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Anagram-Free Colourings of Graphs

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A sequence S is called *anagram-free* if it contains no consecutive symbols $r_1 r_2 \dots r_k r_{k+1} \dots r_{2k}$ such that $r_{k+1} \dots r_{2k}$ is a permutation of the block $r_1 r_2 \dots r_k$. Answering a question of Erdős and Brown, Keränen constructed an infinite anagram-free sequence on four symbols. Motivated by the work of Alon, Grytczuk, Hałuszczak and Riordan [2], we consider a natural generalization of anagram-free sequences for graph colourings. A colouring of the vertices of a given graph G is called *anagram-free* if the sequence of colours on any path in G is anagram-free. We call the minimal number of colours needed for such a colouring the *anagram-chromatic* number of G.

In this paper we study the anagram-chromatic number of several classes of graphs like trees, minor-free graphs and bounded-degree graphs. Surprisingly, we show that there are bounded-degree graphs (such as random regular graphs) in which anagrams cannot be avoided unless we essentially give each vertex a separate colour.

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1. Introduction

The study of non-repetitive colourings was conceived by a famous result of Thue [20] from 1906. He showed that there exists an infinite sequence S on an alphabet of three symbols in which no two adjacent blocks (of any length) are the same. In other words, S contains no sequence of *consecutive* symbols $r_1 r_2 \dots r_{2n}$ with $r_i = r_{i+n}$ for all $i \leqslant n$. Note that it is not *a priori* obvious that the minimal size of the alphabet necessary for an infinite non-repetitive sequence is even finite. Thue's result is interesting in its own right, but it also has influential and surprising applications, the most famous one probably occurring in a solution to the Burnside problem for groups by Novikov and Adjan [18]. Thue-type problems led to the development of Combinatorics on Words, a new area of research with many interesting connections and applications.

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Generalizations of Thue's result have occurred in two directions. Firstly, the setting has been changed from sequences to, for example, the real line, the lattice $\mathbb{Z}^n$, or graphs. Secondly, repetitions as a forbidden structure can be replaced by anagrams, sums, patterns, *etc.* For a formal treatment and references to these problems, we refer the reader to the survey by Grytczuk [14]. Here we focus on graph colourings, and the structure we are avoiding is that of anagrams.

A sequence $r_1r_2\ldots r_nr_{n+1}\ldots r_{2n}$ is called an *anagram* if the second block, $r_{n+1}\ldots r_{2n}$, is a permutation of $r_1r_2\ldots r_n$. A long-standing open question of Erdős [10] and Brown [6] was whether there exists a sequence on $\{0, 1, 2, 3\}$ containing no anagrams. We call such sequences *anagram-free*. It is easy to check that no such sequence on three symbols exists. In 1968 Evdokimov [11] showed that the goal can be achieved with 25 symbols, which was the first finite upper bound. Later Pleasants [19] and Dekking [9] lowered this number to five. Finally, Keränen [16] constructed arbitrarily long anagram-free sequences on four symbols using Thue's idea: given a finite anagram-free sequence S on symbols $\{0,1,2,3\}$, we can replace each symbol by a longer word on the same alphabet in a way that yields a new, longer anagram-free sequence $\overline{S}$. This answered the question of Erdős and Brown, but at the same time opened new avenues for further studies; some of them can be found in [14].

Bean, Ehrenfeucht and McNulty [5] have studied the problem of non-repetitive colourings in a continuous setting. A colouring of the real line is called *square-free* if no two adjacent intervals of the same length are coloured in the same way. More precisely, for any intervals $I = [a,b]$ and $J = [b,c]$ of the same length $L > 0$, there exists a point $x \in I$ whose colour is different from $x+L$. In [5], they showed that there exist square-free two-colourings of the real line. The problem of avoiding anagrams also has a continuous variant. Alon, Grytczuk, Lasoń and Michałek [3] have proved that there exists a measurable 4-colouring of the real line such that no two adjacent segments contain equal measure of every colour.

Alon, Grytczuk, Hałuszczak and Riordan [2] proposed another variation on the non-repetitive theme. Let G be a graph. A vertex colouring $c: V(G) \to \mathcal{C}$ is called *non-repetitive* if any path in G induces a non-repetitive sequence. Define the *Thue number* $\pi(G)$ as the minimal number of colours in a non-repetitive colouring of G. It is easy to see that this number is a strengthening of the classical chromatic number, as well as the star-chromatic number. It turns out that the Thue number is bounded for several interesting classes of graphs, for example $\pi(P_n) \leqslant 3$ for a path P_n of length n (directly from Thue's theorem), and $\pi(T) \leqslant 4$ for any tree T. Using the Lovász Local Lemma, Alon, Grytczuk, Hałuszczak and Riordan [2] showed that $\pi(G) \leqslant c\Delta(G)^2$, where c is a constant and $\Delta(G)$ denotes the maximum degree of G. They also found a graph G with $\pi(G) \geqslant c'\Delta^2/\log\Delta$. Closing the above gap remains an intriguing open question. Another interesting problem is to decide if the Thue number of planar graphs is finite. A survey of Grytczuk [13] lays out some progress in this direction, as well as numerous related questions on non-repetitive graph colourings.

The investigation of anagram-free colourings of graphs, which we do here, was suggested in the concluding remarks of [2]. Let $c: V(G) \to \mathcal{C}$ be a vertex colouring of a graph G. Two vertex sets V_1 and V_2 have *the same colouring* if they have the same number of occurrences of each colour, that is, $|c^{-1}(a) \cap V_1| = |c^{-1}(a) \cap V_2|$ for each $a \in \mathcal{C}$. An *anagram* is a path $v_1v_2\ldots v_{2n}$ in G whose two segments $v_1\ldots v_n$ and $v_{n+1}\ldots v_{2n}$ have the same colouring. We denote the minimum number of colours in an anagram-free colouring of G by $\pi_\alpha(G)$, and call it the *anagram-chromatic number of* G. Clearly $\pi_\alpha(G) \leqslant n$ for any n-vertex graph G. The result of Keränen [16] states that $\pi_\alpha(P_n) \leqslant 4$ for a path P_n of length n, so it is only natural to ask what is π_α for other families of graphs. It turns out that as soon as we move on from paths, the situation becomes very different. We first show that the anagram-chromatic number of a binary tree already increases with the number of vertices.

Proposition 1.1. *Let* T_h *be a perfect binary tree of depth* h*, that is, every non-leaf has two children and there are* 2^h *leaves, all at distance* h *from the root. Then*

$$\sqrt{\frac{h}{\log_2 h}} \leqslant \pi_\alpha(T_h) \leqslant h+1.$$

It follows that the anagram-chromatic number of planar graphs is also unbounded, but it is still interesting to determine how quickly it increases with the number of vertices. We observe that in dealing with a family of graphs which admits small separators (such as H-minor-free graphs), this fact can be used to bound $\pi_\alpha(G)$ from above.

Proposition 1.2. *Let* $h \geqslant 1$ *be an integer, and let* H *be a graph on* h *vertices. Any* n*-vertex graph* G *with no* H*-minor satisfies* $\pi_\alpha(G) \leqslant 10h^{3/2}n^{1/2}$.

In this paper we are particularly interested in anagram-free colourings of graphs of bounded degree. We show that, surprisingly, there are graphs of bounded degree such that to avoid anagrams we essentially need to give every vertex a separate colour. We show this by considering the *random regular graph* $G_{n,d}$, which is chosen uniformly at random from all n-vertex d-regular graphs. Here we write $G_{n,d}$ for the sampled graph as well as the underlying probability space, and we study $G_{n,d}$ for a constant d and $n \to \infty$. We say that an event in this space holds *with high probability* (w.h.p.) if its probability tends to 1 as n tends to infinity over the values of n for which nd is even (so that $G_{n,d}$ is non-empty). Then our main result can be stated as follows.

