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SET SYSTEMS WITH RESTRICTED CROSS-INTERSECTIONS AND THE MINIMUM RANK OF INCLUSION MATRICES*

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Abstract. A set system is L-intersecting if any pairwise intersection size lies in L, where L is some set of s nonnegative integers. The celebrated Frankl–Ray-Chaudhuri–Wilson theorems give tight bounds on the size of an L-intersecting set system on a ground set of size n. Such a system contains at most $\binom{n}{s}$ sets if it is uniform and at most $\sum_{i=0}^{s}\binom{n}{i}$ sets if it is nonuniform. They also prove modular versions of these results.

We consider the following extension of these problems. Call the set systems $\mathcal{A}_1, \ldots, \mathcal{A}_k$ *L-cross-intersecting* if for every pair of *distinct* sets A, B with $A \in \mathcal{A}_i$ and $B \in \mathcal{A}_j$ for some $i \neq j$ the intersection size $|A \cap B|$ lies in L. For any k and for $n > n_0(s)$ we give tight bounds on the maximum of $\sum_{i=1}^{k}|\mathcal{A}_i|$. It is at most $\max\{k\binom{n}{s}, \binom{n}{\lfloor n/2\rfloor}\}$ if the systems are uniform and at most $\max\{k\sum_{i=0}^{s}\binom{n}{i}, (k-1)\sum_{i=0}^{s-1}\binom{n}{i} + 2^n\}$ if they are nonuniform. We also obtain modular versions of these results.

Our proofs use tools from linear algebra together with some combinatorial ideas. A key ingredient is a tight lower bound for the rank of the inclusion matrix of a set system. The *s^*-inclusion matrix* of a set system $\mathcal{A}$ on $[n]$ is a matrix M with rows indexed by $\mathcal{A}$ and columns by the subsets of $[n]$ of size at most s, where if $A \in \mathcal{A}$ and $B \subset [n]$ with $|B| \leq s$, we define M_{AB} to be 1 if $B \subset A$ and 0 otherwise. Our bound generalizes the well-known result that if $|\mathcal{A}| < 2^{s+1}$, then M has full rank $|\mathcal{A}|$. In a combinatorial setting this fact was proved by Frankl and Pach in the study of null t-designs; it can also be viewed as determining the minimum distance of the Reed–Muller codes.

Key words. set systems, restricted intersections, inclusion matrices

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1. Introduction.

Extremal problems on set systems with restricted intersections have been an important part of combinatorics in the last half-century. One of the first such results was obtained by Majumdar [11] and rediscovered by Isbell [8]. Extending earlier results of Fischer, they showed that a set system on $[n] = \{1, \ldots, n\}$ in which the intersection of any pair of sets has the same cardinality t can have at most $n+1$ sets, and if $t \neq 0$ it can have at most n sets. This is commonly known as the nonuniform Fischer inequality. (A set system is *uniform* if all of its sets have the same size.)

Throughout this paper L will denote a set of s nonnegative integers. We say that a set system $\mathcal{A}$ is *L-intersecting* if for every $A, B \in \mathcal{A}$ we have $|A \cap B| \in L$. The nonuniform Fischer inequality was further generalized by Ray-Chaudhuri and Wilson [13] and Frankl and Wilson [7], who obtained tight bounds for L-intersecting set systems, both uniform and nonuniform. They showed that an L-intersecting family on $[n]$ can have at most $\binom{n}{s}$ sets if it is uniform, and at most $\sum_{i=0}^{s}\binom{n}{i}$ sets if it is nonuniform. Frankl and Wilson also proved modular versions of these results. For p prime, they showed that the same bounds hold if the intersection sizes belong to

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L mod p and the sizes of the sets in $\mathcal{A}$ do not belong to L mod p. For an excellent account of this subject and its applications we refer the reader to [2].

In this paper, we consider the following extension of these problems. Call the set systems $\mathcal{A}_1, \ldots, \mathcal{A}_k$ *L-cross-intersecting* if for every pair of *distinct* sets A, B with $A \in \mathcal{A}_i$ and $B \in \mathcal{A}_j$ for some $i \neq j$ we have $|A \cap B| \in L$. We consider the problem of finding L-cross-intersecting systems with total size as large as possible, for each k. This can be thought of as a multicolored version of the Frankl–Ray-Chaudhuri–Wilson theorem in the following sense. We can reformulate the property of being L-intersecting as a forbidden configuration condition: we forbid any pair of sets with intersection size not lying in L. Now suppose we are given a list of set systems $\mathcal{A}_1, \ldots, \mathcal{A}_k$, which we think of as colors. We call another set system $\mathcal{F}$ multicolored if for each $F \in \mathcal{F}$ we can choose a color $\mathcal{A}_i$ containing F in such a way that each $F \in \mathcal{F}$ gets a different color. Suppose we have an integer k and some forbidden configurations $\{\mathcal{F}_\gamma : \gamma \in \Gamma\}$. The multicolored extremal problem is to choose k colors $\mathcal{A}_1, \ldots, \mathcal{A}_k$ with total size $|\mathcal{A}_1| + \cdots + |\mathcal{A}_k|$ as large as possible subject to containing no multicolored forbidden configuration $\mathcal{F}_\gamma$. The L-intersection problem has as forbidden configurations all pairs of sets with intersection sizes not belonging to L. The multicolored version of this is clearly equivalent to the L-cross-intersection problem defined above.

We refer the reader to [9] and [4] for recent results on other multicolored extremal problems and to [14] and [6] for other results on cross-intersecting families.

There are two natural examples of large *L-cross-intersecting* systems that are uniform. One is to take all of the $\mathcal{A}_i$ equal to some fixed maximum uniform L-intersecting set system, which in the case $L = \{0, 1, \ldots, s-1\}$ can have as many as $\binom{n}{s}$ sets. Another is to take one $\mathcal{A}_i$ to be as large as possible, i.e., of size $\binom{n}{\lfloor n/2 \rfloor}$, and then all the other set systems have to be empty. The following theorem shows that one of these constructions is always optimal.

THEOREM 1.1. *Let L be a set of s nonnegative integers, $n > 100s^2 \log(s+1)$, and let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be uniform set systems on $[n]$ that are L-cross-intersecting. Then*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq \max\left\{ k\binom{n}{s}, \binom{n}{\lfloor n/2 \rfloor} \right\}.$$

We get a similar picture in the nonuniform case. Again we have the example where all of the $\mathcal{A}_i$ are equal to some fixed maximum L-intersecting set system, which can have as many as $\sum_{i=0}^{s} \binom{n}{i}$ sets when $L = \{0, 1, \ldots, s-1\}$. Alternatively, if we take one $\mathcal{A}_i$ to be as large as possible, i.e., to contain all 2^n subsets of $[n]$, then the other $\mathcal{A}_i$ can contain only sets whose sizes belong to L (and are also L-cross-intersecting). In the case $L = \{0, 1, \ldots, s-1\}$ we could take one $\mathcal{A}_i$ to contain all sets and take all the other set systems to consist of the subsets of size at most $s-1$. Again we prove that one of these two possibilities is optimal.

THEOREM 1.2. *Let L be a set of s nonnegative integers, $n > 100s^2 \log s$, and let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be set systems on $[n]$ that are L-cross-intersecting. Then*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq \max\left\{ k\sum_{i=0}^{s} \binom{n}{i}, (k-1)\sum_{i=0}^{s-1} \binom{n}{i} + 2^n \right\}.$$

One can ask similar questions in a modular setting. For a prime p, we say that a set system $\mathcal{A}$ is L-intersecting mod p if the sizes of all pairwise intersections of sets belong to L mod p. We define L-cross-intersecting mod p in a similar fashion. The uniform modular Frankl–Ray-Chaudhuri–Wilson theorem states that if $\mathcal{A}$ is an r-uniform set system that is L-intersecting mod p and $r \notin L$ mod p, then $|\mathcal{A}| \le \binom{n}{s}$. The nonuniform modular version is that if $\mathcal{A}$ is L-intersecting mod p and no set in $\mathcal{A}$ has size belonging to L mod p, then $|\mathcal{A}| \le \sum_{i=0}^{s} \binom{n}{i}$. We can show the following cross-intersecting versions of these results.

THEOREM 1.3. *Suppose p is prime, L is a set of $s < p$ residues modulo p, and $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are set systems on $[n]$ that are L-cross-intersecting* mod p *such that every set $A \in \bigcup_{i=1}^{k} \mathcal{A}_i$ has $|A| = r$ for some $r \notin L \bmod p$. Let m be chosen so that $m \notin L \bmod p$ and $|n/2 - m|$ is as small as possible. Then for $n > n(s)$ sufficiently large*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \le \max\left\{ k\binom{n}{s}, \binom{n}{m} \right\}.$$

THEOREM 1.4. *Suppose p is prime, L is a set of $s < p$ residues modulo p, and $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are set systems on $[n]$ that are L-cross-intersecting* mod p *such that every set $A \in \bigcup_{i=1}^{k} \mathcal{A}_i$ has $|A| \notin L \bmod p$. Then for $n > n(s)$ sufficiently large*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \le \max\left\{ k\sum_{i=0}^{s} \binom{n}{i}, \sum_{i \notin L \bmod p} \binom{n}{i} \right\}.$$

Our proofs use two tools from linear algebra that are often useful in problems concerning set systems with restricted intersections: the original inclusion matrix method of Ray-Chaudhuri and Wilson [13] and the polynomial method as used by Alon, Babai, and Suzuki [1]. The *s^*-inclusion matrix* of a set system $\mathcal{A}$ on $[n]$ is a matrix M with rows indexed by $\mathcal{A}$ and columns by the subsets of $[n]$ of size at most s, where if $A \in \mathcal{A}$ and $B \subset [n]$ with $|B| \le s$, we define M_{AB} to be 1 if $B \subset A$ and 0 otherwise. A key ingredient of our proofs is a tight lower bound on the rank of M, which is interesting in its own right.

