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A COUNTEREXAMPLE TO THE ALON–SAKS–SEYMOUR CONJECTURE AND RELATED PROBLEMS

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Consider a graph obtained by taking an edge disjoint union of k complete bipartite graphs. Alon, Saks, and Seymour conjectured that such graphs have chromatic number at most $k+1$. This well known conjecture remained open for almost twenty years. In this paper, we construct a counterexample to this conjecture and discuss several related problems in combinatorial geometry and communication complexity.

1. Introduction

Tools from linear algebra have many striking applications in the study of combinatorial problems. One of the earliest such examples is the theorem of Graham and Pollak [7]. Motivated by a communication problem that arose in connection with data transmission, they proved that the edge set of a complete graph K_k cannot be partitioned into disjoint union of less than $k-1$ complete bipartite graphs. Their original proof used Sylvester's law of inertia. Over the years, this elegant result attracted a lot of attention and by now it has several different algebraic proofs, see [4,15,18,20]. On the other hand, no purely combinatorial proof of this statement is known.

A natural generalization of Graham-Pollak theorem is to ask whether the same estimate holds also for all graphs with chromatic number k. This problem was raised twenty years ago by Alon, Saks, and Seymour, who made the following conjecture (see, e.g., the survey of J. Kahn [9]).

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Conjecture 1. *If the edges of a graph G can be partitioned into k edge disjoint complete bipartite graphs, then the chromatic number of G is at most $k+1$.*

This question is also related to another long-standing open problem by Erdős, Faber, and Lovász. They conjectured that the edge disjoint union of k complete graphs of order k is k-chromatic. Indeed, by replacing cliques in this problem by complete bipartite graphs we obtain the Alon–Saks–Seymour conjecture. The question of Erdős, Faber, and Lovász is still open. On the other hand, Kahn [8] proved the asymptotic version of their conjecture, showing that the chromatic number of edge disjoint union of k complete graphs of order k has chromatic number at most $(1+o(1))k$.

Let $\mathbf{bp}(G)$ be the minimum number of bicliques (i.e., complete bipartite graphs) needed to partition the edges of graph G, and let $\chi(G)$ be the chromatic number of G. The Alon–Saks–Seymour conjecture can be restated as $\mathbf{bp}(G) \geq \chi(G) - 1$. Until recently, there was not much known about this conjecture. Using the folklore result that the chromatic number of a union of graphs is at most the product of their chromatic numbers, one can easily get a lower bound $\mathbf{bp}(G) \geq \log_2 \chi(G)$. In [13], Mubayi and Vishwanathan improved the lower bound to $2^{\sqrt{2\log_2 \chi(G)}}$. This estimate can be also deduced from the well known result of Yannakakis [21] in communication complexity. This connection to communication complexity was discovered by Alon and Haviv [2] (see Section 4 for details). Gao, McKay, Naserasr, and Stevens [6] introduced a reformulation of the Alon–Saks–Seymour conjecture and verified it for graphs with chromatic number $k \leq 9$. The main aim of this paper is to obtain a superlinear gap between chromatic number and biclique partition number, which disproves the Alon–Saks–Seymour conjecture.

Theorem 1.1. *There exist graphs G with arbitrarily large biclique partition number such that $\chi(G) \geq c\big(\mathbf{bp}(G)\big)^{6/5}$, for some fixed constant $c>0$.*

The study of (two-party) communication complexity, introduced by Yao [22], is an important topic in theoretical computer science which has many applications. In the basic model we have two players Alice and Bob who are trying to evaluate a boolean function $f: X \times Y \to \{0,1\}$. Alice only knows x, Bob only knows y, and they want to communicate with each other according to some fixed protocol in order to compute $f(x,y)$. The goal is to minimize the amount of communication during the protocol. The deterministic communication complexity $D(f)$ is the number of bits that need to be exchanged for the worst inputs x,y by the best protocol for f. Let M be a matrix of f, i.e., $M_{x,y} = f(x,y)$ and let $rk(M)$ be the rank of M. It's known that $D(f) \geq \log_2 rk(M)$. Lovász and Saks [12] conjectured that this bound is not very far from being tight. More precisely, their *log-rank conjecture* says that $D(f) \leq (\log_2 rk(M))^{O(1)}$. This problem is directly related to the *rank-coloring conjecture* of Van Nuffelen [19] and Fajtlowicz [5] in graph theory. This conjecture, which was disproved by Alon and Seymour [3], asked whether the chromatic number of a graph G is bounded by the rank of its adjacency matrix A_G. It is known that separation results between $D(f)$ and $\log_2 rk(M)$ give corresponding separation between $\chi(G)$ and $rk(A_G)$. Several authors gave such separation results, e.g., [16,17]. So far, the largest gap was obtained by Nisan and Wigderson [14] who constructed an infinite family of matrices such that $D(f) > (\log_2 rk(M))^{\log_2 3}$.

