On the Asymmetry of Random Regular Graphs and Random Graphs

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ABSTRACT: This paper studies the symmetry of random regular graphs and random graphs. Our main result shows that for all $3 \le d \le n - 4$ the random d-regular graph on n vertices almost surely has no nontrivial automorphisms. This answers an open question of N. Wormald [13]. © 2002 Wiley Periodicals, Inc. Random Struct. Alg., 21: 216–224, 2002

1. INTRODUCTION

The concept of random graphs is one of the central notions in modern Discrete Mathematics. Random graphs have been studied intensively during the last 40 years, with thousands of papers and two excellent monographs by Bollobás [3] and by Janson, Łuczak, and Ruciński [6] devoted to the subject and its diverse applications. Strictly

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speaking, the term "random graph" comprises several models of random graphs which are quite different in many aspects. Usually *asymptotic* properties of random graphs are studied under the condition that the number of vertices n tends to infinity. Putting different probabilities on n-vertex graphs results in different probability spaces. In this paper we are concerned with the two most commonly used models of random graphs: the Erdős-Rényi $G(n, p)$ model and the regular random graph model.

The random graph $G(n, p)$ denotes the probability space whose points are graphs on a fixed set of n vertices, where each pair of vertices forms an edge, randomly and independently, with probability p. The random d-regular graph $G_{n,d}$ is obtained by sampling with the uniform distribution over the set of all possible simple d-regular graphs on n vertices. The vertices of graphs in these two models are labeled, and in this paper we assume that the vertex set is always $V = \{1, 2, \ldots, n\} = [n]$. In both models, we also say that a graph property $\mathcal{P}$ holds *almost surely* (a.s.) if under the correspondent distribution the probability that $\mathcal{P}$ holds tends to 1 as n tends to infinity. In general, p and d can be functions of n, and we will use the notation $G(n, p)$ (and $G_{n,d}$, respectively) to denote both the probability space and a random graph in it.

A permutation $\pi : [n] \to [n]$ of the vertices of graph G belongs to the automorphism group $Aut(G)$ if it satisfies that $(\pi(i), \pi(j))$ is an edge of G if and only if (i, j) is an edge of G. It is clear that for any graph G the identity belongs to $Aut(G)$. We say that $Aut(G)$ is trivial or equivalently G is *asymmetric* if $Aut(G)$ does not contain any permutation other than the identity; otherwise we call G *symmetric.*

The automorphism group was one of the fundamental objects studied by Erdős and Rényi in a sequence of papers, which actually started the theory of random graphs. Erdős and Rényi [5] proved that for $1 - \ln n/n \geq p \geq \ln n/n$, almost surely $G(n, p)$ is asymmetric. In fact, they proved a little bit more, that almost surely one should alter (delete and add) at least $(2 + o(1))np(1 - p)$ edges of $G(n, p)$ to obtain a symmetric graph.

The problem is somewhat more complex for the random regular model. In a sense, this is not totally unexpected as the definition of a random regular graph already guarantees a certain level of symmetry. It is an easy exercise to show that any regular graph with degree $0, 1, 2, n - 1, n - 2$, or $n - 3$ has nontrivial automorphism, due to their simple structures. Almost 20 years ago, Bollobás [4] proved that, for every fixed d, the random d-regular graph $G_{n,d}$ is almost surely asymmetric. McKay and Wormald [10] showed that this remains true for all $d = o(n^{1/2})$. The natural extension of these two results to larger values of d (i.e., $d \leq n - 4$) has been posed by Wormald as one of the open problems in the theory of random regular graphs in his exhaustive survey [13]. The main goal of this paper is to settle this problem and prove the following theorem.

