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MaxCut in *H*-Free Graphs

NOGA ALON,[1†] MICHAEL KRIVELEVICH[2‡] and BENNY SUDAKOV[3§]

[1]Schools of Mathematics and Computer Science, Raymond and Beverly Sackler Faculty of Exact Sciences,
Tel Aviv University, Tel Aviv 69978, Israel
(e-mail: nogaa@post.tau.ac.il)

[2]Department of Mathematics, Raymond and Beverly Sackler Faculty of Exact Sciences,
Tel Aviv University, Tel Aviv 69978, Israel
(e-mail: krivelev@post.tau.ac.il)

[3]Department of Mathematics, Princeton University, Princeton, NJ 08544, USA
(e-mail: bsudakov@math.princeton.edu)

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For Béla Bollobás on his 60th birthday

For a graph G, let $f(G)$ denote the maximum number of edges in a cut of G. For an integer m and for a fixed graph H, let $f(m,H)$ denote the minimum possible cardinality of $f(G)$, as G ranges over all graphs on m edges that contain no copy of H. In this paper we study this function for various graphs H. In particular we show that for any graph H obtained by connecting a single vertex to all vertices of a fixed nontrivial forest, there is a $c(H) > 0$ such that $f(m,H) \geqslant \frac{m}{2} + c(H)m^{4/5}$, and that this is tight up to the value of $c(H)$. We also prove that for any even cycle C_{2k} there is a $c(k) > 0$ such that $f(m,C_{2k}) \geqslant \frac{m}{2} + c(k)m^{(2k+1)/(2k+2)}$, and that this is tight, up to the value of $c(k)$, for $2k \in \{4,6,10\}$. The proofs combine combinatorial, probabilistic and spectral techniques.

1. Introduction

All graphs considered here are finite, undirected and have no loops and no parallel edges, unless otherwise specified. For a graph G, let $f(G)$ denote the maximum number of edges in a cut of G, that is, the maximum number of edges in a bipartite subgraph of G. For a positive integer m let $f(m)$ denote the minimum value of $f(G)$, as G ranges over all

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graphs with m edges. Thus, $f(m)$ is the largest integer f such that *any* graph with m edges contains a bipartite subgraph with at least $f(m)$ edges. Edwards [12, 13] proved that for every m,

$$f(m) \geqslant \frac{m}{2} + \frac{-1+\sqrt{8m+1}}{8},$$

and noticed that this is tight in infinitely many cases, whenever $m = \binom{k}{2}$ for some integer k.

Answering a question of Erdős, it is shown in [2] that the limsup of the difference

$$f(m) - \left(\frac{m}{2} + \frac{-1+\sqrt{8m+1}}{8}\right)$$

tends to infinity as m tends to infinity. In fact, as shown in [2], there are two absolute positive constants c_1, c_2 such that

$$f(m) \geqslant \frac{m}{2} + \sqrt{m/8} + c_1 m^{1/4}$$

for infinitely many values of m, and

$$f(m) \leqslant \frac{m}{2} + \sqrt{m/8} + c_2 m^{1/4}$$

for all m. More information on $f(m)$, including a determination of its precise value for many additional values of m, appears in [4] and [9].

The situation is more complicated if we consider only H-free graphs G, that is, graphs G that contain no copy of a fixed, given graph H. Let $f(m, H)$ denote the minimum possible cardinality of $f(G)$, as G ranges over all H-free graphs on m edges. Similarly, for a set $\mathscr{H}$ of graphs, let $f(m, \mathscr{H})$ denote the minimum possible cardinality of $f(G)$, as G ranges over all graphs on m edges that contain no member of $\mathscr{H}$. It is not difficult to show (see, *e.g.*, [3]) that for every fixed graph H there are some $\epsilon = \epsilon(H) > 0$ and $c = c(H) > 0$ such that $f(m, H) \geqslant \frac{m}{2} + cm^{1/2+\epsilon}$ for all m, but the problem of estimating the error term more precisely is not easy, even for relatively simple graphs H. The case $H = K_3$ in which $f(m, K_3)$ is the minimum possible size of the maximum cut in a triangle-free graph with m edges, has been studied extensively. After a series of papers by various researchers ([14], [20], [21]), the first author proved in [2] that $f(m, K_3) = \frac{m}{2} + \Theta(m^{4/5})$, that is, there are $c_1, c_2 > 0$ such that

$$\frac{m}{2} + c_1 m^{4/5} \leqslant f(m, K_3) \leqslant \frac{m}{2} + c_2 m^{4/5} \tag{1.1}$$

for all m.

