large partial Steiner systems; that is, t - $(n, k, \{0, 1\})$ -designs with as many edges as possible. Since a Steiner system has exactly $\binom{n}{t} / \binom{k}{t}$ edges, they conjectured the existence of partial Steiner systems with $(1 - o(1)) \binom{n}{t} / \binom{k}{t}$ edges. This was first proved by Rödl [8] in 1985, with further refinements [4–6] of the $o(1)$ term, as stated in the following theorem:

Theorem 1 (Rödl). *Let k and t be integers such that $k > t \geq 2$. Then there exists a partial Steiner system with parameters t, k , and n covering all but $o(n^t)$ of the t -sets.*

Theorem 1 can also be rephrased in terms of a covering rather than a packing; that is, it asserts the existence of a system with $(1 + o(1)) \binom{n}{t} / \binom{k}{t}$ edges such that every t -set is covered at least once (see, e.g., [1, page 56]). Nevertheless, some t -sets might be covered multiple times (perhaps even $\omega(1)$ times). It is therefore natural to ask for t - $(n, k, \{1, \dots, r\})$ -designs, where r is as small as possible. The main aim of this short note is to show how to extend Theorem 1 to cover all t -sets at least once but at most twice.

Theorem 2. *Let k and t be integers such that $k > t \geq 2$. Then, for sufficiently large n , there exists a t - $(n, k, \{1, 2\})$ -design.*

Our proof actually gives a stronger result: there exists a t - $(n, k, \{1, 2\})$ -design with $(1 + o(1)) \binom{n}{t} / \binom{k}{t}$ edges.

2. PRELIMINARIES

In this section we present results needed for the proof of Theorem 2.

Given a t - $(n, k, \{0, 1\})$ -design $\mathcal{H} = (X, \mathcal{F})$, we define the *leave hypergraph* $(X, \mathcal{L}_{\mathcal{H}})$ to be the t -uniform hypergraph whose edges are the t -sets $A \subset X$ not covered by any edge $F \in \mathcal{F}$.

Following closely the proof of Theorem 1 appearing in [4], we recover an extended form of the theorem, which is a key ingredient in the proof of our main result.

Theorem 3. *Let k and t be integers such that $k > t \geq 2$. There exists a constant $\varepsilon = \varepsilon(k, t) > 0$ such that for sufficiently large n , there exists a partial Steiner system $\mathcal{H} = (X, \mathcal{F})$ with parameters t, k , and n satisfying the following property:*

- (♣) *For every $0 \leq \ell < t$, every set $X' \subset X$ of size $|X'| = \ell$ is contained in $O(n^{t-\ell-\varepsilon})$ edges of the leave hypergraph.*

We also make use of the following probabilistic tool.

Talagrand’s inequality.

In its general form, Talagrand’s inequality is an isoperimetric-type inequality for product probability spaces. We use the following formulation from [7, pages 232–233], suitable for showing that a random variable in a product space is unlikely to overshoot its expectation under two conditions:

Theorem 4 (Talagrand). *Let $Z \geq 0$ be a nontrivial random variable, which is determined by n independent trials T_1, \dots, T_n . Let $c > 0$ and suppose that the following properties hold:*

- i. (*c*-Lipschitz) *changing the outcome of one trial can affect Z by at most c , and*
- ii. (*Certifiable*) *for any s , if $Z \geq s$ then there is a set of at most s trials whose outcomes certify that $Z \geq s$.*

Then $\Pr[Z > t] < 2 \exp(-t/16c^2)$ for any $t \geq 2\mathbb{E}[Z] + 80c\sqrt{\mathbb{E}[Z]}$.

3. PROOF OF THE MAIN RESULT

In this section we prove Theorem 2.

3.1. Outline

The construction is done in two phases:

- I. Apply Theorem 3 to get a t -($n, k, \{0, 1\}$)-design $\mathcal{H} = (X, \mathcal{F})$ with property (♣) with respect to some $0 < \varepsilon < 1$.
- II. Build another t -($n, k, \{0, 1\}$)-design $\mathcal{H}' = (X, \mathcal{F}')$ that covers the uncovered t -sets $\mathcal{L}_{\mathcal{H}}$.

Combining both designs, we get that every t -set is covered at least once but no more than twice; namely $(X, \mathcal{F} \cup \mathcal{F}')$ is a t -($n, k, \{1, 2\}$)-design, as required.

We now describe how to build \mathcal{H}' . For a set $A \subset X$, denote by $\mathcal{T}_A = \{C \subseteq X : |C| = k \text{ and } A \subseteq C\}$ the family of possible continuations of A to a subset of X of cardinality k . Note that $\mathcal{T}_A = \emptyset$ when $|A| > k$.