Theorem 1.1. *For all d such that* $\ln n \ll d \leq n/2$ *the random graph $G_{n,d}$ is a.s. asymmetric.*

It is easy to see that a graph has a nontrivial automorphism if and only if its complement does. Since, by definition, the complement of random d-regular graph on n vertices is random $(n - 1 - d)$-regular graph, then the above theorem together with the result of McKay and Wormald [10] implies:

Corollary 1.2. *For all* $3 \leq d \leq n - 4$ *the random regular graph $G_{n,d}$ is almost surely asymmetric.*

To conclude this section, let us mention that we actually prove a somewhat stronger statement than Theorem 1.1. Our method, which is totally different from that of McKay and Wormald, also enables us to strengthen Erdős-Rényi's result about the asymmetry of $G(n, p)$.

The rest of the paper is organized as follows. In Section 2, we present the strengthened version of Theorem 1.1 and the key ideas of its proof. The next two sections (Sections 3 and 4) serve as a preparation for the completion of this proof. In Section 3, we prove a stronger version of Erdős-Rényi's result on the model $G(n, p)$ and in Section 4 we prove some facts about random regular graphs with high degrees. We complete our proof in Section 5. The last section of the paper is devoted to concluding remarks.

Throughout the paper, ln denotes the natural logarithm. The neighborhood $N(u)$ of a vertex u is the set of all vertices of G adjacent to it and $d(u) = |N(u)|$ is the degree of u. The *codegree* of two vertices u and v is $codeg(u, v) = |N(u) \cap N(v)|$. We will frequently use the following asymptotic notation: $A \gg B$ means $B = o(A)$ when $n \to \infty$.

2. A STRONGER RESULT AND MAIN IDEAS

Our main idea is to measure the degree of the asymmetry of a graph. The following crucial definition provides the ground for this measurement.

Definition 2.1 *Let $G = (V, E)$ be a graph and let $\pi : V \to V$ be a permutation of the vertices of G. For a vertex $v \in V$ we define a defect of v with respect to π to be*

$$D_\pi(v) = |N(\pi(v)) \triangle \pi(N(v))|.$$

Similarly we define a defect of graph G with respect to this permutation to be $D_\pi(G) = \max_v D_\pi(v)$. Finally we define a defect of graph G to be

$$D(G) = \min_{\pi \neq identity} D_\pi(G).$$

The defect of a graph, in a way, shows how far the graph is from being symmetric (i.e., having a nontrivial automorphism group). It is easy to see that graph G is symmetric if and only if its defect equals zero. The key observation here is that as the defect can take value from a large interval, and this provides a good chance to show that it is not zero. In fact, it turns out that the defect of a random graph (in both models) is concentrated around a trivial upper bound. The definition of defect is motivated by a result of Kim and Vu in a recent paper [7].

Next to the identity, the simplest permutation is the transposition of two vertices (say u and v). The defect of this permutation, with respect to a graph G, is always bounded by $d(u) + d(v) - 2\,codeg(u, v)$. As it has been proved that for a d-regular random graph (with d satisfying the conditions of the theorem), the codegree $codeg(u, v)$ is almost surely at most $(1 + o(1))(d^2/n + d^{1-\delta})$, for some small positive constant δ (see, e.g., [8] or Lemma 4.1), we can conclude that almost surely

$$D(G_{n,d}) \le (2 + o(1))d\left(1 - \frac{d}{n}\right).$$

We shall prove that this upper bound is actually the tight, i.e.,

Theorem 2.2. *For all d satisfying* $\ln n \ll d$ *and* $n - d \gg \ln n$, *we have that almost surely*

$$D(G_{n,d}) = (2 + o(1))d\left(1 - \frac{d}{n}\right).$$

Theorem 2.2 obviously implies Theorem 1.1. In fact, it shows that in certain sense $G_{n,d}$ is being as asymmetric as possible. In the rest of this section we give a brief sketch of our proof. The undefined quantities l, T, and λ (see below) will be specified in Section 5.