Similar to the rank-coloring problem, the Alon–Saks–Seymour conjecture is also closely related to a well known open problem in communication complexity. This communication problem, known as *clique versus independent set* (*CL-IS* for brevity), was introduced by Yannakakis [21] in 1988. In this problem, there is a publicly known graph G, Alice gets a clique C of G and Bob gets an independent set I of G. Their goal is to output $|C \cap I|$, which is clearly either 0 or 1. We will discuss the connection between this problem and the Alon–Saks–Seymour conjecture, and show that our counterexample yields the first nontrivial lower bound on the non-deterministic communication complexity of the *CL-IS* problem.

The rest of this short paper is organized as follows. In the next section we describe a counterexample to the Alon–Saks–Seymour conjecture. In Section 3, we consider minimal coverings of a graph by bicliques, in which every edge of the graph is covered at least once and at most t times, for some parameter t. This more general notion is closely related to the question in combinatorial geometry about a neighborly family of boxes. We show that a natural variant of the Alon–Saks–Seymour conjecture for this more general parameter fails as well. In Section 4, we discuss connections with communication complexity and use our counterexample to obtain a new lower bound on the non-deterministic communication complexity of the clique vs. independent set problem. The final section contains some concluding remarks and open problems.

Notation. The n-dimensional cube Q_n is $\{0,1\}^n$ and two vertices x, y of Q_n are adjacent $x \sim y$ if and only if they differ in exactly one coordinate. A k-dimensional subcube of Q_n is a subset of $\{0,1\}^n$ which can be written as $\{x = (x_1, \dots, x_n) \in Q_n : x_i = a_i, \ \forall i \in T\}$, where T is a set of $n-k$ coordinates (called fixed coordinates), and each a_i is a fixed element in $\{0,1\}$. In addition, we write 1^n and 0^n to represent the all-one and all-zero vectors in Q_n and use Q_n^- to indicate the set $Q_n \setminus \{1^n, 0^n\}$. Given two subsets $X \subset Q_k$ and

$Y \subset Q_\ell$, we denote by $X \times Y$ the subset of the cube $Q_{k+\ell}$ which consists of all binary vectors (x,y) with $x \in X$ and $y \in Y$.

For a graph $G=(V,E)$ with vertex set V and edge set E, we denote by $\chi(G)$, $\alpha(G)$, and $\mathbf{bp}(G)$ the chromatic number, independence number, and biclique partition number, respectively. The collection of all independent sets in G is denoted by $\mathcal{I}(G)$. Similarly, $\mathcal{C}(G)$ stands for the set of all cliques in G. The *OR* product of two graphs G and H is defined as a graph with vertex set equal to the Cartesian product $V(G) \times V(H)$, with adjacency $(g,h) \sim (g',h')$ iff $g \sim g'$ in G or $h \sim h'$ in H. The m-blowup of a graph G is obtained by replacing every vertex v of G with an independent set I_v of size m and by replacing every edge (u,v) of G with the complete bipartite graph whose parts are the independent sets I_u and I_v. We also use the notation $\mathcal{B}(U,W)$ to indicate the biclique with two parts U and W.

To state asymptotic results, we utilize the following standard notations. For two functions $f(n)$ and $g(n)$, write $f(n) = \Omega(g(n))$ if there exists a positive constant c such that $\liminf_{n\to\infty} f(n)/g(n) \geq c$, $f(n) = o(g(n))$ if $\limsup_{n\to\infty} f(n)/g(n) = 0$. Also, $f(n) = O(g(n))$ if there exists a positive constant $C>0$ such that $\limsup_{n\to\infty} f(n)/g(n) \leq C$.

2. Main Result

In this section we describe a counterexample to the Alon–Saks–Seymour conjecture. Our construction is inspired by and somewhat similar to Razborov's counterexample to the rank-coloring conjecture [17]. Consider the following graph $G = (V,E)$. Its vertex set is $V(G) = [n]^7 = \{(x_1,\dots,x_7) : x_i \in [n]\}$. For any two vertices $x=(x_1,\dots,x_7)$, $y=(y_1,\dots,y_7)$ in $V(G)$, let ρ be the comparing function which records all coordinates in which they differ. More precisely, $\rho(x,y)=(\rho_1(x,y),\dots,\rho_7(x,y)) \in Q_7$, such that

$$\rho_i(x,y) = \begin{cases} 1 & \text{if } x_i \neq y_i \\ 0 & \text{if } x_i = y_i \end{cases}$$

Two vertices x and y are adjacent in G if and only if $\rho(x,y) \in S$, where S is the following subset of the cube Q_7

$$S = Q_7 \setminus \left[(1^4 \times Q_3^-) \cup \{0^4 \times 0^3\} \cup \{0^4 \times 1^3\}\right].$$

In the rest of this section we show that this graph G satisfies the assertion of Theorem 1.1.