Theorem 1.3. *There exists a constant* C *such that for sufficiently large* d*, with high probability, the random regular graph* $G_{n,d}$ *satisfies*

$$\left(1-\frac{C\log d}{d}\right)n \leqslant \pi_\alpha(G_{n,d}) \leqslant \left(1-\frac{\log d}{d}\right)n.$$

The rest of this paper is organized as follows. We start with some observations on the anagram-chromatic number for trees and minor-free graphs. Then, we give the proof of Theorem 1.3. We conclude the article with some open questions and conjectures on anagram-free colourings.

We mostly omit floor and ceiling signs for the sake of clarity. The log will denote the base-e logarithm. We will sometimes use standard O-notation for the asymptotic behaviour of the relative order of magnitude of two sequences, depending on a parameter $n \to \infty$.

2. Specific families of graphs

2.1. Bounds for trees

A *binary tree* is a tree in which every vertex has at most two children. Let T_h be a *perfect* binary tree of depth h, that is to say that every non-leaf has two children and there are 2^h leaves, all at distance h from the root. The root is taken to be at depth 0, so a tree consisting of one vertex has depth 0. Colouring each vertex of T_h by its distance from the root shows that $\pi_\alpha(T_h) \leqslant h+1$. In the following section, we will argue that actually any n-vertex tree can be anagram-free coloured with $2\log n$ colours. Proposition 1.1 asserts the lower bound $\pi_\alpha(T_h) \geqslant \sqrt{h/\log_2 h}$, which will be proved in this section.

Let T be a vertex-coloured binary tree and let U be a subtree of T. The *effective* vertices of U are its root (*i.e.* the vertex of U of the smallest depth), leaves, and vertices of degree three. The *effective depth of* U is set to h_1, where h_1+1 is the minimum number of effective vertices on any path from the root to a leaf (that is, the depth of the binary tree obtained by contracting all the internal degree-two vertices of U). Note that if U has effective depth h_1, then it has at least 2^{h_1} leaves. We say that U is *essentially monochromatic* if all its effective vertices carry the same colour.

We will use a Ramsey-type argument to find a large essentially monochromatic subtree of a given tree. A similar result was proved in [7, Lemma 3.4]. In the statement below $H(a_1,a_2,\dots,a_d)$ denotes the minimal number h for which any perfect binary tree T of depth h whose vertices are coloured using colours $1,2,\dots,d$ contains an essentially i-coloured subtree of effective depth a_i, for some $i \in [d]$.

Lemma 2.1. $H(a_1,a_2,\dots,a_d) \leqslant a_1+\cdots+a_d$.

Proof. We use induction on $\sum_{i=1}^{d} a_i$. The base case is $a_1=\cdots=a_d=0$, for which the claim clearly holds.

Let T be a perfect binary tree of depth $a_1+\cdots+a_d$. Suppose that its root v has the colour 1, and call its children v_L and v_R. Consider the subtrees T_L and T_R of depth at least $a_1+\cdots+a_d-1$ rooted at v_L and v_R respectively. If, for some $i \geqslant 2$, T_L contains an essentially i-coloured subtree of effective depth a_i, we are done. The same holds for T_R. Otherwise, using the induction hypothesis, T_L and T_R contain essentially 1-coloured subtrees of effective depth a_1-1. Those two subtrees, together with the root v, form an essentially 1-coloured subtree of T, as required. □

Proof of Proposition 1.1. Let T_h be coloured using $d < \sqrt{h/\log_2 h}$ colours. By Lemma 2.1, it contains an essentially monochromatic subtree U of depth h/d.

Let u be the root of U, and suppose U is essentially red. There are at least $2^{h/d}$ paths from u to the leaves, and the colouring of each path is a multiset of order at most $h+1$. On the other hand, there are at most h^d such multisets. Since $h^d < 2^{h/d}$ for our choice of d, there is a multiset which occurs on two different paths, say P_1 and P_2. Let v be the lowest common vertex of P_1 and P_2, and let ℓ_1 and ℓ_2 be their respective leaves. By construction of U, the vertices v, ℓ_1 and ℓ_2 are red. Hence the segments from ℓ_1 to v, excluding v, and from v to ℓ_2, excluding ℓ_2, have the same colouring.

We conclude that the given colouring of T_h, even restricted to U, contains an anagram. □

2.2. Graphs with an excluded minor

Planar graphs are of special interest when it comes to colouring problems. The Four Colour Theorem is one of the most celebrated results in graph theory. Moreover, the question of whether the Thue-chromatic number of planar graphs is finite has attracted a lot of attention and is still open. We use separator sets to show that for a large class of minor-free graphs the anagram-chromatic number is of order $O(\sqrt{n})$. The crucial ingredient of our argument is the separator theorem, proved by Alon, Seymour and Thomas [4]. It states that for a given h-vertex graph H, in any graph G with n vertices and no H-minor, one can find a set $S \subset V(G)$ of order $|S| \leqslant h^{3/2}n^{1/2}$, whose removal partitions G into disjoint subgraphs each of which has at most $2n/3$ vertices. Such a set S is called a *separator* in G.

Using this theorem, we construct a colouring of any proper minor-closed family of graphs. For convenience of the reader, we restate Proposition 1.2.

Proposition 1.2. *Let $h \geqslant 1$ be an integer, and let H be a graph on h vertices. Any n-vertex graph G with no H-minor satisfies $\pi_\alpha(G) \leqslant 10h^{3/2}n^{1/2}$.*

Proof. The colouring is inductive: suppose the claim holds for graphs on at most $n-1$ vertices. Let G be as in the statement, and let S be a separating set of vertices in G of order at most $h^{3/2}n^{1/2}$ given by the Separator Theorem. Then $G-S$ consists of two vertex-disjoint subgraphs spanned by $A_1 \subset V(G)$ and $A_2 \subset V(G)$, with $|A_i| \leqslant 2n/3$.

The induced subgraphs $G[A_i]$ do not contain H as a minor, so by the inductive hypothesis, we can colour them using $k = 10h^{3/2}\sqrt{2n/3}$ colours $a_1, a_2, \dots, a_k$. Note that the two subgraphs receive colours from the same set. This colouring guarantees that any path containing only vertices from A_1 or A_2 is anagram-free. Furthermore, we assign to each vertex $v_i \in S$ a separate colour b_i, making any path passing through S anagram-free. Hence the colouring is indeed anagram-free. As intended, the number of colours used is at most

$$h^{3/2}n^{1/2}\left(10\sqrt{\frac{2}{3}}+1\right) \leqslant 10h^{3/2}n^{1/2}. \qquad \square$$

Since planar graphs are characterized as graphs containing neither K_5 nor $K_{3,3}$ as a minor, we arrive at the following consequence of the above result (note that the constant 150 can be replaced by 19 if we use the fact that each planar graph has a separator of order $1.84\sqrt{n}$).

Corollary 2.3. *Let G be an n-vertex planar graph. Then $\pi_\alpha(G) \leqslant 150\sqrt{n}$.* $\square$

In fact, any hereditary family of graphs with small separators can be coloured using the argument from Proposition 1.2. For example, it is easy to see that an n-vertex forest F contains a single vertex which separates it into several forests on at most $n/2$ vertices. The same inductive argument implies $\pi_\alpha(F) \leqslant \lceil \log_2 n \rceil$.

As for the lower bound for planar graphs, we only have the following modification of the argument we gave for trees.