Choose a small number l [$l = o(n)$], and let S_1 be the set of the permutations which fix all but at most l vertices; let $S_2 = S_n \backslash S_1$. For a graph G, let $D_i(G) = \min_{\pi \in S_i} D_\pi(G)$. Next, set a threshold $T = (2 - o(1))d(1 - d/n)$. To prove that $\mathbf{P}(D(G_{n,d}) > T) = 1 - o(1)$, it suffices to show that both $\mathbf{P}(D_1(G_{n,d}) \leq T)$ and $\mathbf{P}(D_2(G_{n,d}) \leq T)$ are $o(1)$. Due to the simple structure of the permutations in S_1, the bound on $\mathbf{P}(D_1(G_{n,d}) \leq T)$ follows from the definition of defect and some properties of random regular graphs, which can be verified using elementary arguments. To obtain the bound on $\mathbf{P}(D_2(G_{n,d}) \leq T)$, we need to take a detour via the other random model $G(n, p)$. We first show that for $p = d/n$, with overwhelming probability (say at least $1 - \lambda$, with λ being very small), the Erdős-Rényi random graph $G(n, p)$ has defect $(2 - o(1))np(1 - p) = (2 - o(1))d(1 - d/n)$ with respect to S_2. To complete the proof, we next verify that the probability that $G(n, p)$ is d-regular is much higher than λ. The first step relies on a sharp concentration inequality obtained by Alon, Kim, and Spencer [1]. The latter follows directly from an estimate of Shamir and Upfal [12].

3. ERDŐS-RÉNYI REVISITED

In this section, we prove the following strengthening of Erdős-Rényi's result. Part of the proof will give us a crucial estimate needed for the definition of the parameter λ, discussed in the previous paragraph.

Theorem 3.1. *For all p satisfying* $\ln n/n \ll p$ *and* $1 - p \gg \ln n/n$, *we have that almost surely*

$$D(G(n, p)) = (2 - o(1))np(1 - p).$$

In particular, this theorem implies that a.s. in order to obtain from $G(n, p)$ a symmetric graph, one needs to delete and add at least $(2 - o(1))np(1 - p)$ edges incident to the same vertex. The main tool we use in the proof is the following large deviation inequality [1] (see also [2], Chapter 7).

Let X be a random variable on a probability space generated by a finite set of mutually independent 0/1 choices, indexed by $i \in I$. Let p_i be the probability that choice i is 1, and let c be such that changing any choice i (keeping all other choices of the same) can change X by at most c. Set $\sigma^2 = c^2 \sum_i p_i(1 - p_i)$.

Proposition 3.2. *For all positive* $t < 2\sigma/c$, $\mathbf{P}[|X - E[X]| > t\sigma] \leq 2e^{-t^2/4}$.

Proof of Theorem 3.1. The upper bound on the defect follows easily by considering the transposition of two vertices (see the paragraph proceeding Theorem 2.2). It is well known and easy to prove that if $p = o(1)$ and $np \to \infty$, then almost surely the codegree of any two vertices is $o(np)$ and, if p is a constant, then a.s. the codegree is $(1 + o(1))np^2$.

To prove the lower bound, set $\epsilon = \epsilon(n, p)$ such that $\epsilon = o(1)$ and $\epsilon^2 np(1 - p) \gg \ln n$. This is possible for all p's satisfying the conditions of the theorem. Fix an arbitrary $1 \le k \le n$, and let π be a permutation of vertices of $G(n, p)$ which fixes all but $k > 0$ vertices. Denote the set of vertices $\{u \mid \pi(u) \neq u\}$ by U and by X the random variable $\sum_{u \in U} D_\pi(u)$. By definition, $D_\pi(u) = |N(\pi(u)) \triangle \pi(N(u))|$ is a binomially distributed random variable with expectation either $2(n - 2)p(1 - p)$ or $2(n - 1)p(1 - p)$, depending on whether $\pi(\pi(u)) = u$ or not. Therefore,