Proposition 2.1. *The independence number of G satisfies $\alpha(G)=O(n)$.*

Proof. Let I be an independent set in G. For any set of indices $T = \{i_1,\dots,i_t\} \subset \{1,2,\dots,7\}$, let p_T be the natural projection of $[n]^7$ to $[n]^T$. More precisely, for every vector $x \in [n]^7$, p_T outputs the restriction of x to the coordinates in T, i.e., $p_T(x) = (x_{i_1},\dots,x_{i_t})$. For convenience, we will write p_{1234} instead of $p_{\{1,2,3,4\}}$, etc. If $|p_{1234}(I)| = 1$, it is easy to check from the definition of S, that any two vertices in I are different in each of the last three coordinates. As a result, $|I| = |p_{567}(I)| \le n$. Now suppose that $|p_{1234}(I)| > 1$. Again from the definition of S, it follows that any two vertices $x, y \in G$ which agree on one of the first 4 coordinates and satisfy $p_{1234}(x) \ne p_{1234}(y)$ are adjacent in G. Hence, any two vectors in $p_{1234}(I)$ differ in all their coordinates and therefore $|p_{1234}(I)| \le n$. If in addition, we also have for every element $x \in p_{1234}(I)$, $|p_{1234}^{-1}(x) \cap I| \le 3$, then $|I| \le 3|p_{1234}(I)| = O(n)$ and the proof is complete.

Otherwise, we may assume the existence of $\widetilde{x} \in [n]^4$ and different vertices $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{x}_4 \in I$ such that $p_{1234}(\widetilde{x}_i) = \widetilde{x}$. By the definition of S, it is easy to see that $p_{567}(\widetilde{x}_i)$ differ in every coordinate. Since $|p_{1234}(I)| > 1$, there is a vertex $z \in I$ with $p_{1234}(z)$ different from $\widetilde{x}$. Moreover, by the above discussion $p_{1234}(z)$ and $\widetilde{x}$ differ in every coordinate. As $1^7 \in S$, we also have that any two vertices of G which differ in all 7 coordinates are adjacent. This implies that $p_{567}(z)$ and $p_{567}(\widetilde{x}_i)$ are equal in at least one coordinate. Since the number of coordinates of $p_{567}(z)$ is only 3 and there are 4 vertices $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{x}_4$, we have that two of these vertices agree with $p_{567}(z)$ (and hence with each other) in the same coordinate. This contradicts the fact that $p_{567}(\widetilde{x}_i)$ differ in all coordinates and completes the proof. ∎

Corollary 2.2. *The chromatic number of G is at least $\Omega(n^6)$.*

Proof. Apply Proposition 2.1 together with the well-known fact that $\chi(G) \ge \frac{|V(G)|}{\alpha(G)}$. ∎

Proposition 2.3. *The biclique partition number satisfies* $\mathbf{bp}(G) = O(n^5)$.

Before going into the details of the proof of this statement, we first need the following two lemmas.

Lemma 2.1. *S can be partitioned into the disjoint union $S = \cup_{i=1}^{30} S_i$, where each S_i is a 2-dimensional subcube of Q_7.*

Proof. We start with the following simple observations.

(a) Q_3^- is a disjoint union of 1-dimensional subcubes.
(b) Q_3 can be decomposed into a disjoint union of 2-dimensional subcubes.

(c) For every $R_1 \subset Q_4$, the set $R_1 \times Q_3$ can be decomposed into a disjoint union of 2-dimensional subcubes.

(d) For any $x_1 \sim x_2$ in Q_4 and $y_1 \sim y_2$ in Q_3, the set

$$\{(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2)\}$$

is a 2-dimensional subcube in Q_7.

(e) For any $x_1 \sim x_2$ in Q_4, $(x_1 \times Q_3^-) \cup (x_2 \times Q_3^-)$ can be decomposed into a disjoint union of 2-dimensional subcubes.

To verify (a), note that $Q_3^- = \{(0,0,1),(0,1,1)\} \cup \{(0,1,0),(1,1,0)\} \cup \{(1,0,0),(1,0,1)\}$. Claims (b) and (d) are obvious by the definition of a cube. Claim (c) is an immediate corollary of (b), and claim (e) follows easily from (a) and (d).

Next we can partition the set $S = Q_7 \setminus \left[(1^4 \times Q_3^-) \cup \{0^4 \times 0^3\} \cup \{0^4 \times 1^3\}\right]$ into the following 3 disjoint subsets S', S'', S''', and show that each of them is itself a disjoint union of 2-dimensional subcubes.

$$S' = \begin{cases} (0,0,0,0) \times Q_3^- \cup (0,0,0,1) \times Q_3^- \\ (0,0,1,1) \times Q_3^- \cup (1,0,1,1) \times Q_3^- \\ (0,1,0,1) \times Q_3^- \cup (0,1,1,1) \times Q_3^- \\ (1,1,0,1) \times Q_3^- \cup (1,0,0,1) \times Q_3^- \end{cases}$$

This set can be partitioned into a disjoint union of 2-dimensional subcubes, using claim (e).