For $s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we define functions $f_s : \mathbb{N}_0 \to \mathbb{N}_0$ as follows. For any s we let $f_s(0) = 0$. For any $a > 0$ we let $f_0(a) = 1$. Given $s, a > 0$, write $a = 2^t + c$, where $0 \le c < 2^t$. We define $f_s(a) = \sum_{i=0}^{s} \binom{t}{i} + f_{s-1}(c)$. (Here we let $\binom{t}{i}$ be equal to $\frac{t(t-1)\cdots(t-i+1)}{i!}$ for $t \ge i \ge 1$; for $t \ge 0$ we let $\binom{t}{0} = 1$ and for other values of t and i we set $\binom{t}{i} = 0$.) The following theorem shows that these functions give a tight lower bound for the rank of the s^*-inclusion matrix over any field.

THEOREM 1.5. *If $|\mathcal{A}| = a$ and M is the s^*-inclusion matrix of $\mathcal{A}$, then* $\operatorname{rank}(M) \ge f_s(a)$. *Furthermore, there is a set system $\mathcal{A}$ for which* $\operatorname{rank}(M) = f_s(a)$.

We say that a set system $\mathcal{A}$ is *s^*-independent* if the rows of its s^*-inclusion matrix are linearly independent. It is well known (see, e.g., [2]) that if $|\mathcal{A}| < 2^{s+1}$, then $\mathcal{A}$ is s^*-independent. In a combinatorial setting this fact was proved by Frankl and Pach [5] in the study of null t-designs; it can also be viewed as determining the minimum distance of the Reed–Muller codes (see [10] for background information on codes). One can deduce this statement immediately from the above theorem together with the observation that $f_s(a) = a$ for $a < 2^{s+1}$. This observation can be proved by induction as follows. As before, write $a = 2^t + c$, where $0 \le c < 2^t$. Since $t \le s$, we have $\sum_{i=0}^{s} \binom{t}{i} = 2^t$. Then as $c < 2^s$ we have $f_{s-1}(c) = c$ (by induction), and so $f_s(a) = 2^t + c = a$, as required.

The rest of this paper is organized as follows. In the next section we prove cross-intersecting versions of the oddtown theorem and the nonuniform Fischer inequality. These are special cases of our main theorems, but have the advantage that we can prove them for all n. We set up the linear algebra machinery in section 3 and prove Theorem 1.5. Section 4 contains the proofs of Theorems 1.1 and 1.2. In section 5 we sketch how the proofs may be adapted to give the modular Theorems 1.3 and 1.4. The final section contains some concluding remarks.

We use the following notation throughout the paper. Write $[n] = \{1, \dots, n\}$. The subsets of $[n]$ of size s are denoted by $[n]^{(s)}$, and those of size at most s are denoted by $[n]^{(\le s)}$.

2. Warm-up. In this section we will prove a couple of special cases of our main results, both for illustrative purposes and because in these cases we do not need to impose the condition that n has to be sufficiently large. We recall the oddtown theorem of Berlekamp [3] (see also [2]), which is a special case of the modular Frankl–Ray-Chaudhuri–Wilson theorem. It states that if we have a collection of odd subsets of $[n]$ such that every pairwise intersection has even size, then we can have at most n sets in total. Equality can be achieved by the collection of all singleton sets, for example. We will prove the following cross-intersecting version.

THEOREM 2.1. *Suppose $\mathcal{A}_1, \dots, \mathcal{A}_k$ are set systems on $[n]$ each consisting of odd sets so that every pair of distinct sets A, B with $A \in \mathcal{A}_i$ and $B \in \mathcal{A}_j$ for some $i \neq j$ has intersection of even size. Then $\sum_{i=1}^k |\mathcal{A}_i| \le \max\{kn, 2^{n-1}\}$.*

Proof. Let $\mathcal{A}$ be the subsets of $[n]$ that belong to at least two of the $\mathcal{A}_i$ and let $\mathcal{B}$ be those sets that belong to exactly one of the $\mathcal{A}_i$. Then for any $A \in \mathcal{A}$ and $B \in \mathcal{A} \cup \mathcal{B}$ with $A \neq B$ we have $|A \cap B|$ even. We use boldface letters to indicate the incidence vectors in $\mathbb{F}_2^n$ corresponding to subsets of $[n]$; i.e., if $A \subset [n]$, then $\mathbf{A}$ denotes the vector whose ith coordinate is 1 if $i \in A$ and 0 otherwise. Let $\mathbf{1}$ denote the vector with all coordinates equal to 1. Any $A \in \mathcal{A} \cup \mathcal{B}$ has odd size, i.e., $\mathbf{A} \cdot \mathbf{A} = 1$, and for any $A \in \mathcal{A}$ and $B \in \mathcal{A} \cup \mathcal{B}$ with $A \neq B$ we have $\mathbf{A} \cdot \mathbf{B} = 0$. The sets in $\mathcal{A}$ are linearly independent as vectors, for if $\sum_{A \in \mathcal{A}} c_A \mathbf{A} = 0$, then taking the inner product with $\mathbf{A}$ for any $A \in \mathcal{A}$, we get $c_A = 0$. (In particular $|\mathcal{A}| \le n$.) The sets in $\mathcal{B}$ therefore satisfy $|\mathcal{A}|$ independent homogeneous linear constraints of the form $\mathbf{A} \cdot \mathbf{B} = 0$, as well as the inhomogeneous constraint $\mathbf{1} \cdot \mathbf{B} = 1$ (because they have odd size). If $|\mathcal{A}| = n$, then these constraints are inconsistent. Then $\mathcal{B}$ is empty and we have $\sum_{i=1}^k |\mathcal{A}_i| \le k|\mathcal{A}| \le kn$, so we are done. Otherwise the sets in $\mathcal{B}$ belong to an affine subspace of dimension $n - |\mathcal{A}| - 1$, so $|\mathcal{B}| \le 2^{n-|\mathcal{A}|-1}$ and then $\sum_{i=1}^k |\mathcal{A}_i| \le k|\mathcal{A}| + 2^{n-|\mathcal{A}|-1}$. It is easy to see that $k|\mathcal{A}| + 2^{n-|\mathcal{A}|-1}$ is a convex function of $|\mathcal{A}|$ (e.g., by differentiating twice), so as $0 \le |\mathcal{A}| \le n-1$, it is maximized at either $|\mathcal{A}| = 0$ or $|\mathcal{A}| = n-1$. Either way we have $\sum_{i=1}^k |\mathcal{A}_i| \le \max\{kn, 2^{n-1}\}$, as required. ∎

It is clear from the proof that equality can occur only when either $\mathcal{A}$ or $\mathcal{B}$ is empty. In the first case every odd set appears in exactly one $\mathcal{A}_i$. In fact, one of the $\mathcal{A}_i$ contains all the odd sets, and the other $\mathcal{A}_j$ are empty (assuming that $n \ge 3$). To see this, note that the graph on the odd sets defined by joining sets with odd intersection is connected, so if there are two of the $\mathcal{A}_j$ that are nonempty, we would find an edge of the graph going from one to the other, which is impossible. In the second case $\mathcal{A}$ must be a system of n odd sets with all pairwise intersections of even size, and $\mathcal{A}_1 = \cdots = \mathcal{A}_k = \mathcal{A}$.

We will also prove the following cross-intersecting version of the nonuniform Fischer inequality.

THEOREM 2.2. *Suppose $\mathcal{A}_1, \dots, \mathcal{A}_k$ are set systems on $[n]$ and there is some t so that for every pair of distinct sets A, B with $A \in \mathcal{A}_i$ and $B \in \mathcal{A}_j$ for some $i \neq j$, we have $|A \cap B| = t$. Then $\sum_{i=1}^k |\mathcal{A}_i| \leq \max\{k(n+1), k-1+2^n\}$. Moreover, if $t \neq 0$, then we have $\sum_{i=1}^k |\mathcal{A}_i| \leq \max\{kn, 2^n\}$.*

Proof. Let $\mathcal{A}$ be the subsets of $[n]$ that belong to at least two of the $\mathcal{A}_i$ and let $\mathcal{B}$ be those sets that belong to exactly one of the $\mathcal{A}_i$. Then for any $A \in \mathcal{A}$ and $B \in \mathcal{A} \cup \mathcal{B}$ with $A \neq B$ we have $|A \cap B| = t$.