Consider the leave hypergraph $(X, \mathcal{L}_{\mathcal{H}})$. Our goal is to choose, for every uncovered t -set $A \in \mathcal{L}_{\mathcal{H}}$, a k -set $A' \in \mathcal{T}_A$ such that $|A' \cap B'| < t$ for every two distinct $A, B \in \mathcal{L}_{\mathcal{H}}$. This ensures that the obtained hypergraph $\mathcal{H}' = (X, \{A' : A \in \mathcal{L}_{\mathcal{H}}\})$ is indeed a t -($n, k, \{0, 1\}$)-design.

To this aim, for every $A \in \mathcal{L}_{\mathcal{H}}$ we introduce intermediate lists $\mathcal{R}_A \subseteq \mathcal{S}_A \subseteq \mathcal{T}_A$ that will help us control the cardinalities of pairwise intersections when choosing $A' \in \mathcal{R}_A$. First note that we surely cannot afford to consider continuations that fully contain some other

$B \in \mathcal{L}_{\mathcal{H}}$, so we restrict ourselves to the list

$$\mathcal{S}_A = \mathcal{T}_A \setminus \bigcup \{ \mathcal{T}_B : B \in \mathcal{L}_H \setminus \{A\} \} = \{ C \in \mathcal{T}_A : B \not\subseteq C \text{ for all } B \in \mathcal{L}_H, B \neq A \}.$$

Note that, by definition, the lists \mathcal{S}_A for different A are disjoint. Next, choose a much smaller sublist $\mathcal{R}_A \subseteq \mathcal{S}_A$ by picking each $C \in \mathcal{S}_A$ to \mathcal{R}_A independently at random with probability $p = n^{t-k+\varepsilon/2}$ (we can of course assume here and later that $\varepsilon < 1 \leq k - t$, and thus $0 < p < 1$). Finally, select $A' \in \mathcal{R}_A$ that has no intersection of size at least t with any $C \in \mathcal{R}_B$ for any other $B \in \mathcal{L}_{\mathcal{H}}$. If there is such a choice for every $A \in \mathcal{L}_{\mathcal{H}}$, we get $|A' \cap B'| < t$ for distinct $A, B \in \mathcal{L}_{\mathcal{H}}$, as requested.

3.2. Details

We start by showing that the lists \mathcal{S}_A are large enough.

Claim 5. For every $A \in \mathcal{L}_{\mathcal{H}}$ we have $|\mathcal{S}_A| = \Theta(n^{k-t})$.

Proof. Fix $A \in \mathcal{L}_{\mathcal{H}}$. Obviously $|\mathcal{T}_A| = \binom{n-t}{k-t} = \Theta(n^{k-t})$. Since $\mathcal{S}_A \subseteq \mathcal{T}_A$, it suffices to show that $|\mathcal{T}_A \setminus \mathcal{S}_A| = o(n^{k-t})$.

Writing $\mathcal{L}_H \setminus \{A\}$ as the disjoint union $\bigcup_{\ell=0}^{t-1} \mathcal{B}_\ell$, where $\mathcal{B}_\ell = \{B \in \mathcal{L}_H : |A \cap B| = \ell\}$, we have

$$\begin{aligned} \mathcal{T}_A \setminus \mathcal{S}_A &= \{ C \in \mathcal{T}_A : \exists B \in \mathcal{L}_H \setminus \{A\} \text{ such that } C \in \mathcal{T}_B \} \\ &= \bigcup_{\ell=0}^{t-1} \{ C \in \mathcal{T}_A : \exists B \in \mathcal{B}_\ell \text{ such that } C \in \mathcal{T}_B \} \\ &= \bigcup_{\ell=0}^{t-1} \bigcup_{B \in \mathcal{B}_\ell} \mathcal{T}_{A \cup B}. \end{aligned}$$

Note that for all $0 \leq \ell < t$ and for all $B \in \mathcal{B}_\ell$, $|A \cup B| = 2t - \ell$ and thus $|\mathcal{T}_{A \cup B}| = \binom{n-2t+\ell}{k-2t+\ell} \leq n^{k-2t+\ell}$. Moreover, $|\mathcal{B}_\ell| = \binom{t}{\ell} \cdot O(n^{t-\ell-\varepsilon}) = O(n^{t-\ell-\varepsilon})$ by Property (♣). Thus,

$$|\mathcal{T}_A \setminus \mathcal{S}_A| \leq \sum_{\ell=0}^{t-1} |\mathcal{B}_\ell| n^{k-2t+\ell} = \ell \cdot O(n^{k-t-\varepsilon}) = o(n^{k-t}),$$

establishing the claim. □

Recall that the sublist $\mathcal{R}_A \subseteq \mathcal{S}_A$ was obtained by picking each $C \in \mathcal{S}_A$ to \mathcal{R}_A independently at random with probability $p = n^{t-k+\varepsilon/2}$. The next claim shows that \mathcal{R}_A typically contains many k -sets whose pairwise intersections are exactly A . This will be used in the proof of Claim 7.