$$S'' = \begin{cases} (1,1,1,1) \times 0^3 \cup (1,1,0,1) \times 0^3 \cup (1,0,1,1) \times 0^3 \cup (1,0,0,1) \times 0^3 \\ (1,1,1,1) \times 1^3 \cup (1,1,0,1) \times 1^3 \cup (1,0,1,1) \times 1^3 \cup (1,0,0,1) \times 1^3 \\ (0,1,1,1) \times 0^3 \cup (0,1,0,1) \times 0^3 \cup (0,0,1,1) \times 0^3 \cup (0,0,0,1) \times 0^3 \\ (0,1,1,1) \times 1^3 \cup (0,1,0,1) \times 1^3 \cup (0,0,1,1) \times 1^3 \cup (0,0,0,1) \times 1^3 \end{cases}$$

Note that every line in the definition of S'' describes a 2-dimensional subcube. This shows that S'' is a disjoint union of four 2-dimensional subcubes.

$$S''' = \begin{cases} (0,0,1,0) \times Q_3 \cup (0,1,0,0) \times Q_3 \cup (1,0,0,0) \times Q_3 \cup (0,1,1,0) \times Q_3 \\ (1,0,1,0) \times Q_3 \cup (1,1,0,0) \times Q_3 \cup (1,1,1,0) \times Q_3 \end{cases}$$

To decompose this set into a disjoint union of 2-dimensional subcubes, one can use claim (c).

Finally, it is easy to verify that indeed $S = S' \cup S'' \cup S'''$, and hence S can be partitioned into 2-dimensional subcubes. ∎

Using the decomposition $S = \cup_{i=1}^{30} S_i$ from Lemma 2.1, we can define the following subgraphs $G_i \subset G$. The vertex set $V(G_i) = V(G)$ and two vertices $x, y \in G_i$ are adjacent if and only if $\rho(x,y) \in S_i$. From this definition, it is easy to see that G is the edge disjoint union of subgraphs G_i. Next we will show that every G_i has a small biclique partition number.

Lemma 2.2. $\mathbf{bp}(G_i) \le n^5$.

Proof. Recall that the set S_i, which is used to define the edges of G_i, is a 2-dimensional subcube of Q_7. Therefore there exists a set $T = \{t_1, \dots, t_5\} \subset \{1, \dots, 7\}$ of fixed coordinates and $a_1, \dots, a_5 \in \{0,1\}$, such that $S_i = \big\{z = (z_1, \dots, z_7) : z_{t_j} = a_j,\ \forall 1 \le j \le 5\big\}$. Also note that $a_1, \dots, a_5$ can not all be simultaneously zero, since S does not contain 0^7. Now we define the graph $\widetilde{G}_i$. Its vertex set $V(\widetilde{G}_i) = [n]^5$ and two vertices $\widetilde{x}$ and $\widetilde{y}$ are adjacent in $\widetilde{G}_i$ if and only if $\rho(\widetilde{x}, \widetilde{y}) = (a_1, \dots, a_5)$. It is rather straightforward to see that G_i is a n^2-blowup of $\widetilde{G}_i$.

To complete the proof of this lemma we need two basic facts about the biclique partition number. The first one says that for any graph H, $\mathbf{bp}(H) \le |V(H)| - 1$. Indeed, removing stars rooted at every vertex, one by one, we can partition every graph on h vertices into $h-1$ bicliques. The second one claims that if H is a blowup of $\widetilde{H}$, then $\mathbf{bp}(H) \le \mathbf{bp}(\widetilde{H})$. To prove this, note that the blowup of a biclique is a biclique itself. Therefore the blowup of all the bicliques in a partition of $\widetilde{H}$ becomes a biclique partition of H.

These two statements, together with the fact (mentioned above) that G_i is the blowup of $\widetilde{G}_i$, imply that $\mathbf{bp}(G_i) \le \mathbf{bp}(\widetilde{G}_i) \le |V(\widetilde{G}_i)| - 1 \le n^5$. ∎

Proof of Proposition 2.3. Using that G is the edge disjoint union of G_i together with Lemma 2.2, we conclude that $\mathbf{bp}(G) = \mathbf{bp}(\cup_{i=1}^{30} G_i) \le \sum_{i=1}^{30} \mathbf{bp}(G_i) = O(n^5)$. ∎

Propositions 2.2 and 2.3 show that the graph G, which we constructed, indeed satisfies the assertion of Theorem 1.1 and disproves the Alon–Saks–Seymour conjecture.

3. Neighborly families of boxes and t-biclique covering number

The Alon–Saks–Seymour conjecture deals with the minimum number of bicliques needed to cover all the edges of a given graph G exactly once. It is also very natural to consider a more general problem in which we are allowed to cover the edges of graph at most t times. A *t-biclique covering* of a graph G is a collection of bicliques that cover every edge of G at least once and at most t times. The minimum size of such a covering is called the *t-biclique covering number*, and is denoted by $\mathbf{bp}_t(G)$. In particular, $\mathbf{bp}_1(G)$ is the usual biclique partition number $\mathbf{bp}(G)$.