$$E(X) = \sum_{u \in U} E(D_\pi(u)) = (2 + o(1))knp(1 - p).$$

On the other hand, note that the value of X depends only on the edges of the graph adjacent to the vertices in U. Moreover, adding or deleting any such edge [say (u, v)] can only change the values of at most four terms $D_\pi(u)$, $D_\pi(v)$, $D_\pi(\pi^{-1}(u))$, $D_\pi(\pi^{-1}(v))$ in the sum, each by at most 1. Hence X satisfies the assertion of Proposition 3.2 with parameters $c = 4$ and $\sigma^2 = 16(kn - \binom{k}{2})p(1 - p) = \Theta(knp(1 - p))$. Therefore, Proposition 3.2 yields that for some positive constant α

$$\mathbf{P}[|X - E[X]| > 2\epsilon knp(1 - p)] \le e^{-\alpha\epsilon^2 knp(1-p)}. \tag{1}$$

Thus, with probability at least $1 - e^{-\alpha\epsilon^2 knp(1-p)}$, there is a vertex in U with defect at least

$$\frac{1}{k}(E[X] - 2\epsilon knp(1 - p)) = (2 - \epsilon)np(1 - p) = (2 - o(1))np(1 - p).$$

Thus, we have that for some positive constant α

$$\mathbf{P}(D_\pi(G(n, p)) \le (2 - \epsilon)np(1 - p)) \le e^{-\alpha\epsilon^2 knp(1-p)} = P_k. \tag{2}$$

Finally note that the number of permutations which fixes $n - k$ vertices is at most $\binom{n}{k}k!$. Therefore, the probability that there exists as permutation such that the defect of $G(n, p)$ with respect to it is less than $(2 - \epsilon)np(1 - p)$ is at most

$$\sum_{k=1}^{n} \binom{n}{k} k! P_k \le \sum_{k=1}^{n} n^k e^{-\alpha\epsilon^2 knp(1-p)} = \sum_{k=1}^{n} (e^{-\alpha\epsilon^2 np(1-p) - \ln n})^k. \tag{3}$$

Since $\epsilon^2 np(1 - p) \gg \ln n$, the last sum is $o(1)$, completing the proof. ■

4. FACTS ABOUT RANDOM REGULAR GRAPHS

In this section we prove the following lemma, which summarizes some useful properties of random d-regular graph $G_{n,d}$.

Lemma 4.1.

- **(i)** *Suppose that* $\sqrt{n}\ln n \le d \le n - n/\log n$, *then almost surely* $codeg(u, v) = (1 + o(1))d^2/n$ *for every pair* $u \neq v$ *in* $G_{n,d}$.
- **(ii)** *There exist a constant* $\epsilon > 0$ *such that if* $\ln n \le d \le \sqrt{n}\ln n$ *we have that almost surely for all* u *and* v *in* $[n]$, $codeg(u, v) < d^{1-\epsilon}$.
- **(iii)** *Suppose that* $\ln n \ll d \le n^{3/4}$, *then almost surely every subset of vertices of* $G_{n,d}$ *of size* $a \le nd^{-1/3}$ *spans at most* $da/\ln d$ *edges.*
- **(iv)** *For any fixed* $\delta > 0$ *and* n *sufficiently large, the probability that the random graph* $G(n, d/n)$ *is* d*-regular is at least* $e^{-nd^{1/2+\delta}}$.

Proof. Claims (i) and (ii) were proved by Krivelevich, Sudakov, Vu, and Wormald [8], and we refer interested reader to their paper. Claim (iv) can be proved directly for $d = o(n^{1/2})$, using the well-known estimates on the number of d-regular graphs on n vertices. This gives even better lower bound $e^{-O(n \ln d)}$, on the probability that a random graph $G(n, d/n)$ is d-regular. For other values of d claim (iv) is a corollary of the result of Shamir and Upfal [12] (see [8] for more details). Therefore, it remains to establish validity of claim (iii). To do so we use the so-called switching technique introduced by McKay and Wormald [11] (see also [8]).