Proposition 2.4. *There is an n-vertex planar graph F_n with $\pi_\alpha(F_n) \geqslant \lceil \frac{1}{4}\log_2 n \rceil$.*

Proof. Let F_n be a perfect binary tree with n leaves, plus extra edges between any two vertices on the same level having the same parent. Suppose it is coloured in $k = \lceil \frac{1}{4}\log_2 n \rceil$ colours. The number of shortest paths from the root to the vertices corresponding to leaves is n, whereas the number of possible colourings of these paths is $\binom{\log_2 n + k}{k-1} < n$; hence some two paths have the same colouring. These two paths, minus the shared initial segment, can be made into an anagram. □

3. Bounded-degree graphs

3.1. A four-regular graph with a large anagram-chromatic number

In this section we study the number of colours needed to colour a bounded-degree graph on n vertices so as to avoid all anagrams. The trivial upper bound is n, so we will mainly be interested in lower bounds for the anagram-chromatic number. Keränen's result implies that graphs of maximum degree two satisfy $\pi_\alpha(G) \leqslant 5$. It turns out that there are already 4-regular graphs G for which $\pi_\alpha(G)$ grows rather quickly with the size of the graph.

Proposition 3.1. *For infinitely many values of n, there exists a* 4*-regular n-vertex graph H with* $\pi_\alpha(H) \geqslant \sqrt{n}/\log_2 n$.

Proof. Note that for each even $k \geqslant 4$, there exists a 3-regular k-vertex graph G that is Hamilton-connected, which means that any two vertices of G are joined by a Hamilton path. Indeed, it can be easily checked that for any $m \geqslant 1$, the Cayley graph of $C_2 \times C_{2m+1}$ with canonical generators is Hamilton-connected. For a self-contained proof, we refer the reader to [8]. Let $n = (k+1)k$. Take $k+1$ copies of such G on vertex sets $V_1, V_2, \ldots, V_{k+1}$ with $|V_i| = k$. Furthermore, take a perfect matching M on $V_1 \cup \cdots \cup V_{k+1}$ such that there exists exactly one edge between any two V_i and V_j, for $i \neq j$. To see that such a matching exists, denote $V_i = \{v_{ij} : j \in [k+1] \setminus \{i\}\}$, and take $M = \{\{v_{ij}, v_{ji}\} : 1 \leqslant i < j \leqslant k+1\}$.

Call the resulting graph H. H is 4-regular - any vertex has three adjacent edges belonging to its copy of G and one edge belonging to M. Suppose that the vertices of H are coloured with $\lfloor \sqrt{n}/\log_2 n \rfloor$ colours. Consider the subsets of form $\bigcup_{i\in S} V_i$ for any $S \subset [k+1]$. There are 2^{k+1} such subsets. The colouring of each $\bigcup_{i\in S} V_i$ defines a multiset of order at most n. Given $\lfloor \sqrt{n}/\log_2 n \rfloor$ colours, the number of such multisets is at most $n^{\sqrt{n}/\log_2 n} = 2^{\sqrt{n}} < 2^{k+1}$. Thus, by the pigeonhole principle, there are two distinct sets $S, T \subset [k+1]$ such that $\bigcup_{i\in S} V_i$ and $\bigcup_{i\in T} V_i$ have the same number of occurrences of each colour. The same holds for sets $S' = S \setminus T$ and $T' = T \setminus S$, which are in addition disjoint. Without loss of generality assume $S' = \{V_1, \ldots, V_s\}$ and $T' = \{V_{s+1}, \ldots, V_{2s}\}$. By the choice of M, we can find vertices $v_1, u_1, v_2, u_2, \ldots, v_{2s}, u_{2s}$ such that $v_i, u_i \in V_i$ for $i \in [2s]$, and $u_i v_{i+1}$ are edges in M for $i \in [2s-1]$. Moreover, we can find a Hamilton path in each $H[V_i]$ between u_i and v_i, using Hamilton-connectedness of G. Concatenating these $2s$ paths gives us a path in H which traverses $V_1 \cup V_2 \cdots \cup V_{2s}$ in order. This path forms an anagram in H, so $\pi_\alpha(H) > \lfloor \sqrt{n}/\log_2 n \rfloor$. □

3.2. Random regular graphs

Let us start with a simple observation which slightly improves the trivial upper bound n for the anagram-chromatic number of a graph.

Proposition 3.2. *Let G be an n-vertex graph with an independent set of order m. Then $\pi_\alpha(G) \leqslant n-m+1$.*

Proof. Let S be an independent set inside G of order m. Give each vertex of S the same colour, and each vertex of $V(G)\setminus S$ its own colour. Any path in G contains at least one vertex of $V(G)\setminus S$, so it cannot contain an anagram. This means that our colouring is indeed anagram-free. □

The above bound is essentially optimal for the random regular graph $G_{n,d}$. To recapitulate, Theorem 1.3 states that for sufficiently large d, with high probability, $G_{n,d}$ satisfies

$$\left(1-\frac{2\cdot 10^5 \log d}{d}\right)n \leqslant \pi_\alpha(G_{n,d}) \leqslant \left(1-\frac{\log d}{d}\right)n.$$

The upper bound is an immediate consequence of Proposition 3.2, and the fact that with high probability, $G_{n,d}$ contains an independent set of order asymptotic to $(2\log d/d)n > (\log d/d)n$ (see, for instance, Frieze and Łuczak [12]). We will now outline the proof of the lower bound on $\pi_\alpha(G_{n,d})$, which comprises the remainder of the section. Instead of studying the random d-regular graph $G_{n,d}$, we will consider the union of two random graphs G_{n,d_1} and G_{n,d_2} with $d = d_1 + d_2$. The asymptotic properties of $G_{n,d}$ are contiguous with such a model (see Lemma 9.24 in [15]). Let $G_1 = G_{n,d_1}$, $G_2 = G_{n,d_2}$ and c be a given vertex-colouring of $G = G_1 \cup G_2$. The first step is to find two vertex subsets V_1 and V_2 with the same colouring such that $G_1[V_1]$ and $G_1[V_2]$ have good expansion properties. Then we use the edges of G_2 to extend paths on V_1 and V_2, eventually building Hamilton cycles C_1 in $G[V_1]$ and C_2 in $G[V_2]$. Finally, we can find an edge $v_1v_2 \in G$ with $v_i \in V_i$ and use it to build a single path S which traverses first the vertices of C_1 and then the vertices of C_2. The segments $S[V_1]$ and $S[V_2]$ give an anagram in c.

Before proceeding, let us introduce some notation. For a graph G and $v \in V(G)$, we denote the neighbourhood of v in G by $N_G(v)$. For a vertex set $U \subset V(G)$,

$$N_G(U) = \bigcup_{v\in U} N_G(v) \setminus U.$$

The graph induced on U is $G[U]$, and its edge set is denoted by $E_G(U) = E(G[U])$. For disjoint sets U and T, $E_G(U,T)$ is the set of edges with one endpoint in U and one in T. Finally, the corresponding counts are $e_G(U) = |E_G(U)|$ and $e_G(U,V) = |E_G(U,V)|$. We denote the uniform probability measure on the space of random regular graphs $G_{n,d}$ by $\mathbb{P}$, suppressing the indices. All the inequalities below are assumed to hold only for n large enough.

Edge distribution in the configuration model In analysing $G_{n,d}$, we pass to the *configuration* model of random regular graphs. For nd even, we take a set of nd points partitioned into n cells $v_1, v_2, \ldots, v_n$, each cell containing d points. A perfect matching P on $[nd]$ induces a multigraph $\mathcal{M}(P)$ in which the cells are regarded as vertices and pairs in P as edges. For a fixed degree d and P chosen uniformly from the set of perfect matchings $\mathcal{P}_{n,d}$, the probability that $\mathcal{M}(P)$ is a simple graph is bounded away from zero, and each simple graph occurs with equal probability. Therefore, if an event holds w.h.p. in $\mathcal{M}(P)$, then it holds w.h.p. even when we condition on the event that $\mathcal{M}(P)$ is a simple graph, and therefore it holds w.h.p. in $G_{n,d}$ (for a formal description of the configuration model and its basic properties, see, for instance, Chapter 9 of [15]).

We use the configuration model to get a bound on the edge distribution in $G_{n,d}$ analogous to the Erdős–Rényi model. The uniform probability measure on $\mathcal{P}_{n,d}$ is denoted by $\mathbb{P}_{\mathcal{P}}$. Both indices n and d are kept so that each perfect matching P corresponds to a unique d-regular multigraph $\mathcal{M}(P)$.