In addition to being an interesting parameter to study in its own right, the t-biclique covering number is also closely related to a question in combinatorial geometry about neighborly families of boxes. A finite family $\mathcal{C}$ of d-dimensional convex polytopes is called *t-neighborly* if $d-t \le \dim(C\cap C') \le d-1$ for every two distinct members C and C' of $\mathcal{C}$. One particularly interesting case is when $\mathcal{C}$ consists of d-dimensional boxes with edges parallel to the coordinate axes. This type of box is called a *standard box*. Using the Graham-Pollak theorem, Zaks [23] proved that the maximum possible cardinality of a 1-neighborly family of standard boxes in $\mathbb{R}^d$ is precisely $d+1$. His result was generalized by Alon [1], who proved that $\mathbb{R}^d$ has a t-neighborly family of k standard boxes if and only if the complete graph K_k has a t-biclique covering of size d. This shows that the problem of determining the maximum possible cardinality of a t-neighborly family of standard boxes and the problem of computing the t-biclique covering number of a complete graphs are equivalent.

In his paper [1], Alon gave asymptotic estimates for $\mathbf{bp}_t(K_k)$, showing that

$$(1+o(1))\big(t!/2^t\big)^{1/t}k^{1/t} \le \mathbf{bp}_t(K_k) \le (1+o(1))tk^{1/t}.$$

There is still a gap between these two bounds, and the problem of determining the right constant before $k^{1/t}$ is wide open even for the case $t=2$. Using a different proof, we obtain here a slightly better lower bound of order roughly $\big(t!/2^{t-1}\big)^{1/t}k^{1/t}$. For $t=2$ it improves the above estimate by a factor of $\sqrt{2}$.

Proposition 3.1. *If there exists a t-biclique covering of K_k of size d, then* $k \le 1+\sum_{s=1}^{t} 2^{s-1}\binom{d}{s}$.

Proof. Suppose that the edges of K_k are covered by the bicliques $\{\mathcal{B}(U_j,W_j)\}_{j=1}^{d}$, such that every edge is covered not t-times. For every nonempty subset of indices $S\subset[d]$ of size $|S|\le t$ let $H_S=\cap_{j\in S}\mathcal{B}(U_j,W_j)$, and let A_S be the adjacency matrix of H_S. Let J be the $k\times k$ matrix of ones and let I be the $k\times k$ identity matrix. Then $J-I$ is the adjacency matrix of K_k and it is easy to see, using the inclusion-exclusion principle, that

$$J-I=\sum_{S\subset[d],0<|S|\le t}(-1)^{|S|-1}A_S.$$

Also note that for $|S|=s$, the graph H_S is the disjoint union of at most 2^{s-1} smaller bicliques. Indeed, for every binary vector $z=(z_1,\ldots,z_{s-1})$, consider a complete bipartite graph with parts

$$X_z = \cap_{j,z_j=0} U_j \cap_{j,z_j=1} W_j \cap \; U_s \;\text{ and }\; Y_z = \cap_{j,z_j=0} W_j \cap_{j,z_j=1} U_j \cap W_s.$$

It is not difficult to check that these bicliques are disjoint and their union is H_S. Therefore, for every $S \subset [d], 0 < |S| = s \le t$ we can write $A_S = \sum_i B_{i,S}$, where $B_{i,S}$ is an adjacency matrix of a biclique and $1 \le i \le 2^{s-1}$. Thus we obtain that $J - I$ can be written as a linear combination of at most $m = \sum_{s=1}^{t} 2^{s-1}\binom{d}{s}$ adjacency matrices of complete bipartite graphs.

Now to complete the proof we use the elegant trick of Peck [15] (we can use here other known proofs of the Graham-Pollak theorem as well). For the bipartite graph with adjacency matrix $B_{i,S}$, let $B'_{i,S}$ be the $k \times k$ matrix which contains only ones in positions whose row index lies in the first part of the bipartition and whose column index lies in the second part of the bipartition; the rest of the entries of $B'_{i,S}$ are zeros. Since the corresponding bipartite graph is complete, $B'_{i,S}$ has rank one. Furthermore, the matrix $B_{i,S} - 2B'_{i,S}$ is antisymmetric. As a result we can write $J - I$ as a linear combination of at most m rank one matrices, plus some antisymmetric matrix T. Since an antisymmetric real matrix has only imaginary eigenvalues, $I + T$ must have full rank k. But its rank can not exceed the rank of the linear combination of at most m rank one matrices plus J. As J has rank one as well, this implies that $k \le m + 1 = 1 + \sum_{s=1}^{t} 2^{s-1}\binom{d}{s}$, which completes the proof. ∎

As we already mentioned in the introduction, the motivation for the Alon–Saks–Seymour conjecture comes from the Graham-Pollak theorem which says that $\mathbf{bp}(K_k) \ge k - 1$. Similarly, based on the lower bound of Alon that $\mathbf{bp}_t(K_k) \ge \Omega\big(k^{1/t}\big)$, one can consider the following very natural generalization of this conjecture.