Fix a set $A \subseteq V$ of a vertices. Let $\mathscr{C}_k$ denote the set of all d-regular graph where exactly k edges of the random d-regular graph have both ends in A. For a graph $G \in \mathscr{C}_k$ with $k > 0$, choose an edge uv with both $u, v \in A$ and choose two other edges $u'v'$ and $u''v''$ of G such that v' and u'' are not adjacent and none of the vertices u', v', u'', v'' is in A or is a neighbor of u or v. Delete these three edges and add the edges vu', $v'u''$ and $v''u$. This procedure is called a forward switching if it produces a graph $H \in \mathscr{C}_{k-1}$. The reverse switching is applied to such an H by choosing two edges uv'' and vu' in H, where $u, v \in A$, and an edge $v'u''$, and deleting these three edges and adding edges uv, $u'v'$ and $u''v''$ (provided that a member of $\mathscr{C}_k$ results). Note that for counting purposes we think about the edges as being oriented, i.e., edge uv is different for edge vu. Since $d = o(n)$, the number of ways of applying a forward switching is at least

$$kn^2d^2(1 + O(a/n) + O(d/n)) = (1 + o(1))kn^2d^2.$$

For a reverse switching, it is at most

$$\left(\binom{a}{2} - (k-1)\right)nd^3\left(1 + O\left(\frac{d}{n}\right)\right) \le (1 + o(1))\binom{a}{2}nd^3.$$

It follows that

$$\frac{|\mathscr{C}_k|}{|\mathscr{C}_{k-1}|} \le (1 + o(1))\,\frac{\binom{a}{2}d}{kn} < \frac{2\binom{a}{2}d}{kn}.$$

Now fix $k_0 = \lfloor \binom{a}{2} d/n \rfloor$ and assume $k \ge \lfloor ad/\ln d \rfloor$. Then

$$\frac{|\mathscr{C}_{k+1}|}{|\mathscr{C}_{k_0}|} = \prod_{i=k_0}^{k} \frac{|\mathscr{C}_{i+1}|}{|\mathscr{C}_i|} \le \left(\frac{2\binom{a}{2}d}{n}\right)^{k-k_0} \frac{k_0!}{k!} \le \left(\frac{2\binom{a}{2}d}{n}\cdot\frac{e}{k}\right)^{k} \le \left(e\,\frac{a\ln d}{n}\right)^{ad/\ln d}.$$

Summing over all $a \le nd^{-1/3}$ and $k > ad/\ln d$ and using the fact that $d \gg \ln n$, we conclude that the probability there is at least one set of vertices of $G_{n,d}$ of cardinality a, which spans more than $ad/\ln d$ edges is at most

$$\sum_{a,k}\binom{n}{a}\left(e\,\frac{a\ln d}{n}\right)^{ad/\ln d} \le \sum_{a,k} n^a (d^{-1/4})^{ad/\ln d} \le e^{a\ln n}e^{-\Omega(ad)} = e^{-(1+o(1))\Omega(ad)} = o(1).$$

This completes the proof of the lemma. ∎

5. PROOF OF THEOREM 2.2

Having finished all the necessary preparations, we are now ready to complete the proof of Theorem 2.2, following the sketch given in Section 2.

Note that, by definition, the defects of a graph and its complement are equal. Since the complement of random d-regular graph on n vertices is a random $(n-1-d)$-regular graph, we can assume that $d \le n/2$. The upper bound on the defect in Theorem 2.2 is already discussed in Section 2 so we shall prove the lower bound only. The proof splits into two cases: $n^{3/4} \le d \le n/2$ and $d < n^{3/4}$. The parameters l, T, ϵ will be defined accordingly.

First we consider the case when $n^{3/4} \le d \le n/2$. Set $l = d^{1-\delta} = o(d)$ for some small positive δ ($\delta = 1/100$ is sufficient). Consider a permutation π in S_1 which fixes all but at most l vertices. It is easy to see that the defect of any vertex u such that $\pi(u) \ne u$ is at least

$$(d(u) - l) + (d(\pi(u)) - l) - 2\,codeg(u, \pi(u)) = (1 + o(1))(2d - 2\,codeg(u, \pi(u))).$$

By claim (i) of Lemma 4.1, the codegree is $(1 + o(1))d^2/n$, so we obtain that almost surely $D_1(G_{n,d}) \ge (2 + o(1))d(1 - d/n)$.