Claim 6. Almost surely (i.e., with probability tending to 1 as n tends to infinity), for every $A \in \mathcal{L}_H$, the family \mathcal{R}_A contains a subset $\mathcal{Q}_A \subseteq \mathcal{R}_A$ of size $\Theta(n^{\varepsilon/3})$ such that $C_1 \cap C_2 = A$ for every two distinct $C_1, C_2 \in \mathcal{Q}_A$.

Proof. Fix $A \in \mathcal{L}_H$. Construct \mathcal{Q}_A greedily as follows: start with $\mathcal{Q}_A = \emptyset$; as long as $|\mathcal{Q}_A| < n^{\varepsilon/3}$ and there exists $C \in \mathcal{R}_A \setminus \mathcal{Q}_A$ such that $C \cap C' = A$ for all $C' \in \mathcal{Q}_A$, add C to \mathcal{Q}_A . It suffices to show that this process continues $\lfloor n^{\varepsilon/3} \rfloor$ steps.

If the process halts after $s < \lfloor n^{\varepsilon/3} \rfloor$ steps, then every k -tuple of \mathcal{R}_A intersects one of the s previously chosen sets in some vertex outside A . This means that there exists a subset $X_A \subset X$ of cardinality $|X_A| = kn^{\varepsilon/3}$ (X_A contains the union of these s previously picked sets) such that none of the edges C of \mathcal{S}_A satisfying $C \cap X_A = A$ is chosen into \mathcal{R}_A . For bounding the number of such edges in \mathcal{S}_A from below, we need to subtract from $|\mathcal{S}_A|$ the number of edges $C \in \mathcal{T}_A$ with $C \cap X_A \neq A$. The latter can be bounded (from above) by $\sum_{i=1}^{k-t} \binom{|X_A|}{i} \binom{n-|X_A|}{k-t-i}$ (choose $1 \leq i \leq k-t$ vertices from X_A , other than A , to be in C , and then choose the remaining $k-t-i$ vertices from $X \setminus X_A$). Since $|\mathcal{S}_A| = \Theta(n^{k-t})$, and since

$$\sum_{i=1}^{k-t} \binom{|X_A|}{i} \binom{n-|X_A|}{k-t-i} \leq \sum_{i=1}^{k-t} \Theta(n^{i\varepsilon/3}) \cdot n^{k-t-i} = o(n^{k-t}),$$

we obtain that the number of such edges in \mathcal{S}_A is at least $|\mathcal{S}_A| - \sum_{i=1}^{k-t} \binom{|X_A|}{i} \binom{n-|X_A|}{k-t-i} = \Theta(n^{k-t})$. It thus follows that the probability of the latter event to happen for a given A is at most

$$\binom{n}{kn^{\varepsilon/3}} (1-p)^{\Theta(n^{k-t})}$$

(choose X_A first, and then require all edges of \mathcal{S}_A intersecting X_A only at A to be absent from \mathcal{R}_A). The above estimate is clearly at most

$$n^{n^{\varepsilon/3}} \cdot e^{-\Theta(pn^{k-t})} = \exp\{n^{\varepsilon/3} \ln n - \Theta(n^{\varepsilon/2})\} < \exp\{-n^{\varepsilon/3}\}.$$

Taking the union bound over all ($\leq \binom{n}{t}$) choices of A establishes the claim. □

The last step is to select a well-behaved set $A' \in \mathcal{R}_A$. The next claim shows this is indeed possible.

Claim 7. Almost surely for every $A \in \mathcal{L}_H$ we can select $A' \in \mathcal{R}_A$ such that $|A' \cap C| < t$ for all $C \in \bigcup \{\mathcal{R}_B : B \in \mathcal{L}_H \setminus \{A\}\}$.