Question 3.2. *Is it true that for every fixed integer $t > 0$, there exists a constant $c = c(t)$ such that $\mathbf{bp}_t(G) \ge c\big(\chi(G)\big)^{1/t}$ for all graphs G?*

Recall that in Section 2 we constructed a graph G with $|V(G)| = n^7$ vertices such that $\alpha(G) = O(n)$ and $\mathbf{bp}(G) = O(n^5)$. Consider the OR product (defined in the introduction) of t copies of G. We show that the graph G^t gives a negative answer to the above question for all positive integers t. This follows from the following sequence of claims.

Claim 3.1. $\alpha(G^t) \le \alpha(G)^t = O(n^t)$.

Proof. We only need to prove $\alpha(G \times H) \le \alpha(G)\alpha(H)$ for any two graphs G and H, since then the claim follows by induction on t. To prove this statement, consider a maximum independent set $I \in G \times H$. Let $I' = \{v \in$

$G \mid (v,u) \in I$ for some $u \in H\}$ be the projection of I on $V(G)$. By the definition of the OR product, this is an independent set in G and therefore has size at most $\alpha(G)$. Similarly, if I'' is the projection of I on $V(H)$ then $|I''| \le \alpha(H)$. To complete the proof, note that I is a subset of $I' \times I''$, and therefore its size cannot exceed $\alpha(G)\alpha(H)$. ∎

Corollary 3.3. $\chi(G^t) = \Omega(n^{6t})$.

Proof. By Claim 3.1, $\chi(G^t) \ge \frac{|V(G^t)|}{\alpha(G^t)} \ge \frac{n^{7t}}{\alpha(G)^t} = \Omega(n^{6t})$. ∎

Claim 3.2. $\mathbf{bp}_t(G^t) \le t\mathbf{bp}(G)$.

Proof. Consider graphs H_i, $1 \le i \le t$ with vertex set $V(H_i) = V(G^t)$, where two vertices $(h_1,\dots,h_t)$ and $(h'_1,\dots,h'_t)$ are adjacent in H_i if and only if $h_i \sim h'_i$ in G. Note that H_i is an n^{t-1}-blowup of G and therefore $\mathbf{bp}(H_i) = \mathbf{bp}(G)$. Also, it is easy to see that every edge in G^t is covered by some H_i. Since the number of graphs H_i is t, every edge of G^t is covered at most t times. Then the union of minimum biclique partitions of all H_i gives a t-biclique covering of G. Hence $\mathbf{bp}_t(G^t) \le \sum_{i=1}^{t} \mathbf{bp}(H_i) = t\mathbf{bp}(G)$. ∎

Claim 3.3. $\mathbf{bp}_t(G^t) \le c\big(\chi(G^t)\big)^{\frac{5}{6t}}$ *for some constant* $c = c(t)$.

Proof. By Claims 3.3 and 3.2, $\mathbf{bp}_t(G^t) \le t\mathbf{bp}(G) = O(tn^5) \le c(t)\big(\chi(G^t)\big)^{\frac{5}{6t}}$. ∎

This shows that the answer to Question 3.2 is negative for all natural t.

4. The clique vs. independent set communication problem

In the introduction, we defined the two-party communication model and discussed the concept of deterministic communication complexity. Here we need a few additional notions and definitions (see, e.g., [11] for more details). The *non-deterministic communication complexity* $N^1(f)$ of a function f is the smallest number of bits needed by an all powerful prover to convince Alice and Bob that $f(x,y) = 1$. It is known that $N^1(f) = \lceil \log_2 C^1(f) \rceil$, where $C^1(f)$ is the minimum number of monochromatic combinatorial rectangles needed to cover the 1-inputs of the communication matrix M of f (recall that $M_{x,y} = f(x,y)$). With slight abuse of notation, we will later write $C^1(M)$ instead of $C^1(f)$. The numbers $N^0(f)$, $C^0(f), C^0(M)$ are defined similarly, and the relation $N^0(f) = \lceil \log_2 C^0(f) \rceil$ holds as well.

In this section we consider the communication complexity of the clique versus independent set problem (CL-IS). In this problem, there is a publicly known graph Γ, Alice gets a clique C of Γ and Bob gets an independent set I of Γ. Their goal is to output $|C \cap I|$, which is clearly either 0 or 1. This problem was first introduced by Yannakakis [21], who also proposed the following algorithm to solve it. Given a graph Γ on m vertices, Alice sends to Bob the name of a vertex v in C whose degree in Γ is at most $m/2$. Note that in this case we can reduce the size of the graph by a factor of two by considering only the subgraph Γ' induced by the neighbors of v. In his turn, Bob sends Alice the name of a vertex u in his independent set $I \cap \Gamma'$ which has degree at least $|V(\Gamma')|/2$. In this case we can also reduce the size of the remaining problem by a factor of two. Finally if neither Alice nor Bob can send anything, it is easy to see that $C \cap I = \emptyset$. By repeating this procedure at most $\log_2 m$ rounds, one can show that the deterministic communication complexity satisfies $D(CL\text{-}IS_\Gamma) \leq O(\log_2^2 m)$. However, so far the best lower bound for this problem (see [10]) is only asymptotically $2\log_2 m$.