We first consider the case when there is no set in $\mathcal{A}$ of size t. As in the previous proof we use boldface to denote incidence vectors of sets, which we now think of as belonging to $\mathbb{R}^n$. One can show that the vectors $\{\mathbf{A} : A \in \mathcal{A}\}$ are linearly independent. (This follows from the proof of the nonuniform Fischer inequality given in [2], which we briefly sketch. Let M be the matrix with rows equal to the vectors $\{\mathbf{A} : A \in \mathcal{A}\}$. Then MM^T is the $|\mathcal{A}|$ by $|\mathcal{A}|$ intersection matrix, which has each off-diagonal entry equal to t and each diagonal entry larger than t. It is not hard to show that any such matrix is nonsingular, and therefore M has rank $|\mathcal{A}|$, as required.) It also follows that $|\mathcal{A}| \leq n$.

Now for each $A \in \mathcal{A}$ we consider the linear form $f_A(x) = \mathbf{A} \cdot x - t$ in the variables $x = (x_1, \dots, x_n)$. Then f_A vanishes on all the incidence vectors of members of $\mathcal{A} \cup \mathcal{B}$, except $\mathbf{A}$ itself. Since the incidence vectors of sets $B \in \mathcal{B}$ satisfy $|\mathcal{A}|$ independent constraints $f_A(\mathbf{B}) = 0$, they lie in the intersection of an affine space of dimension $n - |\mathcal{A}|$ with the cube $\{0,1\}^n$. It follows that $|\mathcal{B}| \leq 2^{n-|\mathcal{A}|}$. To see this, pick any $B_0 \in \mathcal{B}$ and consider the vectors $\{\mathbf{B} - \mathbf{B_0} \bmod 2 : B \in \mathcal{B}\}$ in $\mathbb{F}_2^n$. If there are more than $2^{n-|\mathcal{A}|}$ such vectors, then they span an $\mathbb{F}_2$-vector space of dimension at least $n - |\mathcal{A}| + 1$. It follows that the real vectors $\{\mathbf{B} - \mathbf{B_0} : B \in \mathcal{B}\}$ span a real vector space of dimension at least $n - |\mathcal{A}| + 1$ and satisfy $|\mathcal{A}|$ independent constraints, which is impossible. Therefore $\sum_{i=1}^k |\mathcal{A}_i| \leq k|\mathcal{A}| + 2^{n-|\mathcal{A}|} \leq \max\{kn, 2^n\}$ by convexity. This proves both parts of the theorem under the assumption that there is no set in $\mathcal{A}$ of size t.

Now suppose there is some $A_0 \in \mathcal{A}$ with $|A_0| = t$. Then all sets in $\mathcal{A} \cup \mathcal{B}$ contain A_0. Repeating the above argument, we see that the vectors $\{\mathbf{A} : A \in \mathcal{A} \backslash A_0\}$ are linearly independent, so $|\mathcal{A} \backslash A_0| \leq n$ and $|\mathcal{B}| \leq 2^{n-|\mathcal{A} \backslash A_0|}$. If $|\mathcal{A} \backslash A_0| = n$, then $\mathcal{B}$ must be empty. For if there is $B \in \mathcal{B}$, then $\mathcal{A} \cup B$ contains $n+2$ sets with all pairwise intersections having size t, which is impossible. In this case we have $\sum_{i=1}^k |\mathcal{A}_i| \leq k|\mathcal{A}| \leq k(n+1)$. In the case $\mathcal{A} = \{A_0\}$ we have $|\mathcal{B}| \leq 2^n - 1$ (since $A_0 \notin \mathcal{B}$) so $\sum_{i=1}^k |\mathcal{A}_i| \leq k|\mathcal{A}| + |\mathcal{B}| \leq k + 2^n - 1$. Otherwise we have $2 \leq |\mathcal{A}| \leq n$ and

$$\sum_{i=1}^k |\mathcal{A}_i| \leq k|\mathcal{A}| + 2^{n-|\mathcal{A}|+1} \leq \max\{2k + 2^{n-1}, kn + 2\}$$

by convexity. Now $kn + 2 \leq k(n+1)$ for $k \geq 2$, and if $2k + 2^{n-1} > k(n+1)$, we have $k < 2^{n-1}/(n-1)$, so $(k - 1 + 2^n) - (2k + 2^{n-1}) = 2^{n-1} - (k+1) \geq 0$. (We are ignoring the case $n = 1$, for which the theorem is trivially true.) We deduce that $\sum_{i=1}^k |\mathcal{A}_i| \leq \max\{k(n+1), k-1+2^n\}$, which is the first part of the theorem.

To get the improvement when $t \neq 0$, consider the set systems $\mathcal{A}_i' = \{A \backslash A_0 : A \in \mathcal{A}_i\}$. These are defined on a set of size $n - t$, and for every pair of distinct sets A, B

with $A \in \mathcal{A}'_i$ and $B \in \mathcal{A}'_j$ for some $i \neq j$ we have $|A \cap B| = 0$. By the first part of the theorem we have

$$\sum_{i=1}^{k} |\mathcal{A}_i| = \sum_{i=1}^{k} |\mathcal{A}'_i| \leq \max\{k(n-t+1), k-1+2^{n-t}\} \leq \max\{kn, 2^n\}$$

and we are done. $\square$

3. Tools from linear algebra. This section contains the linear algebra components of our argument, which are a tight lower bound on the rank of the s^*-inclusion matrix and the polynomial method.

3.1. The rank of the inclusion matrix. For a set system $\mathcal{A}$ on $[n]$, the *s^*-inclusion matrix* M has rows indexed by $\mathcal{A}$ and columns indexed by the subsets of $[n]$ of size at most s (including the empty set), where if $A \in \mathcal{A}$ and $B \subset [n]$ with $|B| \leq s$ we define M_{AB} to be 1 if $B \subset A$ and 0 otherwise. In this subsection we will prove a tight lower bound for the rank of this matrix, which is of interest in its own right.

Let $V = \mathbb{F}^{\sum_{i=0}^{s} \binom{n}{i}}$, where $\mathbb{F}$ is some field, and denote its standard basis by e_Z, where Z ranges over subsets of $[n]$ of size at most s. Given a set $A \in \mathcal{A}$, we define the *s^*-inclusion vector*

$$v_A^s = \sum_{Z \subset A, |Z| \leq s} e_Z.$$

These are the row vectors of the s^*-inclusion matrix. We define $V_{\mathcal{A}}^s$ to be the row space, i.e., the subspace of V spanned by the vectors $\{v_A^s : A \in \mathcal{A}\}$. Note that the rank of the s^*-inclusion matrix is equal to the dimension of $V_{\mathcal{A}}^s$.

Throughout we adopt the following standard convention for binomial coefficients. We let $\binom{t}{i}$ be equal to $\frac{t(t-1)\cdots(t-i+1)}{i!}$ for $t \geq i \geq 1$; for $t \geq 0$ we let $\binom{t}{0} = 1$ and for other values of t and i we set $\binom{t}{i} = 0$. We will use the following well-known identities, which follow easily from the fact that $\binom{t+1}{s} = \binom{t}{s} + \binom{t}{s-1}$:

$$\sum_{i=0}^{s} \binom{t+1}{i} = \sum_{i=0}^{s} \binom{t}{i} + \sum_{i=0}^{s-1} \binom{t}{i}, \tag{1}$$

$$\binom{t+1}{s} = \sum_{i=0}^{s} \binom{t-i}{s-i}. \tag{2}$$

For $s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we define functions $f_s : \mathbb{N}_0 \to \mathbb{N}_0$ as follows. For any s we let $f_s(0) = 0$. For any $a > 0$ we let $f_0(a) = 1$. Given $s, a > 0$, write $a = 2^t + c$, where $0 \leq c < 2^t$. We define $f_s(a) = \sum_{i=0}^{s} \binom{t}{i} + f_{s-1}(c)$. We will show that if $|\mathcal{A}| = a$, then $\dim V_{\mathcal{A}}^s \geq f_s(a)$. First we need some inequalities for the functions f_s.

LEMMA 3.1. *If $a < 2^t$, then $f_s(a) - f_{s-1}(a) \leq \binom{t}{s}$.*

Proof. Write $a = 2^{t_1} + 2^{t_2} + \cdots$, where $t > t_1 > t_2 > \cdots$. Then $t_i \leq t - i$. We have

$$f_s(a) - f_{s-1}(a) = \sum_{i \geq 1} \binom{t_i}{s+1-i} \leq \sum_{i \geq 1} \binom{t-i}{s+1-i} = \binom{t}{s},$$

where we use (2). $\square$

LEMMA 3.2. *If $a \geq b$, then $f_s(a+b) \leq f_s(a) + f_{s-1}(b)$ for $s \geq 1$.*

Proof. We argue by induction on $a+b$ and s. Write $a = 2^t + c$, where $0 \leq c < 2^t$. First we check the base cases of the induction. The statement is trivial when $b = 0$, so we can suppose $b > 0$. When $s = 1$ we have two cases. First suppose that $c = 0$. Then $f_1(a) = t+1$. Since $0 < b \leq a = 2^t$ we have $2^t < a+b \leq 2^{t+1}$, so $f_1(a+b) = t+2$. Since $f_0(b) = 1$ we have $f_1(a+b) = f_1(a) + f_0(b)$. Next suppose that $c > 0$. Then $f_1(a) = t+2$. Since $b \leq a < 2^{t+1}$ we have $a+b < 2^{t+2}$, so $f_1(a+b) \leq t+3 = f_1(a) + f_0(b)$, as required.