Lemma 3.3. *Let $V_1 \subset [n]$, and let B be a set of pairs of vertices from V_1. Let E be another set of pairs of vertices from $[n]$ with $|E| \leqslant \min\{|B|d/(4|V_1|), nd/20\}$. For a fixed positive integer d and $P \in \mathcal{P}_{n,d}$ chosen uniformly at random,*

$$\mathbb{P}_{\mathcal{P}}[\mathcal{M}(P) \supset E \text{ and } \mathcal{M}(P) \cap B = \emptyset] \leqslant \left(\frac{2d}{n}\right)^{|E|} e^{-(2|B|d)/(5n)}.$$

The lemma also holds for more general configurations of B and E, but we state it in the form which is suited to our purpose. The proof is given after Lemma 3.5, which is a bound on the probability that $G_{n,d}$ does not intersect a given set of edges. A crucial ingredient is the following estimate of Alon and Friedland [1], which is a simple corollary of the Brègman bound on the permanent of a $(0,1)$-matrix.

Theorem 3.4 ([1]). *Let H be a graph on $[N]$. Let $r_1, r_2, \ldots, r_N$ be the degrees of the vertices in H. Furthermore, denote $r = \frac{1}{N}\sum_{i=1}^{N} r_i$. Then the number of perfect matchings in H is at most*

$$\prod_{i=1}^{N} (r_i!)^{1/(2r_i)} \leqslant (r!)^{N/(2r)}. \qquad \square$$

Lemma 3.5. *For each even number N, let $F = F(N)$ be a graph on $[N]$ consisting of at least βN^2 edges. Let $G_{N,1}$ denote a random matching on $[N]$. Then*

$$\mathbb{P}[G_{N,1} \cap F = \emptyset] \leqslant e^{-(8\beta N/9)}.$$

Proof. Let $\mathcal{P}(F)$ be the set of perfect matchings on $[N]$ which do not intersect our graph F. Since the number of perfect matchings on $[N]$ is exactly $N!/(2^{N/2}(N/2)!)$, we need to show that

$$|\mathcal{P}(F)| \leqslant e^{-(8\beta N/9)} \frac{N!}{2^{N/2}(N/2)!}.$$

Consider the complement $\overline{F}$ of F. The matchings in $\mathcal{P}(F)$ are exactly perfect matchings in $\overline{F}$. We apply Theorem 3.4 directly to the graph $\overline{F}$ with $r = N - 2\beta N$, and use Stirling's formula to reach the final result. Indeed, we have

$$|\mathcal{P}(F)| \leqslant ((N-2\beta N)!)^{1/(2(1-2\beta))}$$
$$\mathbb{P}[G_{N,1} \cap F = \phi] \leqslant ((N-2\beta N)!)^{1/(2(1-2\beta))} \cdot \frac{(N/2)! \cdot 2^{N/2}}{N!}$$
$$= O(\sqrt{N})\left(\frac{(1-2\beta)N}{e}\right)^{N/2}\left(\frac{e}{N}\right)^{N/2} = O(\sqrt{N})e^{-\beta N}.$$

Here we use the fact that

$$\frac{N!}{(N/2)!\cdot 2^{N/2}} = \Theta(1)\left(\frac{N}{e}\right)^{N/2},$$

as well as the inequality $1-2\beta \leqslant e^{-2\beta}$. Absorbing the error term into the constant, we get, for N large enough,

$$\mathbb{P}[G_{N,1}\cap F = \emptyset] \leqslant e^{-(8\beta N/9)}. \qquad \square$$

Proof of Lemma 3.3. We will restate the event $\{\mathcal{M}(P)\supset E \text{ and } \mathcal{M}(P)\cap B=\emptyset\}$ in terms of P, rather than $\mathcal{M}(P)$. For a matching $M\subset\binom{[nd]}{2}$, we denote the induced multigraph on $V=\{v_i\}_{i\in[n]}$ by $\mathcal{M}(M)$. To save on notation, we write $\mathcal{M}(M)$ for both the graph and its edge set. Conversely, if $e=\{v_i,v_j\}$ is a pair of vertices from V, we denote its corresponding pairs in $\binom{[nd]}{2}$ by $\tilde{e}=\{\{x,y\}: x\in v_i, y\in v_j\}$. Finally, for a set $E\subset\binom{V}{2}$, we put $\tilde{E}=\bigcup_{e\in E}\tilde{e}$.

Assume that $\mathcal{M}(P)\supset E$. Then we can find a matching $M\subset P$ such that $|M|=|E|=m$ and $\mathcal{M}(M)=E$. Conditioning over the possible choices of M, we have

$$\mathbb{P}_{\mathcal{P}}[\mathcal{M}(P)\supset E \wedge \mathcal{M}(P)\cap B=\emptyset] \leqslant \sum_M \mathbb{P}_{\mathcal{P}}[P\supset M]\mathbb{P}_{\mathcal{P}}[P\cap\tilde{B}=\emptyset \mid P\supset M].$$

We bound the two probabilities separately. Fix a choice $M=\{\{x_i,y_i\}: i\in[m]\}$, and let $W=[nd]\setminus\{x_1,\ldots,x_m,y_1,\ldots,y_m\}$.

Claim 1. $\mathbb{P}_{\mathcal{P}}[P\supset M]\leqslant 2/((nd-2m)^m)$

To show this, we just count perfect matchings. The total number of perfect matchings P is

$$\frac{(nd)!}{(nd/2)!2^{nd/2}}.$$

The points from W can be paired in

$$\frac{(nd-2m)!}{((nd/2)-m)!2^{(nd/2)-m}}$$

ways. Altogether, using Stirling's formula, we get

$$\begin{aligned}
\mathbb{P}_{\mathcal{P}}[M\subset P] &\leqslant \frac{(nd-2m)!(nd/2)!}{(nd)!((nd/2)-m)!2^{-m}}\\
&= (1+o(1))\left(\frac{nd-2m}{nd}\right)^{nd}\left(\frac{nd-2m}{e}\right)^{-2m}\left(\frac{nd}{nd-2m}\right)^{nd/2}\left(\frac{nd-2m}{e}\right)^{m}\\
&= (1+o(1))\left(1-\frac{2m}{nd}\right)^{nd/2}\left(\frac{e}{nd-2m}\right)^{m} \leqslant \frac{2}{(nd-2m)^m}.
\end{aligned}$$

Here we used the fact that since $1-x\leqslant e^{-x}$, we have

$$\left(1-\frac{2m}{nd}\right)^{nd/2}\leqslant e^{-m}.$$

Claim 2. *For* $|B| = \beta n^2$, $\mathbb{P}_{\mathcal{P}}[P \cap \tilde{B} = \emptyset \mid P \supset M] \leqslant e^{-(2\beta/5)nd}$.