For the non-deterministic communication complexity of clique vs. independent set problem, it is easy to see that $N^1(CL\text{-}IS_\Gamma)$ is always $\log m$. Indeed, for every vertex $v \in \Gamma$ consider the rectangle R_v formed by all cliques and all independent sets containing v. By definition, these m rectangles cover all 1-inputs of the communication matrix M of $CL\text{-}IS_\Gamma$. On the other hand, determining the correct order of magnitude of $N^0(CL\text{-}IS_\Gamma)$ is wide open except for the trivial lower bound $\log_2 m$. This lower bound follows from the simple fact that taking all single vertices as cliques vs. the same vertices as independent sets shows that the $m \times m$ identity matrix is a submatrix of M. Next we discuss the connection between the Alon–Saks–Seymour conjecture and the $CL\text{-}IS$ problem which was discovered by Alon and Haviv [2]. This connection together with our counterexample gives the first nontrivial lower bound for the non-deterministic communication complexity of the clique vs. independent set problem. It implies that there exists a graph Γ such that $N^0(CL\text{-}IS_\Gamma) \geq 6/5 \log_2 m - O(1)$.

Suppose we have a graph $G = (V, E)$, $V(G) = [n]$, $\mathbf{bp}(G) = m$, and a partition of $E(G)$ into a disjoint union of bicliques $\{\mathcal{B}(U_i, W_i)\}_{i=1}^m$. Define the characteristic vector v_i of each biclique to be $v_i = (v_{i1}, \dots, v_{in}) \in \{0, 1, *\}^n$, so that

$$v_{ij} = \begin{cases} 0 & \text{if } j \in U_i \\ 1 & \text{if } j \in W_i \\ * & \text{otherwise} \end{cases}$$

Using the notations above, we create a new graph Γ with vertex set $[m]$. Two vertices i and i' are adjacent in Γ if there exists a $j \in [n]$ such that $v_{ij} = v_{i'j} = 1$. Two vertices i and i' are nonadjacent if there exists a $j' \in [n]$ such that $v_{ij'} = v_{i'j'} = 0$. In every other case, arbitrarily assign an edge or non-edge between i and i'. If there are two indices j, j' such that $v_{ij} = v_{i'j} = 1$ and $v_{ij'} = v_{i'j'} = 0$, then $j \in W_i \cap W_{i'}$ and $j' \in U_i \cap U_{i'}$. Therefore the edge (j', j) is covered by two bicliques, which is impossible since $\cup_{i=1}^m \mathcal{B}(U_i, W_i)$ is an edge partition of G. This shows that Γ is well defined.

Now consider the CL-IS problem on Γ. Define $C_j = \{q \in [m] : v_{qj} = 1\}$ and $I_j = \{q \in [m] : v_{qj} = 0\}$. By definition of Γ, it is easy to see that $\{C_j\}$ are cliques and $\{I_j\}$ are independent sets in this graph. Denote the matrix of CL-IS_Γ by M. Let M' be the submatrix of M corresponding to the rows determined by $\{C_j\}_{j=1}^n$ and columns determined by $\{I_j\}_{j=1}^n$. Obviously $N^0(M) \geq N^0(M') = \log_2 C^0(M')$. Assume that we have a covering of 0-entries of M' by monochromatic rectangles, and let $R_1, \ldots, R_t$ be the rectangles which cover the diagonal entries of M'. Note that $M'_{pp} = M'_{qq} = 0$ by definition. If M'_{pp} and M'_{qq} are both covered by R_i, then $M'_{pq} = M'_{qp} = 0$ and thus $C_p \cap I_q$ and $C_q \cap I_p$ are both empty. This implies that (p, q) is not an edge in graph G, since otherwise there must exist an index i such that $v_{ip} = 0$, $v_{iq} = 1$ or $v_{ip} = 1$, $v_{iq} = 0$. Then either $i \in I_p \cap C_q$ or $i \in C_p \cap I_q$, which gives a contradiction. In particular, the family of rectangles $\{R_i\}_{i=1}^t$ corresponds to a covering of graph G by independent sets, and therefore $\chi(G) \leq t$. Thus we have that

$$N^0(M) \geq N^0(M') = \log_2 C^0(M') \geq \log_2 t \geq \log_2 \chi(G).$$

This estimate, together with the existence of a graph G (from Section 2) which has $\mathbf{bp}(G) = O(\chi(G)^{5/6})$, proves the following theorem.