Now suppose that $s > 1$ and that the statement is true with s replaced by $s' < s$ and also for the same s applied to a pair a', b' with $a' + b' < a + b$. Note in particular that for any $s' < s$ and any x, y we have $f_{s'}(x+y) \leq f_{s'}(x) + f_{s'}(y)$. For we may suppose that $x \geq y$, and then by induction $f_{s'}(x+y) \leq f_{s'}(x) + f_{s'-1}(y) \leq f_{s'}(x) + f_{s'}(y)$.

Consider first the case that $b < 2^t - c$. Then $a + b < 2^{t+1}$. We have $f_s(a) = \sum_{i=0}^{s} \binom{t}{i} + f_{s-1}(c)$ and $f_s(a+b) = \sum_{i=0}^{s} \binom{t}{i} + f_{s-1}(b+c)$, so $f_s(a) + f_{s-1}(b) - f_s(a+b) = f_{s-1}(b) + f_{s-1}(c) - f_{s-1}(b+c) \geq 0$, by the observation in the previous paragraph.

Next we consider the case that $b \geq 2^t$, say $b = 2^t + d$, where $0 \leq d \leq c < 2^t$. Then $f_{s-1}(b) = \sum_{i=0}^{s-1} \binom{t}{i} + f_{s-2}(d)$. Since $2^{t+1} \leq a + b < 2^{t+2}$ we have $f_s(a+b) = \sum_{i=0}^{s} \binom{t+1}{i} + f_{s-1}(c+d)$. Using (1) we get $f_s(a) + f_{s-1}(b) - f_s(a+b) = f_{s-1}(c) + f_{s-2}(d) - f_{s-1}(c+d) \geq 0$ by induction (since $c \geq d$).

Finally, we are left with the case $2^t - c \leq b < 2^t$. We have $2^{t+1} \leq a+b < 2^{t+2}$, so $f_s(a+b) = \sum_{i=0}^{s} \binom{t+1}{i} + f_{s-1}(b+c-2^t)$. Since $2^t \leq b+c < 2^{t+1}$ we have $f_s(b+c) = \sum_{i=0}^{s} \binom{t}{i} + f_{s-1}(b+c-2^t)$, so $f_s(a+b) - f_s(b+c) = \sum_{i=0}^{s} \binom{t+1}{i} - \sum_{i=0}^{s} \binom{t}{i} = \sum_{i=0}^{s-1} \binom{t}{i}$. Then $f_s(a) + f_{s-1}(b) - f_s(a+b) = \sum_{i=0}^{s} \binom{t}{i} + f_{s-1}(c) + f_{s-1}(b) - f_s(b+c) - \sum_{i=0}^{s-1} \binom{t}{i} = f_{s-1}(b) + f_{s-1}(c) - f_s(b+c) + \binom{t}{s}$. If $b \geq c$, then by Lemma 3.1 we have $f_{s-1}(b) + \binom{t}{s} \geq f_s(b)$, so $f_s(a) + f_{s-1}(b) - f_s(a+b) \geq f_s(b) + f_{s-1}(c) - f_s(b+c) \geq 0$ by induction (since $b + c < a + b$). Similarly, if $c \geq b$ we have $f_s(a) + f_{s-1}(b) - f_s(a+b) \geq f_s(c) + f_{s-1}(b) - f_s(b+c) \geq 0$. In all cases we are done. □

Proof of Theorem 1.5. We argue by induction on a and s. The result is trivial if $a = 0$, $a = 1$, or $s = 0$, so we suppose that $a \geq 2$ and $s \geq 1$. Let $\mathcal{A}$ be a set system on a set X with $|\mathcal{A}| = a$. Pick $x \in X$ and let $\mathcal{A}_x = \{A \in \mathcal{A} : x \in A\}$, $\mathcal{A}_{\overline{x}} = \mathcal{A} \backslash \mathcal{A}_x$. Write $a_x = |\mathcal{A}_x|$ and $a_{\overline{x}} = |\mathcal{A}_{\overline{x}}|$. We can choose x so that $0 < a_x, a_{\overline{x}} < a$.

Let M be the matrix whose rows consist of the s^*-inclusion vectors of sets in $\mathcal{A}$, with the following order of rows and columns. The rows are ordered in such a way that those corresponding to sets in $\mathcal{A}_{\overline{x}}$ precede those in $\mathcal{A}_x$. The columns are ordered into three groups; the first group is those columns given by entries in the s^*-inclusion vectors corresponding to sets in $X^{(\leq s-1)}$ not containing x, the second group is those corresponding to sets in $X^{(s)}$ not containing x, and the third group is those corresponding to sets in $X^{(\leq s)}$ that contain x; each of the three groups is ordered lexicographically. Thus M has the structure

$$\begin{pmatrix} M_1 & M_2 & 0 \\ M_3 & M_4 & M_3 \end{pmatrix}$$

for some matrices M_1, M_2, M_3, M_4. Note that $rk(M) = \dim V^s_{\mathcal{A}}$.

Consider the system $\mathcal{A}' = \{A \Delta \{x\} : A \in \mathcal{A}\}$, where Δ denotes symmetric difference. Since $\mathcal{A}'_x = \{A \cup \{x\} : A \in \mathcal{A}_{\overline{x}}\}$ and $\mathcal{A}'_{\overline{x}} = \{A \backslash \{x\} : A \in \mathcal{A}_x\}$, the matrix corresponding to $\mathcal{A}'$ (with respect to the same order on rows and columns) is

$$M' = \begin{pmatrix} M_3 & M_4 & 0 \\ M_1 & M_2 & M_1 \end{pmatrix}.$$

Note that M' can be obtained from M by row and column operations. In terms of the block structure, we swap the two rows, subtract the first column from the third column, then multiply the third column by -1. This shows that $rk(M') = rk(M)$, i.e., $\dim V^s_{\mathcal{A}'} = \dim V^s_{\mathcal{A}}$. Therefore, we can suppose without loss of generality that $a_{\overline{x}} \geq a_x$.

Now note that

$$\dim V^s_{\mathcal{A}} = rk(M) \geq rk\Big(M_1 \;\; M_2 \Big) + rk(M_3) = \dim V^s_{\mathcal{A}_{\overline{x}}} + \dim V^{s-1}_{\mathcal{A}_x}.$$

Since $0 < a_x, a_{\overline{x}} < a$ we can apply induction to get $\dim V^s_{\mathcal{A}_{\overline{x}}} \geq f_s(a_{\overline{x}})$ and $\dim V^{s-1}_{\mathcal{A}_x} \geq f_{s-1}(a_x)$. Since $a_{\overline{x}} \geq a_x$ and $a_{\overline{x}} + a_x = a$, by Lemma 3.2 we have $\dim V^s_{\mathcal{A}} \geq f_s(a_{\overline{x}}) + f_{s-1}(a_x) \geq f_s(a)$. This proves the first part of the theorem.

Finally we note that the bound on dimension is tight. To show this, we prove by induction on a and s that if $a = 2^t + c$, with $c < 2^t$, then there is a set system $\mathcal{A}$ on $[t+1]$ with $|\mathcal{A}| = a$ and $\dim V^s_{\mathcal{A}} = f_s(a)$. This is clear if $a = 1$ or if $s = 0$, so we suppose $a \geq 2$ and $s \geq 1$. By induction we can find a set system $\mathcal{B}$ on $[t]$ with c sets so that $\dim V^{s-1}_{\mathcal{B}} = f_{s-1}(c)$. Let $\mathcal{B}' = \big\{ B \cup \{t+1\} \,:\, B \in \mathcal{B} \big\}$ and $\mathcal{A} = \mathcal{P}([t]) \cup \mathcal{B}'$, i.e., $\mathcal{A}$ consists of all subsets of $[t]$ together with each set of $\mathcal{B}$ with the element $t+1$ added. In the s^*-inclusion matrix for $\mathcal{A}$, the block with rows corresponding to $\mathcal{P}([t])$ and columns corresponding to $[t]^{(\leq s)}$ has full rank $\sum_{i=0}^s \binom{t}{i}$. Any extra rank in the matrix can come only from the block with rows corresponding to $\mathcal{B}'$ and columns corresponding to $\big\{ X \cup \{t+1\} \,:\, X \in [t]^{(\leq s-1)} \big\}$, and this has rank $\dim V^{s-1}_{\mathcal{B}} = f_{s-1}(c)$. Therefore $\dim V^s_{\mathcal{A}} = \sum_{i=0}^s \binom{t}{i} + f_{s-1}(c) = f_s(a)$, as required. □

We note the following properties of the function $f_s(a)$ for future reference:

$$f_s(a) \geq \sum_{i=0}^{s} \binom{\lfloor \log_2 a \rfloor}{i}; \tag{3}$$

$$\text{If } 2^n - 2^{n-s} < a \leq 2^n, \text{ then } f_s(a) = f_s(2^n) = \sum_{i=0}^{s} \binom{n}{i}. \tag{4}$$

To see the second property note that we can write $a = 2^{n-1} + 2^{n-2} + \cdots + 2^{n-s} + b$, with $0 < b < 2^{n-s}$, and so $f_s(a) = \sum_{j=0}^s \sum_{i \geq 0} \binom{n-1-j}{s-j-i} = \sum_{i \geq 0} \binom{n}{s-i}$, where we use (2).