Let W be as before, and denote $N = |W|$. Using the assumption $m \leqslant nd/20$, we get $N = nd - 2m \geqslant 9nd/10$. Let $B_W = \tilde{B}[W]$, that is, the set of pairs contained in W which would induce B. By putting the matching M aside, we have lost some pairs from $\tilde{B}$, namely those touching the vertices of M. Each vertex of M is contained in at most $|V_1|d$ pairs from $\tilde{B}$, so the hypothesis $m \leqslant (1/(4|V_1|))|B|d$ implies

$$|B_W| \geqslant |B|d^2 - 2m|V_1|d = \beta n^2 d^2 \left(1 - \frac{2m|V_1|}{\beta n^2 d}\right) \geqslant \frac{1}{2}\beta n^2 d^2 \geqslant \frac{1}{2}\beta N^2.$$

A random matching P conditioned on $P \supset M$ corresponds to a random matching on W, that is, an element of $G_{N,1}$. Hence we can apply Lemma 3.5 with $|B_W| \geqslant \frac{1}{2}\beta N^2$, and $N = |W| \geqslant 9nd/10$.

$$\mathbb{P}_{\mathcal{P}}[P \cap B_W = \emptyset \mid P \supset M] \leqslant \mathbb{P}[E(G_{N,1}) \cap B_W = \emptyset] \leqslant e^{-(8/9)\cdot(1/2)\beta N} \leqslant e^{-(2\beta/5)nd}.$$

Claim 1 and 2 hold for any choice of the matching M with $\mathcal{M}(M) = E$. Putting them together, and using the fact that there are at most d^{2m} such matchings M, we get

$$\mathbb{P}_{\mathcal{P}}[\mathcal{M}(P) \supset E \text{ and } \mathcal{M}(P) \cap B = \emptyset] \leqslant d^{2m} \cdot \frac{2}{(nd-2m)^m} \cdot e^{-(2\beta/5)nd}.$$

Using $m \leqslant nd/20$,

$$\mathbb{P}_{\mathcal{P}}[\mathcal{M}(P) \supset E \text{ and } \mathcal{M}(P) \cap B = \emptyset] \leqslant d^{2m} \cdot \left(\frac{2}{nd}\right)^m e^{-(2\beta/5)nd} = \left(\frac{2d}{n}\right)^m e^{-(2\beta/5)nd}. \qquad \square$$

Expansion properties and Pósa rotations Recall that we will be working with the union of random graphs G_{n,d_1} and G_{n,d_2}. First we focus on the expansion properties of $G_1 = G_{n,d_1}$, which will allow us to do rotations in G_1. The aim is to identify large sets of vertex pairs, called boosters, which could increase the length of the longest path in G_1. Hence all the lemmas in this section will later be applied with d replaced by d_1. The following lemma says that edges in G_{n,d_1} are uniformly distributed.

Lemma 3.6. *For sufficiently large d, with high probability $G_{n,d}$ has the following two properties:*

(P1) *any vertex set U with*

$$|U| \leqslant \frac{30 \log d}{d} n$$

satisfies $e_G(U) \leqslant 100|U| \log d$,

(P2) *for any two disjoint vertex subsets T and U with*

$$|T| \geqslant \frac{10 \log d}{d} n \quad \text{and} \quad |U| \geqslant \frac{100 \log d}{d} n,$$

we have

$$e_G(T, U) \geqslant |T||U| \frac{d}{20n}. \tag{3.1}$$

Proof. We prove Lemma 3.6 for G sampled according to the configuration model, that is, we take $G = \mathcal{M}(P)$, where P is a random element of $\mathcal{P}_{n,d}$. We start with (P2). Take vertex sets T and U in G with $|T| = t$ and $|U| = u$. We need to bound the probability of the event

$$D_{T,U} = \left\{ e_G(T,U) < \frac{d}{20n} tu \right\}.$$

For a fixed set of m edges E with $m \leqslant (d/(20n))tu$, the probability that $E_G(T,U) = E$ is at most

$$\left(\frac{2d}{n}\right)^m e^{-2d(tu-m)/(5n)}.$$

This bound is a direct application of Lemma 3.3 to the edge set E and its bipartite complement $(T \times U) \setminus E$. Taking the union bound over all sets E, we get

$$\mathbb{P}_{\mathcal{P}}[D_{T,U}] \leqslant \sum_{m=0}^{dtu/(20n)} \binom{tu}{m} \left(\frac{2d}{n}\right)^m e^{-2d(tu-m)/(5n)} \leqslant \sum_{m=0}^{dtu/(20n)} \left(\frac{etu}{m} \cdot \frac{2d}{n}\right)^m e^{-2d(tu-m)/(5n)}.$$

The summand is increasing in m, so we bound it using the largest term, $m = M = dtu/(20n)$.

$$\mathbb{P}_{\mathcal{P}}[D_{T,U}] \leqslant M \left(\frac{etu \cdot 20n}{dtu} \cdot \frac{2d}{n}\right)^{dtu/(20n)} e^{-2d(tu-M)/(5n)} = M \cdot (40e)^{dtu/(20n)} e^{-2d(tu-M)/(5n)}$$
$$= Me^{(dtu/n)((5/20)-(2/5)+(d/n))} \leqslant e^{-dtu/(8n)}.$$

Finally, we take the union bound over all sets T and U of order at least $t_0 = (10 \log d/d)n$ and $u_0 = (100 \log d/d)n$ respectively.

$$\mathbb{P}_{\mathcal{P}}[G \text{ violates (P2)}] \leqslant \sum_{t=t_0}^{n} \sum_{u=u_0}^{n} \binom{n}{t} \binom{n}{u} e^{-dtu/(8n)}.$$

We use the bound $\binom{n}{t} \leqslant d^t = e^{t \log d}$ valid for $t \geqslant t_0$ and large enough d:

$$\mathbb{P}_{\mathcal{P}}[G \text{ violates (P2)}] \leqslant \sum_{t=t_0}^{n} \sum_{u=u_0}^{n} e^{t \log d + u \log d} e^{-dtu/(8n)}.$$

For $t \geqslant (10 \log d/d)n$, we get $u \log d \leqslant dtu/(10n)$. Similarly, since $u \geqslant (100 \log d/d)n$, it holds that $t \log d \leqslant dtu/(100n)$.

$$\mathbb{P}_{\mathcal{P}}[G \text{ violates (P2)}] \leqslant O(n^2) \exp\left(\left(\frac{1}{100} + \frac{1}{10} - \frac{1}{8}\right) \frac{n \log^2 d}{d}\right) = o(1).$$

We deduce (P1) from the following more general statement.

Claim 3. *Fix the constants A_1 and A_2 satisfying $(eA_1/A_2)^{A_2} \leqslant e^{-2}$. Then with high probability any vertex set $U \subset V(G)$ with $|U| \leqslant (A_1 \log d/d)n$ satisfies $e_G(U) \leqslant A_2 |U| \log d$.*

Introducing $A_1 = 30$ and $A_2 = 100$, which satisfy $(30e/100)^{100} \leqslant e^{-2}$, gives exactly (P1).

To prove the claim, we fix a set U of order u, and use Lemma 3.3 to establish that the probability of some $A_2 u \log d$ edges occurring in U is at most

$$\binom{u^2/2}{A_2 u \log d} \left(\frac{2d}{n}\right)^{A_2 u \log d} \leqslant \left(\frac{eu}{2A_2 \log d} \cdot \frac{2d}{n}\right)^{A_2 u \log d}.$$

Let D_u denote the event that some subset U with $|U| = u$ spans more than $A_2 u \log d$ edges. We have

$$\mathbb{P}_{\mathcal{P}}[D_u] \leqslant \binom{n}{u} \left(\frac{eud}{A_2 n \log d} \right)^{A_2 u \log d} \leqslant \left[\frac{ne}{u} \left(\frac{eud}{A_2 n \log d} \right)^{A_2 \log d} \right]^u. \tag{3.2}$$

The term in square brackets is increasing in u, so for $u \leqslant (A_1 \log d/d)n$,

$$\mathbb{P}_{\mathcal{P}}[D_u] \leqslant \left[\frac{ed}{A_1 \log d} \left(\frac{eA_1}{A_2} \right)^{A_2 \log d} \right]^u \leqslant \left[\frac{ed}{A_1 \log d} e^{-2 \log d} \right]^u < d^{-u}.$$

Here we used the condition $(eA_1/A_2)^{A_2} \leqslant e^{-2}$. For $u \leqslant \sqrt{n}$ we use (3.2) to get a stronger bound

$$\mathbb{P}_{\mathcal{P}}[D_u] \leqslant (O(n^{\frac{1}{2}(1-A_2 \log d)}))^u < n^{-1},$$

valid for large d. Putting the two bounds together,

$$\mathbb{P}_{\mathcal{P}}[G \text{ violates (P1)}] \leqslant \sum_{u=1}^{\sqrt{n}} \mathbb{P}_{\mathcal{P}}[D_u] + \sum_{u=\sqrt{n}}^{(A_1 \log d/d)n} \mathbb{P}_{\mathcal{P}}[D_u] \leqslant \sqrt{n} \cdot n^{-1} + \sum_{u=\sqrt{n}}^{(A_1 \log d/d)n} d^{-u} = o(1),$$

completing the proof of Claim 3. To recapitulate, applying the claim for $A_1 = 30$ and $A_2 = 100$ gives that $G = \mathcal{M}(P)$ satisfies (P1) with high probability.