Now consider a permutation $\pi \in S_2$ which fixes only $n - k$ vertices, where $k > l$. Set $p = d/n$ and $T = (2-\epsilon)np(1-p) = (2-\epsilon)d(1-d/n)$, where, as in Section 3, $\epsilon = \epsilon(n) = o(1)$, $\epsilon^2 np(1-p) \gg \ln n$ and, furthermore, it is also chosen to satisfy $\epsilon \ge d^{-\delta}$ (again it is easy to verify that one can find such an ϵ). Note that, in the proof of Theorem 3.1 [see bound (2)] we shown that the probability that the defect of the random graph $G(n, p)$ with respect to π is less than T is at most $P_k = e^{-\alpha\epsilon^2 knp(1-p)}$, where α is a positive constant. As there are at most $\binom{n}{k}k!$ permutations which fix $n - k$ vertices, the probability that $D_2(G(n, d/n)) \le T$ is at most

$$\sum_{k=l+1}^{n}\binom{n}{k}k!P_k \le \sum_{k=l+1}^{n} n^k P_k \le \sum_{k=l+1}^{n}\left(e^{-\alpha\epsilon^2 np(1-p)-\ln n}\right)^k = \lambda. \tag{4}$$

Recall that ϵ was chosen such that $\epsilon^2 np(1 - p) \gg \ln n$; thus the last sum in the previous line is dominated by the first term, and we can conclude that for some positive constant α, $\lambda \leq e^{-\alpha l \epsilon^2 np(1-p)}$. On the other hand, by claim (iv) of Lemma 4.1, we have that for $\delta = 1/100$, the probability that $G(n, d/n)$ is d-regular is at least $e^{-nd^{1/2+\delta}} \gg \lambda$ (with room to spare), provided that $p = d/n \leq 1/2$, $d \geq n^{3/4}$, $l \geq d^{1-\delta}$, and $\epsilon \geq d^{-\delta}$. This shows that the probability that $D_2(G(n, d/n)) \leq T$ is $o(1)$ and completes the proof of the first case when $n/2 \geq d \geq n^{3/4}$.

Next, we treat the case when $\ln n \ll d \leq n^{3/4}$. Set $l = nd^{-1/3}$ and define sets of permutation S_1 and S_2 accordingly. Similarly as above we can argue that almost surely $D_2(G_{n,d}) \geq (2 - o(1))d(1 - d/n)$ (the inequality $e^{-nd^{1/2+\delta}} \gg \lambda$ still holds for the new value of l).

To handle S_1, assume that π is permutation which fixes all but at most $l = nd^{-1/3}$ vertices and denote the set of vertices $\{u \mid \pi(u) \neq u\}$ by U. The size of U is at most $nd^{-1/3}$, and by claim (iii) of Lemma 4.1 we have that almost surely this set spans at most $d|U|/\ln d$ edges in $G_{n,d}$. Therefore, all but at most $o(1)$ fraction of the vertices of U have $o(d)$ neighbors inside U. Since in a set of ordered pairs $(u, \pi(u))$, $u \in U$, every vertex of U appears exactly twice, we conclude that almost surely there exist a vertex $u \in U$ such that both u and $\pi(u)$ have at most $o(d)$ neighbors inside U. Since the set $V \backslash U$ is fixed by π, then, by definition, it is easy to see that the defect of u with respect to π is at least

$$(d(u) - d_U(u)) + (d(\pi(u)) - d_U(\pi(u))) - 2\, codeg(u, \pi(u)) = (2 + o(1))d - 2\, codeg(u, \pi(u)) = (2 + o(1))d.$$

Here we use the fact that a.s. $codeg(u, \pi(u)) = o(d)$, which follows from claims (i) and (ii) of Lemma 4.1. This implies that almost surely $D_1(G_{n,d})$ is $(2 - o(1))d$ and completes the proof of the theorem. ∎