3.2. The polynomial method. In this subsection we summarize the particular application of the polynomial method that we need in the following lemma.

LEMMA 3.3. (i) *Suppose $\mathcal{A}$ is an L-intersecting family of sets and that $|A| \notin L$ for all $A \in \mathcal{A}$. Then the s^*-inclusion vectors $\{v^s_A : A \in \mathcal{A}\}$ are linearly independent over $\mathbb{R}$.*

(ii) *Suppose also that $\mathcal{B}$ is a set system such that $|A \cap B| \in L$ for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Then no vector v^s_B with $B \in \mathcal{B}$ lies in $V^s_{\mathcal{A}}$.*

Proof. We use boldface to denote the incidence vector corresponding to a subset of $[n]$. For a set A we define the polynomial $f_A(x) = \prod_{\ell \in L}(x \cdot \mathbf{A} - \ell)$. We will restrict $x = (x_1, \ldots, x_n)$ to range over $\{0,1\}$-vectors, so by repeatedly replacing any occurrence of x_i^2 by x_i we can represent $f_A(x)$ by a multilinear polynomial $\sum_{X \in [n]^{(\leq s)}} c_{A,X} \prod_{i \in X} x_i$. Let $w_A = \sum_{X \in [n]^{(\leq s)}} c_{A,X} e_X$, where we recall that $\{e_X : X \in [n]^{\leq s}\}$ denotes the standard basis of $V = \mathbb{R}^{\sum_{i=0}^s \binom{n}{i}}$. Then by definition we have $f_A(\mathbf{B}) = w_A \cdot v^s_B$.

Note that $f_A(\mathbf{B}) = 0$ if and only if $|A \cap B| \in L$, so for $A, B \in \mathcal{A}$ we have $w_A \cdot v^s_B = f_A(\mathbf{B}) = 0$ if and only if $A \neq B$. Now if $\sum_{A \in \mathcal{A}} t_A v^s_A = 0$, then taking the inner product of this identity with w_A for each $A \in \mathcal{A}$ we obtain that $t_A = 0$ for every A, which proves part (i) of the lemma. Also, if $B \in \mathcal{B}$ and $v^s_B = \sum_{A\in\mathcal{A}} t_A v^s_A$, then taking the inner product with w_A, we again see that $t_A = 0$ for each $A \in \mathcal{A}$. This gives $v^s_B = \mathbf{0}$, a contradiction that proves part (ii) of the lemma. ∎

The same proof shows that this result holds with $\mathbb{R}$ replaced by the field with p elements (for some prime p) provided that $|A| \notin L \bmod p$ for all $A \in \mathcal{A}$.

4. Proofs of the main theorems. We start by proving Theorem 1.1, which we recall states that if $n > 100s^2 \log(s+1)$ and $\mathcal{A}_1, \dots, \mathcal{A}_k$ are uniform set systems on $[n]$ that are L-cross-intersecting, then $\sum_{i=1}^k |\mathcal{A}_i| \le \max\{k\binom{n}{s}, \binom{n}{\lfloor n/2\rfloor}\}$. First we need the following estimate on the middle binomial coefficients.

LEMMA 4.1. $\frac{2^n}{2\sqrt{n}} \le \binom{n}{\lfloor n/2\rfloor} \le \frac{2^n}{\sqrt{n}}$.

Proof. Let $g(n) = 2^{-n}\sqrt{n}\binom{n}{\lfloor n/2\rfloor}$. We want to prove that $1/2 \le g(n) \le 1$. This is easily verified for $n = 1$ and $n = 2$. We see that $g(n+2) > g(n)$: for even n we have $\frac{g(2m)}{g(2m-2)} = \left(1 - \frac{1}{2m}\right)\sqrt{\frac{m}{m-1}} > 1$, as $\left(1 - \frac{1}{2m}\right)^2 - \frac{m-1}{m} = \frac{1}{4m^2} > 0$, and for odd n we have $\frac{g(2m+1)}{g(2m-1)} = (1 - \frac{1}{2(m+1)})\sqrt{\frac{2m+1}{2m-1}} > 1$, as $(1-\frac{1}{2(m+1)})^2 - \frac{2m-1}{2m+1} > \frac{1}{4(m+1)^2} > 0$. Now $g(n) \ge 1/2$ follows for all n by induction. For the upper bound we use the Stirling approximation $n! \sim \sqrt{2\pi n}\, n^n e^{-n}$, from which it follows that $g(n) \to \sqrt{2/\pi}$ as $n \to \infty$. Since $g(2m)$ and $g(2m+1)$ are increasing sequences we have $g(n) \le \sqrt{2/\pi} < 1$. ∎

Proof of Theorem 1.1. Let $k_c = \lfloor \binom{n}{\lfloor n/2\rfloor} / \binom{n}{s} \rfloor$. Then for $k \le k_c$ we want to show that $\sum_{i=1}^k |\mathcal{A}_i| \le \binom{n}{\lfloor n/2\rfloor}$ and for $k > k_c$ we want to show that $\sum_{i=1}^k |\mathcal{A}_i| \le k\binom{n}{s}$. Note that it suffices to prove these two statements in the specific cases $k = k_c$ and $k = k_c+1$. Then the case $k = k_c$ clearly implies that for $k \le k_c$ we have $\sum_{i=1}^k |\mathcal{A}_i| \le \binom{n}{\lfloor n/2\rfloor}$. Also the case $k > k_c + 1$ follows by induction. If we ignore the smallest $\mathcal{A}_i$ we are left with $k-1$ L-cross-intersecting set systems, which have total size at most $(k-1)\binom{n}{s}$, so the total size of all k systems is at most $\frac{k}{k-1} \cdot (k-1)\binom{n}{s} = k\binom{n}{s}$.

By the above remark we can assume that $k = k_c$ or $k = k_c + 1$. Suppose that $\mathcal{A}_1, \dots, \mathcal{A}_k$ are L-cross-intersecting r-uniform set systems with $\sum_{i=1}^k |\mathcal{A}_i| \ge \max\{k\binom{n}{s}, \binom{n}{\lfloor n/2\rfloor}\}$. Note that we can assume $r \notin L$. Let $\mathcal{A}$ be the subsets of $[n]$ that belong to at least two of the $\mathcal{A}_i$ and let $\mathcal{B}$ be those subsets that belong to exactly one of the $\mathcal{A}_i$. Since the $\mathcal{A}_i$ are L-cross-intersecting, for any $A \in \mathcal{A}$ and $B \in \mathcal{A} \cup \mathcal{B}$ we have $|A \cap B| \in L$. It follows from the Ray-Chaudhuri–Wilson theorem that $|\mathcal{A}| \le \binom{n}{s}$, and if $\mathcal{B} \ne \emptyset$, then $|\mathcal{A}| < \binom{n}{s}$ (as we can add one set from $\mathcal{B}$ to $\mathcal{A}$ and still have an L-intersecting family). From Lemma 3.3 we know that the s^*-inclusion vectors $\{v^s_A : A \in \mathcal{A}\}$ are linearly independent over $\mathbb{R}$, i.e., they form a basis of $V^s_{\mathcal{A}}$, and we also see that no vector v^s_B with $B \in \mathcal{B}$ lies in $V^s_{\mathcal{A}}$. We conclude that

$$|\mathcal{A}| + \dim V^s_{\mathcal{B}} \le \sum_{i=0}^{s} \binom{n}{i}. \tag{5}$$

Note that we can assume that both $\mathcal{A}$ and $\mathcal{B}$ are nonempty. For if $\mathcal{A} = \emptyset$ we have $\sum_{i=1}^k |\mathcal{A}_i| \le |\mathcal{B}| \le \binom{n}{r} \le \binom{n}{\lfloor n/2\rfloor}$ and if $\mathcal{B} = \emptyset$ we have $\sum_{i=1}^k |\mathcal{A}_i| \le k|\mathcal{A}| \le k\binom{n}{s}$; in either case we are done. Thus we cannot have $|\mathcal{A}| = \binom{n}{s}$ (for then $\mathcal{B} = \emptyset$), so we have $|\mathcal{A}| \le \binom{n}{s} - 1$. Since

$$k\binom{n}{s} \le \sum_{i=1}^{k} |\mathcal{A}_i| \le k|\mathcal{A}| + |\mathcal{B}| \le k\left(\binom{n}{s} - 1\right) + |\mathcal{B}|,$$

we have $|\mathcal{B}| \ge k > (\binom{n}{\lfloor n/2 \rfloor}/\binom{n}{s}) - 1$. By Lemma 4.1 we have $|\mathcal{B}| > 2^n/n^{s+1}$, so by Theorem 1.5 and (3)

$$\dim V_{\mathcal{B}}^s \ge f_s(|\mathcal{B}|) \ge \sum_{i=0}^{s} \binom{\lfloor \log_2 |\mathcal{B}| \rfloor}{i} \ge \sum_{i=0}^{s} \binom{\lfloor n - (s+1)\log_2 n \rfloor}{i}.$$

