Subgraphs with a Large Cochromatic Number

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Abstract: The cochromatic number of a graph $G = (V, E)$ is the smallest number of parts in a partition of V in which each part is either an independent set or induces a complete subgraph. We show that if the chromatic number of G is n, then G contains a subgraph with cochromatic number at least $\Omega(\frac{n}{\ln n})$. This is tight, up to the constant factor, and settles a problem of Erdős and Gimbel. J Graph Theory **26:** 295–297, 1997

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1. INTRODUCTION

All graphs considered here are finite and simple. For a graph G, let $\chi(G)$ denote the chromatic number of G. The *cochromatic number* of $G = (V, E)$ is the smallest number of sets into which the vertex set V can be partitioned so that each set is either independent or induces a complete graph. We denote by $z(G)$ the cochromatic number of G.

The cochromatic number was originally introduced by L. Lesniak and H. Straight [6] and is related to coloring problems and to Ramsey theory. The subject has been studied by various

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researches (see [8] for several references). A natural question is to find a connection between the chromatic and the cochromatic numbers of a graph. A complete graph on n vertices shows that a graph G with a high chromatic number may have a low cochromatic number. Thus to get a nontrivial result one should consider subgraphs of G. P. Erdős and J. Gimbel [3] studied this question and proved that if $\chi(G) = n$, then G contains a subgraph whose cochromatic number is at least $\Omega(\sqrt{n/\ln n})$. They conjectured (see also [7] and [8], pp. 262–263) that the square root can be omitted. In this note we prove the following theorem which settles this conjecture.

Theorem 1.1. *Let G be a graph with chromatic number n, then G contains a subgraph with cochromatic number at least $(\frac{1}{4} + o(1))\frac{n}{\log_2 n}$.*

Note that the result of the above theorem is best possible up to a constant factor, as shown by a clique on n vertices together with the simple result of [2], [4] that the cochromatic number of any graph on n vertices is at most $(2 + o(1))\frac{n}{\log_2 n}$.

2. THE PROOF

In this section we prove the main result. This is done using a probabilistic argument. Throughout, we assume that n is sufficiently large. To simplify the presentation, we omit all floor and ceiling signs whenever these are not crucial. Let $G = (V, E)$ be a graph with chromatic number n. We can assume that G does not contain a clique of size n. Otherwise, by the known results about Ramsey numbers (see, e.g., [5], [1]), G contains an n-vertex subgraph with neither a clique nor an independent set of size at least $2\log_2 n$, whose cochromatic number is at least $\frac{n}{2\log_2 n}$, as needed.

As the first step we reduce the size of the problem. More precisely, we prove that it is enough to consider graphs with at most n^2 vertices. This can be done by the following lemma.

Lemma 2.1. *Let $G = (V, E)$ be a graph with chromatic number n. Then either $z(G) \geq n/\ln n$ or G contains a subgraph $G_1 = (V_1, E_1)$, such that $\chi(G_1) = (1 + o(1))n$ and $|V_1| \leq n^2$.*

Proof. Suppose that $z(G) < n/\ln n$. Let $V = \bigcup_{i=1}^{k} U_i \cup \bigcup_{j=1}^{l} W_j$ be a partition of the set of vertices of G into independent sets U_i and cliques W_j, such that $k + l < n/\ln n$. Define $V_1 = \bigcup_{j=1}^{l} W_j$. Since G has no clique of size n, $|V_1| \leq n^2/\ln n < n^2$. Let G_1 be the subgraph of G induced on the set V_1. Then any coloring of G_1 together with the sets U_i forms a coloring of G. Thus

$$n = \chi(G) \leq \chi(G_1) + k \leq \chi(G_1) + n/\ln n.$$

Therefore $\chi(G_1) \geq n - n/\ln n = (1 + o(1))n$. ∎

Theorem 1.1 is now a straightforward consequence of the following lemma.

Lemma 2.2. *Let $G_1 = (V_1, E_1)$ be a graph on at most n^2 vertices with $\chi(G_1) = (1+o(1))n$. Let H be a subgraph of G_1, obtained by choosing each edge of G_1 randomly and independently with probability $1/2$. Then almost surely*

$$z(H) \geq \left(\frac{1}{4} + o(1)\right)\frac{n}{\log_2 n}.$$

Proof. The probability that H contains a clique of size $4\log_2 n$ is clearly at most

$$\binom{n^2}{4\log_2 n}\left(\frac{1}{2}\right)^{\binom{4\log_2 n}{2}} \leq \left(\frac{en^2}{4\log_2 n}\right)^{4\log_2 n}\left(\frac{1}{2}\right)^{8(\log_2 n)^2-2\log_2 n}$$
$$\leq \left[\frac{n^2}{\log_2 n}\frac{\sqrt{2}}{n^2}\right]^{4\log_2 n} = o(1),$$

where here we used the estimate $\binom{m}{k} \leq (\frac{em}{k})^k$ which is valid for all m and k.

The probability that there exists a subset $V_0 \subseteq V_1$ such that the induced subgraph $G_1[V_0]$ of G_1 on V_0 has minimum degree at least $4\log_2 n$ and V_0 becomes an independent set in H, is at most

$$\sum_{k=4\log_2 n}^{n^2}\binom{n^2}{k}\left(\frac{1}{2}\right)^{\frac{4k\log_2 n}{2}} \leq \sum_{k=4\log_2 n}^{n^2}\left[\frac{en^2}{k}\frac{1}{n^2}\right]^k$$
$$= \sum_{k=4\log_2 n}^{n^2}\left(\frac{e}{k}\right)^k \leq n^2\left(\frac{1}{\log_2 n}\right)^{\log_2 n} = o(1).$$

This implies that almost surely (that is, with probability tending to 1 as n tends to infinity) any independent set V_0 in H induces a subgraph of G_1 with chromatic number at most $4\log_2 n$. Indeed, if $\chi(G_1[V_0]) > 4\log_2 n$, then $G_1[V_0]$ contains a color-critical subgraph $G_2 = (V_2, E_2)$ with $\chi(G_2) = 4\log_2 n + 1$ which must have minimum degree at least $4\log_2 n$. Since V_2 is independent in H this almost surely does not happen, by the above argument.

Now, let $V_1 = \bigcup_{i=1}^k U_i \cup \bigcup_{j=1}^l W_j$ be a partition of the vertex set of H into independent sets U_i and cliques W_j, satisfying $k + l = z(H)$. Then almost surely

$$(1+o(1))n = \chi(G_1) \leq \sum_{i=1}^k \chi(G_1[U_i]) + \sum_{j=1}^l |W_j| \leq k\cdot 4\log_2 n + l\cdot 4\log_2 n$$
$$= (k+l)4\log_2 n = z(H)4\log_2 n,$$

implying $z(H) \geq \frac{1+o(1)}{4}\frac{n}{\log_2 n}$. ∎

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