Proof. Fix $A \in \mathcal{L}_H$ and fix all the random choices that determine the list \mathcal{R}_A such that it satisfies Claim 6. Let $\mathcal{Q}_A \subseteq \mathcal{R}_A$ be as provided by Claim 6 and let $\mathcal{R} = \bigcup \{\mathcal{R}_B : B \in \mathcal{L}_H \setminus \{A\}\}$ be the random family of all obstacle sets. Define the random variable Z to be the number of sets $A' \in \mathcal{Q}_A$ for which $|A' \cap C| \geq t$ for some $C \in \mathcal{R}$. Since \mathcal{S}_A is disjoint from $\mathcal{S} = \bigcup \{\mathcal{S}_B : B \in \mathcal{L}_H \setminus \{A\}\}$, we can view \mathcal{R} as a random subset of \mathcal{S} , with each element selected to \mathcal{R} independently with probability $p = n^{t-k+\varepsilon/2}$. Thus Z is determined by $|\mathcal{S}|$ independent trials. We wish to show that Z is not too large via Theorem 4; for this, Z has to satisfy the two conditions therein.

1. If $C \in \mathcal{S}$ satisfies $|A' \cap C| \geq t$ for some $A' \in \mathcal{Q}_A$ then $C \setminus A$ must intersect $A' \setminus A$ (since $A \not\subseteq C$). However, the $(k-t)$ -sets $\{A' \setminus A : A' \in \mathcal{Q}_A\}$ are pairwise disjoint (by the definition of \mathcal{Q}_A) so each C cannot rule out more than k different sets $A' \in \mathcal{Q}_A$. Thus Z is k -Lipschitz.

2. Assume that $Z \geq s$. Then, by definition, there exist distinct sets $A'_1, \dots, A'_s \in \mathcal{Q}_A$ and (not necessarily distinct) sets $C_1, \dots, C_s \in \mathcal{R}$ such that $|A'_i \cap C_i| \geq t$ for $i = 1, \dots, s$. These are at most s trials whose outcomes ensure that $Z \geq s$; that is, Z is certifiable.

Let us now calculate $\mathbb{E}[Z]$. Fix $A' \in \mathcal{Q}_A$ and let $Z_{A'}$ be the indicator random variable of the event $E_{A'} = \{\exists C \in \mathcal{R} : |A' \cap C| \geq t\}$. The only set in \mathcal{L}_H fully contained in A' is A , so we can write $\mathcal{L}_H \setminus \{A\}$ as the disjoint union $\bigcup_{\ell=0}^{t-1} \mathcal{B}'_\ell$, where $\mathcal{B}'_\ell = \{B \in \mathcal{L}_H : |A' \cap B| = \ell\}$. For any $0 \leq \ell < t$ and $B \in \mathcal{B}'_\ell$, the number of bad sets (i.e., sets that will trigger $E_{A'}$) in \mathcal{S}_B is

$$\begin{aligned} |\{C \in \mathcal{S}_B : |A' \cap C| \geq t\}| &\leq |\{C \in \mathcal{T}_B : |A' \cap C| \geq t\}| \\ &= |\{C \in \mathcal{T}_B : |(A' \cap C) \setminus B| \geq t - \ell\}| \\ &= \sum_{i=t-\ell}^{k-t} \binom{k-\ell}{i} \binom{n-k-t+\ell}{k-t-i} = O(n^{k-2t+\ell}), \end{aligned}$$

since C contains B , $|B| = t$, together with $i \geq t - \ell$ elements from $A' \setminus B$ and the rest from $X \setminus (A' \cup B)$. Each such bad set ends up in \mathcal{R}_B with probability $p = n^{\varepsilon/2+t-k}$, so the expected number of bad sets in \mathcal{R}_B is $O(n^{\varepsilon/2-t+\ell})$. By Property (♣) we have $|\mathcal{B}'_\ell| = O(n^{t-\ell-\varepsilon})$ and thus the total expected number of bad sets in \mathcal{R} is $\ell \cdot O(n^{\varepsilon/2-\varepsilon}) = O(n^{-\varepsilon/2})$. By Markov's inequality we have

$$\Pr[E_{A'}] = O(n^{-\varepsilon/2}).$$

Now, Z is a sum of $|\mathcal{Q}_A| = \Theta(n^{\varepsilon/3})$ random variables $Z_{A'}$ and thus

$$\mathbb{E}[Z] = |\mathcal{Q}_A| \cdot \mathbb{E}[Z_{A'}] = |\mathcal{Q}_A| \cdot \Pr[E_{A'}] = O(n^{-\varepsilon/6}) = o(1).$$

Applying Theorem 4 with $t = |\mathcal{Q}_A| = \Theta(n^{\varepsilon/3})$ and $c = k = O(1)$, we get that

$$\Pr[\text{all } A' \in \mathcal{R}_A \text{ are ruled out}] \leq \Pr[Z \geq |\mathcal{Q}_A|] < 2 \exp(-\Omega(n^{\varepsilon/3})).$$

Taking the union bound over all $|\mathcal{L}_H| \leq \binom{n}{t}$ choices of $A \in \mathcal{L}_H$ establishes the claim. \square

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