Now from (5) we get

$$\begin{aligned} |\mathcal{A}| &\le \sum_{i=0}^{s} \binom{n}{i} - \dim V_{\mathcal{B}}^s \le \sum_{i=0}^{s} \left(\binom{n}{i} - \binom{n - \lceil (s+1)\log_2 n \rceil}{i} \right) \\ &\le \lceil (s+1)\log_2 n \rceil \sum_{i=0}^{s-1} \binom{n-1}{i}, \end{aligned}$$

where we use the inequality $\binom{n}{i} - \binom{n-t}{i} = \sum_{j=1}^{t} \left(\binom{n+1-j}{i} - \binom{n-j}{i} \right) = \sum_{j=1}^{t} \binom{n-j}{i-1} \le t\binom{n-1}{i-1}$. Therefore

$$\begin{aligned} |\mathcal{B}| &\ge \sum_{i=1}^{k} |\mathcal{A}_i| - k|\mathcal{A}| \ge \binom{n}{\lfloor n/2 \rfloor} - \left(\frac{\binom{n}{\lfloor n/2 \rfloor}}{\binom{n}{s}} + 1 \right) \lceil (s+1)\log_2 n \rceil \sum_{i=0}^{s-1} \binom{n-1}{i} \\ &> \left(1 - \frac{3s(s+1)\log_2 n}{2n} \right) \binom{n}{\lfloor n/2 \rfloor}. \end{aligned} \tag{6}$$

In particular we easily see that $|\mathcal{B}| > \binom{n}{\lfloor n/3 \rfloor}$, so $n/3 < r < 2n/3$. Recalling that $\mathcal{A} \ne \emptyset$, we now consider any $A \in \mathcal{A}$. For any $B \in \mathcal{B}$ the size of its intersection with A belongs to L, so we get

$$\begin{aligned} |\mathcal{B}| &\le \sum_{\ell \in L} \binom{r}{\ell}\binom{n-r}{r-\ell} \le s\binom{r}{\lfloor r/2 \rfloor}\binom{n-r}{\lfloor (n-r)/2 \rfloor} \\ &< s \cdot \frac{2^r}{\sqrt{r}} \cdot \frac{2^{n-r}}{\sqrt{n-r}} < \frac{2^n s}{n/3} < \frac{6s}{\sqrt{n}} \binom{n}{\lfloor n/2 \rfloor}, \end{aligned}$$

where we use Lemma 4.1. Comparing with (6) we get

$$\frac{6s}{\sqrt{n}} > |\mathcal{B}| / \binom{n}{\lfloor n/2 \rfloor} > 1 - \frac{3s(s+1)\log_2 n}{2n}.$$

Since $n > 100s^2 \log(s+1)$, this gives the required contradiction. □

It is clear from the proof that equality can occur only when either $\mathcal{A}$ or $\mathcal{B}$ is empty. In the first case every set of size $\lfloor n/2 \rfloor$ appears in exactly one $\mathcal{A}_i$. In fact, one of the $\mathcal{A}_i$ contains all the sets of size $\lfloor n/2 \rfloor$, and the other $\mathcal{A}_j$ are empty (which can be proved as in the remark after Theorem 2.1). In the second case $\mathcal{A}$ must be a maximum uniform L-intersecting family, and $\mathcal{A}_1 = \cdots = \mathcal{A}_k = \mathcal{A}$.

Next we prove Theorem 1.2, which we recall states that if $n > 100s^2 \log s$ and $\mathcal{A}_1, \dots, \mathcal{A}_k$ are set systems on $[n]$ that are L-cross-intersecting, then $\sum_{i=1}^{k} |\mathcal{A}_i| \le \max\{k \sum_{i=0}^{s} \binom{n}{i}, (k-1)\sum_{i=0}^{s-1} \binom{n}{i} + 2^n\}$.

Proof of Theorem 1.2. We will assume that $s > 1$, as the case $s = 1$ is covered by Theorem 2.2. Suppose that $\mathcal{A}_1, \dots, \mathcal{A}_k$ are L-cross-intersecting set systems with

$$\sum_{i=1}^{k} |\mathcal{A}_i| \ge \max\left\{ k \sum_{i=0}^{s} \binom{n}{i}, (k-1)\sum_{i=0}^{s-1} \binom{n}{i} + 2^n \right\}.$$

Let $\mathcal{A}$ be the sets that belong to at least two of the $\mathcal{A}_i$ and let $\mathcal{B}$ be those sets that belong to exactly one of the $\mathcal{A}_i$.

Write $k_c = \lfloor \frac{2^n - \sum_{i=0}^{s-1}\binom{n}{i}}{\binom{n}{s}} \rfloor$. Then for $k \le k_c$ we want to show that $\sum_{i=1}^{k} |\mathcal{A}_i| \le (k-1)\sum_{i=0}^{s-1}\binom{n}{i} + 2^n$ and for $k > k_c$ we want to show that $\sum_{i=1}^{k} |\mathcal{A}_i| \le k\sum_{i=0}^{s}\binom{n}{i}$. Note that it suffices to prove these two statements in the specific cases $k = k_c$ and $k = k_c + 1$. As for Theorem 1.1, the case $k > k_c + 1$ follows by induction. We can prove the case $k < k_c$ by induction on k (decreasing from k_c) with the following argument. Suppose for the sake of contradiction that we have $\sum_{i=1}^{k} |\mathcal{A}_i| > (k-1)\sum_{i=0}^{s-1}\binom{n}{i} + 2^n$. Then clearly $|\mathcal{A}| > \sum_{i=0}^{s-1}\binom{n}{i}$. Let $\mathcal{A}_{k+1} = \mathcal{A}$. Then $\mathcal{A}_1, \dots, \mathcal{A}_{k+1}$ are L-cross-intersecting and $\sum_{i=1}^{k+1} |\mathcal{A}_i| > k\sum_{i=0}^{s-1}\binom{n}{i} + 2^n$, which contradicts our induction hypothesis. Therefore we can assume that $k = k_c$ or $k = k_c + 1$.

Since $\mathcal{A}$ is L-intersecting we have $|\mathcal{A}| \le \sum_{i=0}^{s}\binom{n}{i}$ by the Frankl–Wilson theorem, and if $\mathcal{B} \neq \emptyset$, then $|\mathcal{A}| < \sum_{i=0}^{s}\binom{n}{i}$ (similar to the previous theorem). Let $\mathcal{A}_L = \{A \in \mathcal{A} : |A| \in L\}$ and $\mathcal{A}_{\overline{L}} = \{A \in \mathcal{A} : |A| \notin L\}$. Let ℓ be the largest element of L. Then $\mathcal{A}_L$ is $(L\backslash\ell)$-intersecting, so $|\mathcal{A}_L| \le \sum_{i=0}^{s-1}\binom{n}{i}$. Exactly as in the proof of Theorem 1.1 we see that the s^*-inclusion vectors $\{v_A^s : A \in \mathcal{A}_{\overline{L}}\}$ form a basis of $V^s_{\mathcal{A}_{\overline{L}}}$ and no vector v_B^s with $B \in \mathcal{B}$ lies in $V^s_{\mathcal{A}_{\overline{L}}}$. This shows that $|\mathcal{A}_{\overline{L}}| + \dim V^s_{\mathcal{B}} \le \sum_{i=0}^{s}\binom{n}{i}$.

We can assume that $\mathcal{B}$ is nonempty, for otherwise $\sum_{i=1}^{k} |\mathcal{A}_i| \le k|\mathcal{A}| \le k\sum_{i=0}^{s}\binom{n}{i}$, and we are done. We can also assume that $\mathcal{A}_{\overline{L}}$ is nonempty, for otherwise we have $|\mathcal{A}| = |\mathcal{A}_{\mathcal{L}}| \le \sum_{i=0}^{s-1}\binom{n}{i}$ and so

$$\sum_{i=1}^{k} |\mathcal{A}_i| \le k|\mathcal{A}| + |\mathcal{B}| \le k|\mathcal{A}| + (2^n - |\mathcal{A}|) \le (k-1)\sum_{i=0}^{s-1}\binom{n}{i} + 2^n,$$

and again we are done. We cannot have $|\mathcal{A}| = \sum_{i=0}^{s}\binom{n}{i}$ (for then $\mathcal{B} = \emptyset$) so we have $|\mathcal{A}| \le \sum_{i=0}^{s}\binom{n}{i} - 1$. It follows that $|\mathcal{B}| \ge k > \frac{2^n}{\binom{n}{s}} - 2 > \frac{2^n}{n^s}$, and so by Theorem 1.5 $\dim V^s_{\mathcal{B}} > \sum_{i=0}^{s}\binom{\lfloor n - s\log_2 n\rfloor}{i}$. Now we get

$$\begin{aligned} |\mathcal{A}_{\overline{L}}| &\le \sum_{i=0}^{s}\binom{n}{i} - \dim V^s_{\mathcal{B}} < \sum_{i=0}^{s}\left(\binom{n}{i} - \binom{n - \lceil s\log_2 n\rceil}{i}\right) \\ &\le \lceil s\log_2 n\rceil \sum_{i=0}^{s}\binom{n-1}{i-1} < \frac{2s^2\log_2 n}{n}\binom{n}{s}. \end{aligned}$$

Choose an integer t so that $2^{-(t+1)} \le |\mathcal{A}_{\overline{L}}|/\binom{n}{s} \le 2^{-t}$. Since $n \ge 100s^2\log s$ and $s \ge 2$, from the above inequality we have $t \ge 2$. Also, since $\mathcal{A}_{\overline{L}}$ is nonempty we have $t \le \log_2\binom{n}{s} < s\log n$.