Since the random graph $G_{n,d}$ is contiguous to G, we conclude that for large enough d, $G_{n,d}$ satisfies (P1) and (P2). □

The next step is to build subsets of $[n]$ which will later give us the required anagram. In everything that follows, take $\alpha = 10^5$. Given a d-regular graph G, we say that a subset $V_1 \subset [n]$ is *G-dense* if

$$\frac{\alpha \log d}{2d} n \geqslant |V_1| \geqslant \frac{\alpha \log d}{4d} n,$$

and any vertex $v \in V_1$ has at least $\alpha \log d/160$ neighbours in V_1.

Lemma 3.7. *Suppose we are given a d-regular graph G on $[n]$ with properties* (P1) *and* (P2), *and a vertex colouring $c : [n] \to \mathcal{C}$ with*

$$|\mathcal{C}| = \left(1 - \frac{\alpha \log d}{d}\right) n$$

colours. For sufficiently large d and n, there exist two disjoint G-dense sets of vertices $V_1, V_2 \subset [n]$ which have the same colouring.

Proof. Let c be a colouring of the vertices of G into $(1 - (\alpha \log d/d))n$ colours.

Claim 4. *There exists a subset $Z \subset V(G)$ satisfying*

$$\frac{\alpha \log d}{d} n \geqslant |Z| \geqslant \frac{\alpha \log d}{2d} n$$

such that each colour appears in Z an even number of times, and for all $v \in Z$, $|N_G(v) \cap Z| \geqslant (\alpha/40) \log d$.

We construct Z using the following algorithm. Let $V(G)=[n]$, and denote $\delta=\alpha\log d/40$. Let X contain one vertex from each colour class with an odd number of colours, so

$$|X|\leqslant\left(1-\frac{\alpha\log d}{d}\right)n.$$

We assume

$$|X|=\left(1-\frac{\alpha\log d}{d}\right)n,$$

by discarding more pairs of vertices of the same colour if necessary. Furthermore, set $R_0=\hat{R}_0=\emptyset$. Note that from this step onwards, all colour classes in $[n]\setminus(X\cup R_i\cup\hat{R}_i)$ will have even order. For $i\geqslant 0$, we form $R_{i+1}:=R_i\cup\{v\}$, $\hat{R}_{i+1}=\hat{R}_i\cup\{w\}$, where v is the smallest vertex with fewer than δ neighbours in $[n]\setminus(X\cup R_i\cup\hat{R}_i)$, and w is the smallest vertex with $c(v)=c(w)$. When there are no such vertices v, set $Z=[n]\setminus(X\cup R_i\cup\hat{R}_i)$ and terminate the algorithm. We claim that this occurs after at most $(10\log d/d)n$ steps.

Suppose that it is not the case and G satisfies (P1) and (P2), but the algorithm continues beyond $t=(10\log d/d)n$ steps. Look at the sets R_t and $Z_t=[n]\setminus(X\cup R_t\cup\hat{R}_t)$. Each vertex from R_t has fewer than δ neighbours in Z_t, so

$$e_G(R_t,Z_t)<\delta|R_t|=\frac{\alpha}{40}|R_t|\log d.$$

On the other hand, since

$$|R_t|=t=\frac{10\log d}{d}n\quad\text{and}\quad|Z_t|=\frac{\alpha\log d}{d}n-2t\geqslant\frac{\alpha\log d}{2d}n,$$

property (P2) gives

$$e_G(R_t,Z_t)\geqslant|R_t||Z_t|\frac{d}{20n}\geqslant|R_t|\cdot\frac{\alpha\log d}{2d}\cdot\frac{d}{20n}=\frac{\alpha}{40}|R_t|\log d.$$

We have reached a contradiction, so indeed we have the desired set Z with $|Z|\geqslant(\alpha\log d/2d)n$.

We show the existence of the required partition of Z into sets V_1 and V_2 using a probabilistic argument. Partition each colour class $c^{-1}(a)$ into ordered pairs arbitrarily, and denote the collection of pairs by Q. For each pair $(v,w)\in Q$, randomly and independently put v into V_1 and w into V_2 or vice versa, with probability $1/2$. This guarantees that V_1 and V_2 have the same colouring.

Claim 5. *With positive probability, the partition satisfies* $\deg_{G[V_i]}(v)\geqslant\alpha\log d/160$ *for all* $v\in V_i$ *and* $i\in\{1,2\}$.

We use the Local Lemma. Fix a vertex v; without loss of generality $v\in V_1$. Let B_v be the event that fewer than $(\alpha/160)\log d$ neighbours of v in Z have ended up in V_1. Let S be a set of $(\alpha/40)\log d$ neighbours of v in Z, and let $T\subset S$ be the set of vertices whose match according to Q does not lie in S. Note that S contains exactly $y=\frac{1}{2}(|S|-|T|)$ pairs of Q. If B_v occurs, then $|S\cap V_1|\leqslant(\alpha/160)\log d$, and therefore

$$\frac{1}{2}|S|-|S\cap V_1|\geqslant\frac{\alpha}{160}\log d.$$

But this implies

$$\frac{1}{2}|T| - |T \cap V_1| = \frac{1}{2}(|S| - 2y) - (|S \cap V_1| - y) \geqslant \frac{\alpha}{160}\log d.$$

Here $|T \cap V_1|$ is a random variable with distribution $B(|T|, 1/2)$, so Chernoff bounds (as stated in [15, Remark 2.5]) give

$$\mathbb{P}[B_v] = \mathbb{P}\left[\frac{1}{2}|T| - |T \cap V_1| \geqslant \frac{\alpha}{160}\log d\right] \leqslant e^{-(2/|T|)((\alpha/160)\log d)^2} \leqslant e^{-3\log d} = d^{-3}.$$

We used $|T| \leqslant \alpha \log d/40$ and $\alpha = 10^5$.

Two events B_v and B_w are dependent only if v and w share a neighbour, or if some two neighbours of v and w are paired. In such a dependency graph, the event B_v has degree at most $2d^2$. Since, for sufficiently large d,

$$e(2d^2 + 1)d^{-3} < 1,$$

the Lovász Local Lemma grants that there is a splitting avoiding all the bad events B_v. This is exactly the required splitting. It concludes the proof of Claim 5 and Lemma 3.7. □

We say a graph G is a p-expander if it is connected, and $|N_G(U)| \geqslant 2|U|$ for $|U| \leqslant p$.

Lemma 3.8. *Let G be a d-regular graph on vertex set $[n]$ satisfying* (P1) *and* (P2)*, and let $V_1 \subset [n]$ be a G-dense subset of $[n]$. Then $G[V_1]$ is a $(|V_1|/4)$-expander.*

Proof. Denote $H = G[V_1]$. To show expansion, suppose for the sake of contradiction that $|N_H(U)| < 2|U|$, and first assume that $|U| \leqslant (10\log d/d)n$. We can apply (P1) to $T = U \cup N_H(U) \subset V_1$, using the assumption $|T| \leqslant (30\log d/d)n$. This gives $e(G[T]) \leqslant 100|T|\log d$. Counting all the edges with an endpoint in U, which certainly lie inside T, we get

$$\frac{1}{2}\cdot\frac{\alpha}{160}|U|\log d \leqslant e(G[T]).$$

The two inequalities imply

$$|T| \geqslant \frac{1}{100}\cdot\frac{\alpha}{320}|U| = \frac{10^5}{32000} > 3|U|,$$

which contradicts our assumption.