6. CONCLUDING REMARKS

- Our proof is an instance of the following method that might be useful for other problems. Let f be a graph function, whose value is either 0 or 1 [in our case $f(G) = 1$ if G is asymmetric and 0 otherwise]. Assume that we want to prove that for certain model of random graphs that f is almost surely 1. This is usually hard as it is difficult to apply sharp concentration results to a function with only two values. We can avoid this by introducing an auxiliary function F which agrees with f on the bad instances ($F(G) = 0$ if $f(G) = 0$), but takes values in a large interval otherwise (in our case F is the defect). Defining F properly, we have a better chance of showing that F is concentrated above 0, which yields the desired conclusion.
- Erdős and Rényi [5] proved that every graph G of order n with average degree d contains a pair of vertices u, v such that the number of vertices of G joined precisely to one of u, v is at most $2d(1 - d/(n - 1))$. Let π be a transposition of u and v. Then we obtain that

$$D(G) \leq D_\pi(G) \leq 2d\left(1 - \frac{d}{n-1}\right).$$

On the other hand, our Theorem 2.2 shows that there are graphs for which this bound is asymptotically tight.

- It is clear that our method works, without any significant modification, for more general models such as random hypergraphs and random regular hypergraphs. The interested reader is invited to work out the details.
- Another direction of generalization is to consider random graphs with given degree sequence. For a sequence $d_1, \ldots, d_n$, we choose a graph randomly with the uniform distribution from the set of all graphs with this degree sequence. Our argument could be repeated to prove a similar result for any sequence $d_1, \ldots, d_n$, where the d_i's do not deviate to much from some value d, which satisfies the conditions of Theorem 2.2. However, the situation for an arbitrary sequence is not absolutely clear.
- For $p \leq \ln n/n$, $G(n, p)$ is symmetric by a trivial reason since it has almost surely more than one isolated vertex. However, one might ask about the symmetry of the largest connected component or the symmetry of k-core of $G(n, p)$ for $k \geq 2$. These questions were studied by Łuczak in [9], and we refer interested reader to his paper.

REFERENCES

[1] N. Alon, J. H. Kim, and J. Spencer, Nearly perfect matchings in regular simple hypergraphs, Isr J Math 100 (1997), 171–187.

[2] N. Alon and J. Spencer, The probabilistic method, 2nd ed, Wiley, New York, 2000.

[3] B. Bollobás, Random graphs, Academic, London, 1985.

[4] B. Bollobás, The asymptotic number of unlabelled regular graphs, J Lond Math Soc 26 (1982), 201–206.

[5] P. Erdős and A. Rényi, Asymmetric graphs, Acta Math Acad Sci Hung 14 (1963), 295–315.

[6] S. Janson, T. Łuczak, and A. Ruciński, Random graphs, Wiley, New York, 2000.

[7] J. H. Kim and V. H. Vu, Sandwiching random graphs, manuscript.

[8] M. Krivelevich, B. Sudakov, V. Vu, and N. Wormald, Random regular graphs of high degree, Random Struct Alg 18 (2001), 346–363.

[9] T. Łuczak, The automorphism group of random graphs with a given number of edges, Math Proc Camb Phil Soc 104 (1988), 441–449.

[10] B. D. McKay and N. C. Wormald, Automorphisms of random graphs with specified degrees, Combinatorica 4 (1984), 325–338.

[11] B. D. McKay and N. C. Wormald, Uniform generation of random regular graphs of moderate degree, J Alg 11 (1991), 52–67.

[12] E. Shamir and E. Upfal, "Large regular factors in random graphs," Convexity and graph theory (Jerusalem 1981), North Holland Math Stud 87, North Holland, Amsterdam 1984, pp. 271–282.

[13] N. C. Wormald, "Models of random regular graphs," Eds. J. D. Lamb and D. A. Preece, Surveys in combinatorics, Cambridge Univ Press, Cambridge, 1999, pp. 239–298.