Since $|\mathcal{A}| = |\mathcal{A}_{\overline{L}}| + |\mathcal{A}_L| \le 2^{-t}\binom{n}{s} + \sum_{i=0}^{s-1}\binom{n}{i}$ we see that

$$\begin{aligned} |\mathcal{B}| &\ge \sum_{i=1}^{k} |\mathcal{A}_i| - k|\mathcal{A}| > (k-1)\sum_{i=0}^{s-1}\binom{n}{i} + 2^n - k\left(2^{-t}\binom{n}{s} + \sum_{i=0}^{s-1}\binom{n}{i}\right) \\ &> 2^n - \sum_{i=0}^{s-1}\binom{n}{i} - \left(\frac{2^n}{\binom{n}{s}} + 1\right)2^{-t}\binom{n}{s} > 2^n - 2^{n-t+1}, \end{aligned}$$

where for the last inequality we use the upper bound on t. We cannot have $t \ge s+1$, for then (4) gives $\dim V^s_{\mathcal{B}} = \sum_{i=0}^{s}\binom{n}{i}$ and then $\mathcal{A}_{\overline{L}}$ must be empty, which is a contradiction. We deduce that $t \le s$. Now by Theorem 1.5 and (2) we have

$$\begin{aligned}\dim V_{\mathcal{B}}^s > f_s(2^n - 2^{n-t+1}) &= f_s(2^{n-1} + 2^{n-2} + \cdots + 2^{n-t+1}) \\ &= \sum_{j=0}^{t-2} \sum_{i \ge 0} \binom{n-1-j}{s-j-i} \\ &= \sum_{i\ge 0}\left(\sum_{j\ge 0} \binom{n-1-j}{s-i-j} - \sum_{j \ge 0} \binom{n-t-j}{s-i-t+1-j}\right) \\ &= \sum_{i \ge 0} \left(\binom{n}{s-i} - \binom{n-t+1}{s-i-t+1} \right).\end{aligned}$$

Therefore

$$2^{-(t+1)}\binom{n}{s} \le |\mathcal{A}_{\overline{L}}| \le \sum_{i=0}^{s} \binom{n}{i} - \dim V_{\mathcal{B}}^s \le \sum_{i \ge 0} \binom{n-t+1}{s-i-t+1} \le 2\binom{n-t+1}{s-t+1}.$$

We deduce that $2^{t+2} \ge \binom{n}{s}/\binom{n-t+1}{s-t+1} \ge (n/s)^{t-1}$, and so $n/s \le 2^{1+3/(t-1)} \le 16$, which gives the required contradiction. □

From the proof we see that equality can occur only when either $\mathcal{B}$ or $\mathcal{A}_{\overline{L}}$ is empty. In the first case we have $\mathcal{A}_i = \mathcal{A}$ equal to an L-intersecting family of size $\sum_{i=0}^{s} \binom{n}{i}$. It was shown by Qian and Ray-Chaudhuri [12] that this is only possible when $L = \{0, 1, \ldots, s-1\}$ and $\mathcal{A} = [n]^{(\le s)}$. In the second case $|\mathcal{B}| = 2^n$ and $\mathcal{A} = \mathcal{A}_L$ must have size $\sum_{i=0}^{s-1} \binom{n}{i}$ and be $(L \backslash \ell)$-intersecting (where ℓ is the largest element of L), so again using the result of [12] we must have $L\backslash \ell = \{0, 1, \ldots, s-2\}$ and $\mathcal{A} = [n]^{(\le s-1)}$. Therefore one of the $\mathcal{A}_i$ contains all subsets of $[n]$, and the others are all equal to $[n]^{(\le s-1)}$.

5. The modular versions. The modular versions of the theorems proved in the last section have very similar proofs. The main ideas are the same, but the computations are significantly different and more involved, so we feel obliged to present them separately. We will be brief on those points of similarity to avoid excessive repetition, and we make no effort to obtain a bound on the smallest n for which the results hold. This section may be omitted on a first reading of this paper.

First we recall the statement of Theorem 1.3. Suppose p is prime, let L be a set of $s < p$ residues modulo p, and let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be set systems on $[n]$ that are L-cross-intersecting mod p such that every set $A \in \bigcup_{i=1}^{k} \mathcal{A}_i$ has $|A| = r$, for some $r \notin L \bmod p$. Let m be chosen so that $m \notin L \bmod p$ and $|n/2 - m|$ is as small as possible. The theorem claims that for $n > n(s)$ sufficiently large $\sum_{i=1}^{k} |\mathcal{A}_i| \le \max\left\{k\binom{n}{s}, \binom{n}{m}\right\}$.

We define all vectors and polynomials over $\mathbb{F}_p$ (the field with p elements) instead of $\mathbb{R}$.

Proof of Theorem 1.3. Let $k_c = \left\lfloor \binom{n}{m}/\binom{n}{s} \right\rfloor$. We can assume that $k = k_c$ or $k = k_c + 1$. Suppose that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are set systems that are L-cross-intersecting mod p such that every set $A \in \bigcup_{i=1}^{k} \mathcal{A}_i$ has $|A| = r$, for some $r \notin L \bmod p$, and suppose that $\sum_{i=1}^{k} |\mathcal{A}_i| \ge \max\left\{k\binom{n}{s}, \binom{n}{m}\right\}$. Let $\mathcal{A}$ be the subsets of $[n]$ that belong to at least two of the $\mathcal{A}_i$ and let $\mathcal{B}$ be those sets that belong to exactly one of the $\mathcal{A}_i$.

Since the $\mathcal{A}_i$ are L-cross-intersecting mod p, for any $A \in \mathcal{A}$ and $B \in \mathcal{A} \cup \mathcal{B}$ we have $|A \cap B| \in L \bmod p$. It follows from the modular Frankl–Wilson theorem that $|\mathcal{A}| \le \binom{n}{s}$, and if $\mathcal{B} \neq \emptyset$, then $|\mathcal{A}| < \binom{n}{s}$. From the remark after Lemma 3.3 we know that the s^*-inclusion vectors $\{v_A^s : A \in \mathcal{A}\}$ form a basis of the $\mathbb{F}_p$-vector space $V_{\mathcal{A}}^s$, and we also see that no vector v_B^s with $B \in \mathcal{B}$ lies in $V_{\mathcal{A}}^s$. We conclude that $|\mathcal{A}| + \dim V_{\mathcal{B}}^s \le \sum_{i=0}^{s} \binom{n}{i}$.

We can assume that both $\mathcal{A}$ and $\mathcal{B}$ are nonempty. Then $|\mathcal{A}| \le \binom{n}{s} - 1$, so $|\mathcal{B}| \ge k > \left(\binom{n}{m}/\binom{n}{s}\right) - 1$. By definition of m we have $|m - n/2| \le s$ so $\binom{n}{m} = (1+o(1))\binom{n}{\lfloor n/2 \rfloor}$ and then by Lemma 4.1 we have $|\mathcal{B}| > 2^n/n^{s+1}$. Following the proof of Theorem 1.1 we get the inequalities $\dim V_{\mathcal{B}}^s \ge \sum_{i=0}^{s} \binom{\lfloor n-(s+1)\log_2 n \rfloor}{i}$ and $|\mathcal{A}| \le \lceil (s+1)\log_2 n \rceil \sum_{i=0}^{s-1} \binom{n-1}{i}$. Then

$$|\mathcal{B}| \ge \sum_{i=1}^{k} |\mathcal{A}_i| - k|\mathcal{A}| \ge \binom{n}{m} - \left(\frac{\binom{n}{m}}{\binom{n}{s}} + 1\right) \lceil (s+1)\log_2 n \rceil \sum_{i=0}^{s-1} \binom{n-1}{i}$$

$$> \left(1 - \frac{3s(s+1)\log_2 n}{2n}\right)\binom{n}{m}. \tag{7}$$

In particular $|\mathcal{B}| = (1+o(1))\binom{n}{\lfloor n/2 \rfloor}$ so $|r - n/2| = o(\sqrt{n})$. Recalling that $\mathcal{A} \neq \emptyset$, we now consider any $A \in \mathcal{A}$. For any $B \in \mathcal{B}$ the size of its intersection with A belongs to $L \bmod p$. We can choose x so that $|x - r/2| = o(\sqrt{n})$ and $x \notin L \bmod p$. Any set of size r which intersects A in x points cannot belong to $\mathcal{B}$, and there are at least $\binom{r}{x}\binom{n-r}{r-x}$ of these. Now $\binom{r}{x} = (1+o(1))\binom{r}{\lfloor r/2 \rfloor}$ and $r - x = (n-r)/2 + o(\sqrt{n})$, so $\binom{n-r}{r-x} = (1+o(1))\binom{n-r}{\lfloor (n-r)/2 \rfloor}$. Therefore we can choose n large enough that $\binom{r}{x} > 2^r/3\sqrt{r}$ and $\binom{n-r}{r-x} > 2^{n-r}/3\sqrt{n-r}$. We deduce that $|\mathcal{B}| < \binom{n}{m} - 2^n/9n < (1 - 1/10\sqrt{n})\binom{n}{m}$. For $n > n(s)$ sufficiently large this contradicts (7), which completes the proof. ∎

Next we recall the statement of Theorem 1.4. Suppose p is prime, L is a set of $s < p$ nonnegative integers, and $\mathcal{A}_1, \dots, \mathcal{A}_k$ are set systems on $[n]$ that are L-cross-intersecting mod p such that every set $A \in \bigcup_{i=1}^{k} \mathcal{A}_i$ has $|A| \notin L \bmod p$. The theorem claims that for $n > n(s)$ sufficiently large

$$\sum_{i=1}^{k} |\mathcal{A}_i| \le \max\left\{ k \sum_{i=0}^{s} \binom{n}{i}, \sum_{i \notin L \bmod p} \binom{n}{i} \right\}.$$

First we need the following lemma.