Theorem 4.1. *There exists an infinite collection of graphs Γ, such that*

$$N^0(CL\text{-}IS_\Gamma) \geq \frac{6}{5} \log_2 |V(\Gamma)| - O(1).$$

In addition, the combination of the inequality $N^0(CL\text{-}IS_\Gamma) \geq \log_2 \chi(G)$ we just proved, and the result of Yannakakis that $D(CL\text{-}IS_\Gamma) \leq O(\log_2^2 m)$, immediately gives a different derivation of the following result of Mubayi and Vishwanathan. It shows that if $\mathbf{bp}(G) = m$, then

$$\chi(G) \leq 2^{N^0(CL\text{-}IS_\Gamma)} \leq 2^{D(CL\text{-}IS_\Gamma)} \leq 2^{O(\log_2^2 m)}.$$

From the above discussions, we know that any separation result between $\chi(G)$ and $\mathbf{bp}(G)$ gives corresponding separation between $N^0(CL\text{-}IS)$ and the trivial lower bound $\log_2 |V(\Gamma)|$. We do not yet know whether the converse is also true. However, a weaker converse does exist, as was observed by Alon and Haviv [2]. More precisely, the gap between $N^0(CL\text{-}IS_\Gamma)$ and $\log_2 |V(\Gamma)|$

implies a gap between $\chi(H)$ and the 2-biclique covering number $\mathbf{bp}_2(H)$ for some graph H.

Let $\Gamma=(V,E)$ be a graph with vertices $V=\{v_1,\dots,v_m\}$ and consider the following graph H. The vertices of H are all the pairs (C,I) such that C is a clique and I is an independent set in Γ, and $C\cap I=\emptyset$. Two vertices (C,I) and (C',I') are adjacent if $C\cap I'\neq\emptyset$ or $C'\cap I\neq\emptyset$. For every vertex v_i in Γ, we define two subsets $U_i=\{(C,I): v_i\in C\}$ and $W_i=\{(C,I): v_i\in I\}$ of H. These subsets have the following properties.

1. U_i and W_i are disjoint.
2. (U_i,W_i) is a complete bipartite subgraph of H.
3. $G'=\cup_{i=1}^m \mathcal{B}(U_i,W_i)$ and each edge of H is covered at most two times.

Property (1) holds since $C\cap I=\emptyset$ for every vertex (C,I) of H. To verify (2), consider two vertices $(C,I)\in U_i$ and $(C',I')\in W_i$. Then $v_i\in C\cap I'$, which means $C\cap I'\neq\emptyset$ and thus (C,I) and (C',I') are adjacent in H. To prove (3), note that by definition, any edge $(C,I)\sim(C',I')$ in G' either satisfies $C\cap I'\neq\emptyset$ or $C'\cap I\neq\emptyset$ or both. If $C\cap I'\neq\emptyset$, then there is a unique i (since $|C\cap I'|\le 1$) such that $v_i\in C$ and $v_i\in I'$, which means that this edge belongs to $\mathcal{B}(U_i,W_i)$. A similar conclusion holds in the case when $C'\cap I\neq\emptyset$. Thus every edge of H is covered by $\{\mathcal{B}(U_i,W_i)\}_{i=1}^m$ either once or twice. This shows that $\mathbf{bp}_2(H)\le m=|V(\Gamma)|$.

Next we bound the chromatic number of H from below by a function of $N^0(CL\text{-}IS_\Gamma)$. Denote the matrix of $CL\text{-}IS_\Gamma$ by M. By definition, an independent set $I'=\{(C_1,I_1),\dots,(C_l,I_l)\}$ in H corresponds to an all-zero submatrix of M, whose rows and columns are indexed by $C_1,\dots,C_l$ and $I_1,\dots,I_l$ respectively. Thus a proper coloring of H corresponds to a covering of the 0-entries of M by monochromatic rectangles. Therefore $\chi(H)\ge C^0(M)=C^0(CL\text{-}IS_\Gamma)\ge 2^{N^0(CL\text{-}IS_\Gamma)}$, and hence we established the following claim.

Claim 4.1. *For every graph Γ there exists a graph H such that*

$$\mathbf{bp}_2(H)\le |V(\Gamma)| \quad \textit{and} \quad \chi(H)\ge 2^{N^0(CL\text{-}IS_\Gamma)}.$$

5. Concluding remarks

In this paper we constructed a graph which has a polynomial gap between its chromatic number and its biclique partition number, thereby disproving the Alon–Saks–Seymour conjecture. A very interesting problem which remains widely open is to determine how large this gap can be. In communication complexity it is a long standing open problem to prove an $\Omega(\log^2 N)$

lower bound on the complexity of the clique vs. independent set problem for graphs on N vertices. Since, as we already explained in the previous section, this problem is closely related to the Alon–Saks–Seymour conjecture, it is plausible to believe that one can obtain a corresponding gap between chromatic and biclique partition numbers. We conjecture that there exists a graph G with biclique partition number k and chromatic number at least $2^{c\log^2 k}$, for some constant $c > 0$. The existence of such a graph will also resolve the complexity of the clique vs. independent set problem.

Another intriguing question which deserves further study is to determine the t-biclique covering numbers of complete graphs. This will also solve the problem of the maximum possible cardinalities of t-neighborly families of standard boxes in finite dimensional Euclidean spaces. Even the asymptotics of $\mathbf{bp}_t(K_k)$ are only known up to a multiplicative constant factor. In the first open case when $t=2$, the current best bounds are $(1+o(1))k^{1/2} \le \mathbf{bp}_2(K_k) \le (1+o(1))2k^{1/2}$, and it would be interesting to close this gap.

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