Secondly, for

$$\frac{|V_1|}{4} \geqslant |U| \geqslant \frac{10\log d}{d}n \quad \text{and} \quad |N_G(U)| < 2|U|,$$

we have

$$|V_1 \setminus (U \cup N_H(U))| \geqslant \frac{|V_1|}{4} \geqslant \frac{10^4\log d}{d}n.$$

This puts us in a position to apply (P2) and claim that G contains edges between U and $V_1 \setminus (U \cup N_H(U))$, contradicting the definition of $N_H(U)$. Hence sets of order up to $|V_1|/4$ indeed expand in H.

Finally, assume that H is not connected. Let its smallest component be spanned by $S \subset V_1$, that is, $|S| \leqslant |V_1|/2$ and $N_H(S) = \emptyset$. We have already shown that certainly $|S| > |V_1|/4$. But then the fact that $E_G(S, V_1 \setminus S) = \emptyset$ contradicts (P2). □

We will use these expansion properties to build long paths and ultimately a Hamilton cycle in $G[V_i]$. Our approach is based on the rotation–extension technique originally developed by Pósa. Given a graph G, denote the length (number of edges) of the longest path in G by $\ell(G)$. We say that a non-edge $\{u,v\} \notin E(G)$ is a *booster* with respect to G if $G + \{u,v\}$ is Hamiltonian or $\ell(G + \{u,v\}) > \ell(G)$. We denote the set of boosters in G by $B(G)$. Pósa's rotation technique guarantees that there exist plenty of boosters in G (see, for instance, Corollary 2.10 from [17]).

Lemma 3.9. *Let p be a positive integer. Let $G = (V,E)$ be a p-expander. Then $|B(G)| \geqslant p^2/2$.* □

Using G_2 to hit boosters in G_1 Now we move on to $G_2 = G_{n,d_2}$. Recall that we would like to add its edges to $G_1[V_1]$ and complete a cycle on V_1. However, we have to argue carefully because the choice of a G_1-dense set V_1 will depend on the given vertex colouring.

Lemma 3.10. *Let G_1 be a d_1-regular graph on $[n]$ with properties* (P1) *and* (P2)*, for sufficiently large d_1, and let $d_1/150 \leqslant d_2 \leqslant d_1/100$. With high probability, $G_2 = G_{n,d_2}$ has the property that for any G_1-dense subset $V_1 \subset [n]$, $(G_1 \cup G_2)[V_1]$ is Hamiltonian.*

The proof of Lemma 3.10 consists of two parts. First we identify a deterministic property that is sufficient to make $(G_1 \cup G_2)[V_1]$ Hamiltonian, and then, using the configuration model, we show that G_{n,d_2} possesses this property with high probability.

Lemma 3.11. *Let H_1 and H_2 be graphs on vertex set V_1. Suppose that for any edge set $E' \subset E(H_2)$ with $|E'| \leqslant |V_1|$,*

$$H_1 \cup E' \text{ is Hamiltonian}, \quad \text{or} \quad B(H_1 \cup E') \cap E(H_2) \neq \emptyset.$$

Then the graph $H_1 \cup H_2$ is Hamiltonian.

Proof. We will build a subset of $E(H_2)$ such that its addition to H_1 creates a Hamiltonian graph. Start with $E_0 = \emptyset$. Assume that E_i is a subset of i edges in $E(H_2)$. If the graph $H_1 \cup E_i$ is Hamiltonian, we are done. Otherwise, by hypothesis, $E(H_2) \cap B(H_1 \cup E_i)$ contains an edge e, so we set $E_{i+1} = E_i \cup \{e\}$.

In each step i, we have $\ell(H_1 \cup E_{i+1}) > \ell(H_1 \cup E_i)$, so the process terminates after at most $|V_1|$ steps, with a Hamiltonian graph $H_1 \cup E_i$. □

Lemma 3.12. *Let G_1 be a d_1-regular graph on V with properties* (P1) *and* (P2)*, where $|V| = n$ and d_1 is sufficiently large. Let $G_2 = G_{n,d_2}$ for $d_1/150 \leqslant d_2 \leqslant d_1/100$. We say that $G_2 \in A_{G_1}$ (or A_{G_1} occurs) if there exists a G_1-dense subset $V_1 \subset V$, and an edge set $E \subset \binom{V_1}{2}$, $|E| \leqslant |V_1|$, such that G_2 contains E and does not intersect $B((G_1 + E)[V_1])$. It holds that $\mathbb{P}[A_{G_1}] = o(1)$.*

Proof. We will prove the claim for G_2 sampled according to the configuration model, which is contiguous to the uniform model G_{n,d_2}. This allows us to apply Lemma 3.3, which gives us a precise estimate on the probability of (non-)occurrence of certain edge sets. Let $P \in \mathcal{P}_{n,d_2}$ be chosen uniformly at random. We will actually bound the probability that the induced multigraph $\mathcal{M}(P)$ is in A_{G_1}, denoted by $\mathbb{P}_{\mathcal{P}}[A_{G_1}]$, with a slight abuse of notation for not renaming the event A_{G_1} itself.

Fix a G_1-dense subset $V_1 \subset V$ with $|V_1| = \xi n$, and $E \subset \binom{V_1}{2}$ with $|E| = m \leqslant |V_1|$. Recall that, since V_1 is G_1-dense,

$$\xi n \geqslant \frac{\alpha \log d_1}{4d_1} n = \frac{10^5 \log d_1}{4d_1} n.$$

Note that the graph $G_1 + E$ is a $(|V_1|/4)$-expander, so we apply Lemma 3.9, which says that the set of boosters $B = B((G_1+E)[V_1])$ contains at least $2^{-5}\xi^2 n^2$ edges.

Applying Lemma 3.3 to E and B, we get

$$\mathbb{P}_{\mathcal{P}}[\mathcal{M}(P) \supset E \text{ and } \mathcal{M}(P) \cap B = \emptyset] \leqslant \left(\frac{2d_2}{n}\right)^m e^{-(2/5\cdot 2^5)\xi^2 n d_2}.$$

Now we can take the union bound over all choices of E and V_1. We crudely bound the number of ways to choose V_1 by $n\binom{n}{\xi n}$.

$$\mathbb{P}_{\mathcal{P}}[A_{G_1}] \leqslant n\binom{n}{\xi n} \sum_{m=1}^{\xi n} \binom{\xi^2 n^2/2}{m} \left(\frac{2d_2}{n}\right)^m e^{-(1/5\cdot 2^4)\xi^2 n d_2}.$$

The term

$$\binom{\xi^2 n^2/2}{m} \left(\frac{2d_2}{n}\right)^m \leqslant \left(\frac{e\xi^2 n d_2}{m}\right)^m$$

is increasing in m in the given range, and hence

$$\mathbb{P}_{\mathcal{P}}[A_{G_1}] \leqslant n \cdot \xi n \cdot (e\xi^{-1} \cdot e\xi d_2 \cdot e^{-(1/5\cdot 2^4)\xi d_2})^{\xi n}.$$

Introducing the value of ξ, the term in brackets is at most

$$e^2 d_2 e^{-(10^5/(4\cdot 5\cdot 2^4))(d_2 \log d_1/d_1)} \leqslant e^2 d_2 d_1^{-300 d_2/d_1}.$$

For $d_2 \in [d_1/150, d_1/100]$ the term above is upper-bounded by $d_1^{1-300/150}$, so

$$\mathbb{P}_{\mathcal{P}}[A_{G_1}] \leqslant \xi n^2 e^{-\Omega(\xi n)} = e^{-\Omega(\xi n)},$$

as claimed. □

Proof of Lemma 3.10. Since G_1 satisfies (P1) and (P2), for $G_2 = G_{n,d_2}$ it holds with high probability that $G_2 \notin A_{G_1}$. Hence, given a G_1-dense set $V_1 \subset V$, we can apply Lemma 3.11 to $G_1[V_1]$ and $G_2[V_1]$ to find a Hamilton cycle in $(G_1 \cup G_2)[V_1]$, as required. □

We are now ready to put together the proof.