Lemma 5.1. *Suppose $|L| = s$ and $x > x(s)$ is sufficiently large. Then*

$$\sum_{i \notin L \bmod p} \binom{x}{i} > \frac{2^x}{3s}.$$

Proof. We will restrict attention to those i that lie in the interval $I = [x/2 - x^{2/3}, x/2 + x^{2/3}]$, as the sum of $\binom{x}{i}$ for i outside this interval is $o(2^x)$ by the Chernoff bound. By slightly altering I if necessary we may suppose that $|I|$ is divisible by $s+1$, and we partition it into subintervals $\{J_\phi : \phi \in \Phi\}$ with $|J_\phi| = s+1$ for every $\phi \in \Phi$. Note that any J_ϕ contains at least one $i \notin L \bmod p$. (Since $p \ge s+1$, each element of J_ϕ gives a distinct residue mod p, so not every element of J_ϕ can belong to L mod p.) It is easy to see that $\binom{x}{j_1} = (1+o(1))\binom{x}{j_2}$ for any $j_1, j_2 \in J_\phi$, so $\sum_{i\in J_\phi,\, i\notin L \bmod p}\binom{x}{i} > \frac{1+o(1)}{s+1}\sum_{i\in J_\phi}\binom{x}{i}$. Therefore

$$\sum_{i\notin L \bmod p}\binom{x}{i} \geq \sum_{J_\phi}\sum_{i\in J_\phi,\, i\notin L \bmod p}\binom{x}{i} > \frac{1+o(1)}{s+1}\sum_{J_\phi}\sum_{i\in J_\phi}\binom{x}{i}$$
$$= \frac{1+o(1)}{s+1}\sum_{i\in I}\binom{x}{i} > \frac{2^x}{3s},$$

as required. □

Proof of Theorem 1.4. Write $k_c = \lfloor \frac{\sum_{i\notin L \bmod p}\binom{n}{i}}{\sum_{i=0}^{s}\binom{n}{i}} \rfloor$. We can assume that $k = k_c$ or $k = k_c+1$. Suppose that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are set systems that are L-cross-intersecting mod p such that every set $A \in \bigcup_{i=1}^{k}\mathcal{A}_i$ has $|A| \notin L \bmod p$, and suppose that $\sum_{i=1}^{k}|\mathcal{A}_i| \geq \max\{k\sum_{i=0}^{s}\binom{n}{i}, \sum_{i\notin L \bmod p}\binom{n}{i}\}$. Let $\mathcal{A}$ be the sets that belong to at least two of the $\mathcal{A}_i$ and let $\mathcal{B}$ be those sets that belong to exactly one of the $\mathcal{A}_i$. Since $\mathcal{A}$ is L-intersecting mod p we have $|\mathcal{A}| \leq \sum_{i=0}^{s}\binom{n}{i}$ by the Frankl–Wilson theorem, and if $\mathcal{B} \neq \emptyset$, then $|\mathcal{A}| < \sum_{i=0}^{s}\binom{n}{i}$. The s^*-inclusion vectors $\{v_A^s : A \in \mathcal{A}\}$ form a basis of $V_{\mathcal{A}}^s$ over $\mathbb{F}_p$ and no vector v_B^s with $B \in \mathcal{B}$ lies in $V_{\mathcal{A}}^s$. This shows that $|\mathcal{A}| + \dim V_{\mathcal{B}}^s \leq \sum_{i=0}^{s}\binom{n}{i}$.

We can assume that both $\mathcal{A}$ and $\mathcal{B}$ are nonempty. Then $|\mathcal{A}| \leq \sum_{i=0}^{s}\binom{n}{i} - 1$, so $|\mathcal{B}| \geq k > (\sum_{i\notin L \bmod p}\binom{n}{i}/\sum_{i=0}^{s}\binom{n}{i}) - 1 > 2^n/n^{s+1}$, by Lemma 5.1. Again we get the inequalities $\dim V_{\mathcal{B}}^s \geq \sum_{i=0}^{s}\binom{\lfloor n-(s+1)\log_2 n\rfloor}{i}$ and $|\mathcal{A}| \leq \lceil (s+1)\log_2 n\rceil \sum_{i=0}^{s-1}\binom{n-1}{i}$. Then

$$|\mathcal{B}| \geq \sum_{i=1}^{k}|\mathcal{A}_i| - k|\mathcal{A}|$$
$$\geq \sum_{i\notin L \bmod p}\binom{n}{i} - \left(\frac{\sum_{i\notin L \bmod p}\binom{n}{i}}{\sum_{i=0}^{s}\binom{n}{i}} + 1\right)\lceil (s+1)\log_2 n\rceil \sum_{i=0}^{s-1}\binom{n-1}{i}$$
$$> \left(1 - \frac{3s(s+1)\log_2 n}{2n}\right)\sum_{i\notin L \bmod p}\binom{n}{i}. \tag{8}$$

Recalling that $\mathcal{A} \neq \emptyset$, we now consider any $A \in \mathcal{A}$. For any $B \in \mathcal{B}$ the size of its intersection with A belongs to $L \bmod p$. Let $\mathcal{C} = \{C \subset [n] : |C| \notin L \bmod p \text{ and } |A \cap C| \notin L \bmod p\}$. Then we have $|\mathcal{B}| \leq \sum_{i\notin L \bmod p}\binom{n}{i} - |\mathcal{C}|$. Fix a number $m > 10s$ so that Lemma 5.1 holds for all $x \geq m$.

Suppose first that $|A| < m$. Note that $\mathcal{C}$ contains all sets of the form $C = A \cup D$, where $D \cap A = \emptyset$ and $|D| \notin L' = \{\ell - |A| \bmod p : \ell \in L\}$. By applying Lemma 5.1 to L' there are at least $2^{n-m}/3s > 2^{n-2m}$ such sets D, so $|\mathcal{C}| \geq 2^{n-2m}$. Next suppose that $|A| > n-m$. Since $\mathcal{C}$ contains all sets C such that $C \subset A$ and $|C| \notin L \bmod p$, we again have $|\mathcal{C}| > 2^{n-m}/3s > 2^{n-2m}$. Finally suppose that $m \leq |A| \leq n-m$. There are at least $2^{|A|}/3s$ sets $D \subset A$ such that $|D| \notin L \bmod p$. For each such D there are at least $2^{n-|A|}/3s$ sets $E \subset [n]\backslash A$ such that $|E| \notin \{\ell - |D| \bmod p : \ell \in L\}$. We obtain at least $2^n/9s^2 > 2^{n-2m}$ sets $D \cup E \in \mathcal{C}$. In all cases we see that $|\mathcal{C}| \geq 2^{n-2m}$. Therefore $|\mathcal{B}| \leq \sum_{i\notin L \bmod p}\binom{n}{i} - 2^{n-2m} < (1-2^{-2m})\sum_{i\notin L \bmod p}\binom{n}{i}$. For $n > n(s)$ sufficiently large this contradicts (8), which completes the proof. □

6. Concluding remarks.

- It would be interesting to determine the minimum value of n for which our results hold.
- The bounds that we give are tight when $L = \{0, 1, \ldots, s-1\}$, but one could consider a variant of this problem in which the set L is fixed to be some different set. It seems plausible that the following should be true.

  (1) To maximize the total size of uniform L-cross-intersecting systems $\mathcal{A}_1, \ldots, \mathcal{A}_k$ on $[n]$ one should either take all $\mathcal{A}_i$ equal to a maximum uniform L-intersecting system or take one $\mathcal{A}_i$ equal to all sets of size $\lfloor n/2 \rfloor$ and the others empty.

  (2) To maximize the total size of nonuniform L-cross-intersecting systems $\mathcal{A}_1, \ldots, \mathcal{A}_k$ on $[n]$ one should either take all $\mathcal{A}_i$ equal to a maximum nonuniform L-intersecting system or take one $\mathcal{A}_i$ to consist of all subsets of $[n]$ and the others equal to a maximum L-intersecting system in which the sizes of all sets also belong to L.

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