Proof of Theorem 1.3. For a given d, set $d_2 = 2\cdot\lceil d/300\rceil$ and $d_1 = d - d_2$. Let d be large enough so that $d_2 \leqslant \frac{1}{100} d_1$, and for Lemma 3.6 and Lemma 3.10 to hold. Moreover, by choosing

d_2 to be even, we have ensured that whenever nd is even (so that $G_{n,d}$ is non-empty), nd_1 and nd_2 are also even.

Generate $G_1 = G_{n,d_1}$ and $G_2 = G_{n,d_2}$ on vertex set V. Suppose that G_1 has properties (P1) and (P2), and G_2 satisfies the conclusion of Lemma 3.10. By Lemma 3.6 and Lemma 3.10, this holds with high probability. We claim that in this case

$$\pi_\alpha(G_1 \cup G_2) > \left(1 - \frac{\alpha \log d_1}{d_1}\right) n,$$

where $\alpha = 10^5$ as before. Let

$$c : V \to \left[\left(1 - \frac{\alpha \log d_1}{d_1}\right) n\right]$$

be a given colouring.

We first use Lemma 3.7 to find G_1-dense sets $V_1, V_2 \subset V$ with the same colouring. Then by Lemma 3.10, we conclude that the graphs $(G_1 \cup G_2)[V_i]$ are Hamiltonian, for $i = 1, 2$. Let C_1 and C_2 be Hamilton cycles on V_1 and V_2. G_1 satisfies (P2), which implies that it contains an edge between some two vertices $v_1 \in V_1$ and $v_2 \in V_2$. We form the required path S by going along C_1, using $v_1 v_2$ to skip to V_2 and then going along C_2. The segments $S[V_1]$ and $S[V_2]$ give an anagram in c, as required.

It remains to express the bound in terms of d. Note that d_1 lies between $(1 - 1/100)d$ and d, so

$$\frac{\alpha \log d_1}{d_1} \leqslant \frac{10^5 \log d}{d_1} \leqslant \frac{2 \cdot 10^5 \log d}{d}.$$

Hence

$$\pi_\alpha(G_1 \cup G_2) > \left(1 - \frac{2 \cdot 10^5 \log d}{d}\right) n,$$

and by contiguity, the same holds for $G_{n,d}$ with high probability. □

4. Concluding remarks

In this paper we have studied anagram-free colourings of graphs, and showed that there are very sparse graphs in which anagrams cannot be avoided unless we basically give each vertex a separate colour. Our research suggests several interesting questions, some of which we mention here.

The first question concerns the lower bound on the anagram-chromatic number for trees. Is there a family of trees $T(n)$ on n vertices for which $\pi_\alpha(T(n)) \geqslant \varepsilon \log n$ for some positive constant $\varepsilon > 0$? We remark that this is the case for the analogous problem of finding the *anagram-chromatic index* of a tree. Indeed, a simple counting argument (see Proposition 2.4) shows that if instead of vertex colourings, we colour edges of a graph, then to avoid anagrams in the complete binary tree of depth h, we need to use at least $\lceil \frac{1}{4} h \rceil$ colours.

In estimating the anagram-chromatic number of planar graphs we relied only on the fact that they have small separators. It would be interesting to know a better lower bound on $\pi_\alpha(G)$ for such graphs. In particular, we wonder if there exists a family H_n of planar graphs on n vertices such that $\pi_\alpha(H_n) \geqslant n^\varepsilon$ for some absolute constant $\varepsilon > 0$.

Let $G(n,d)$ denote the graph with the largest anagram-chromatic number among all graphs G on n vertices with $\Delta(G) \leqslant d$. Our main result shows that if d is large enough then

$$\pi_\alpha(G(n,d)) \geqslant n\left(1 - C\frac{\log d}{d}\right),$$

while for $d = 4$ we can only provide a construction which gives

$$\pi_\alpha(G(n,4)) \geqslant \frac{\sqrt{n}}{2\log n}.$$

We believe that there exist cubic graphs for which the anagram-chromatic number grows linearly with the order of the graph.

It would be nice to know how fast the function $f(d) = 1 - \limsup_{n\to\infty} \pi_\alpha(G(n,d))/n$ decreases with d. Let us recall that from Proposition 3.2 and Theorem 1.3 it follows that

$$\frac{1}{d} \leqslant f(d) = O\left(\frac{\log d}{d}\right).$$

We do not know the correct bound, but we have good reasons to believe that the upper bound can be improved. Indeed, consider a graph which is a union of $2n/d$ cliques of size $d/2$ and a random n-vertex $d/2$-regular graph. We think that using such a construction one can show that $f(d) \leqslant (\log d)^{1/2+o(1)}/d$, but the proof looks quite involved and would probably not be worth the effort since it is by no means clear whether it would give the right order of $f(d)$.

Finally, Lemma 3.10 motivates questions on Hamiltonicity of small induced subgraphs of $G_{n,d}$. Pursuing our proof outline, we can prove the following.

Claim 6. *There is a constant C such that, with high probability $G = G_{n,d}$ has the following property. For any vertex set $V_1 \subset [n]$ of order at least $C\, n\sqrt{(\log d/d)}$, if the graph $H = G[V_1]$ has minimum degree at least $(d/10n)|V_1|$, then H is Hamiltonian.*

To see this, take $G_2 = G_{n,d_2}$ for $d_2 = (C/20)\sqrt{d\log d}$, and $G_1 = G_{n,d_1}$ for $d_1 = d - d_2$. Consider $G = G_{n,d_1} \cup G_{n,d_2}$. Let V_1 and $H = G[V_1]$ satisfy the hypothesis, and denote $|V_1| = \xi n$ with $\xi \geqslant C\sqrt{\log d/d}$. Since the graph $G[V_1]$ has minimum degree at least $\xi d/10$, and we ensured $d_2 \leqslant \xi d/20$, $G_1[V_1]$ has minimum degree at least $\xi d/20$. This guarantees that $G_1[V_1]$, as well as any graph on V_1 containing it, has $\Theta(\xi^2 n^2)$ boosters. On the other hand, the condition $d_2 e^{-\Omega(\xi d_2)} < 1$ implies that G_2 hits those boosters with high probability (see the calculation in Lemma 3.12). Hence $G[V_1]$ is Hamiltonian for any such V_1, and by contiguity, $G_{n,d}$ satisfies the claim.

The above discussion leads to the natural question, what is the smallest possible lower bound on $|V_1|$ in Claim 6? Note that $|V_1| = C\, n\sqrt{\log d/d}$ is the best we can get from our approach. Namely, the above-mentioned conditions require

$$\Omega\left(\frac{1}{\xi}\log\left(\frac{1}{\xi}\right)\right) = d_2 \leqslant \frac{\xi d}{20},$$

that is,

$$\xi = \Omega\left(\sqrt{\frac{\log d}{d}}n\right).$$

We also give a lower bound on $|V_1|$. Using independent sets in $G_{n,d}$, one can find an induced unbalanced bipartite subgraph of order $(\log d/d)n$ with high minimum degree, which is obviously non-Hamiltonian. This observation implies that we need at least $|V_1| \geqslant (\log d/d)n$. We wonder how tight this estimate is.

Note added in proof

After this paper was written and submitted, we learned that Wilson and Wood [21] independently studied graph colourings avoiding *abelian squares*, which are equivalent to anagrams. In particular, our Proposition 1.1, which shows that complete binary trees have an unbounded anagram-free chromatic number, answers one of their questions.

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