In a recent joint paper with Bollobás [3], we studied the value of $f(m, \mathscr{H}_r)$ for the families $\mathscr{H}_r = \{C_3, \ldots, C_{r-1}\}$ of all cycles of length less than r. Here $f(m, \mathscr{H}_r)$ is the minimum possible size of a maximum cut in a graph with m edges and girth at least r. The problem of estimating $f(m, \mathscr{H}_r)$ had in fact been considered much earlier by Erdős in [14]. He conjectured that for every $r \geqslant 4$ there exists a constant $c_r > 0$ such that, for every $\epsilon > 0$,

$$\frac{m}{2} + m^{c_r - \epsilon} \leqslant f(m, \mathscr{H}_r) \leqslant \frac{m}{2} + m^{c_r + \epsilon}$$

provided $m > m(\epsilon)$. He also mentioned that together with Lovász they proved that

$$\frac{m}{2} + c_2 m^{c''_r} \leqslant f(m, \mathscr{H}_r) \leqslant \frac{m}{2} + c_1 m^{c'_r},$$

where $1/2 < c''_r < c'_r < 1$ for all $r > 3$, and c''_r (as well as c'_r) tend to one as r tends to infinity.

In [3] it is proved that for every $r \geqslant 4$ there is a $c(r) > 0$ such that

$$f(m, \mathscr{H}_r) \geqslant \frac{m}{2} + c(r) m^{\frac{r}{r+1}}$$

for all m. It is further shown that this is tight, up to the value of $c(r)$, for $r = 5$ (note that it is tight for $r = 4$ as well, by (1.1)). The authors of [3] conjecture that this is in fact tight for all $r \geqslant 4$.

In the present paper we study the value of $f(m, H)$ for several additional graphs H. Although we are unable to determine the asymptotic behaviour of the error term of this function (that is, the behaviour of $f(m, H) - m/2$) up to a constant factor for general graphs H, we can do it for infinitely many graphs H. Our main new results are the following.

Theorem 1.1. *Let H be a tree with $h > 1$ edges.*
(i) *If h is even, then $f(m, H) \geqslant \frac{m}{2} + \frac{m}{2h-2}$.*
(ii) *If h is odd, then $f(m, H) \geqslant \frac{m}{2} + \frac{m}{2h}$.*
Both bounds hold as equalities if $\binom{h}{2}$ divides m.

Theorem 1.2. *For every odd integer $r > 3$ there is a $c(r) > 0$ such that*

$$f(m, C_{r-1}) \geqslant \frac{m}{2} + c(r) m^{r/(r+1)}$$

for all m. This is tight, up to the value of $c(r)$, for $r \in \{5, 7, 11\}$.

Note that the lower bounds in the last theorem strengthens the lower bound for $f(m, \mathscr{H}_r)$, for odd r. It seems plausible that the above estimate holds for even values of r as well, but this remains open for all $r > 4$.

Theorem 1.3. *Let H be a graph obtained by connecting a single vertex to all vertices of a fixed nontrivial forest. Then there is a $c = c(H) > 0$ such that*

$$f(m, H) \geqslant \frac{m}{2} + c m^{4/5}$$

for all m. This is tight (up to the value of c) for each such H.

Theorem 1.4. *Let $K_{t,s}$ denote the complete bipartite graph with classes of vertices of size t and s.*
(i) *For each $s \geqslant 2$ there is a $c(s) > 0$ such that*

$$f(m, K_{2,s}) \geqslant \frac{m}{2} + c(s) m^{5/6},$$

for all m, and this is tight up to the value of $c(s)$.

(ii) *For each $s \geqslant 3$ there is a $c'(s) > 0$ such that*

$$f(m, K_{3,s}) \geqslant \frac{m}{2} + c'(s)m^{4/5},$$

for all m, and this is tight up to the value of $c'(s)$.

We can handle some additional forbidden graphs H, as well as certain families of graphs $\mathscr{H}$. Our proofs combine combinatorial, probabilistic and spectral techniques, including extensions of ideas that appear in [21], [2], [3]. Throughout the paper, we omit all floor and ceiling signs whenever these are not crucial, to simplify the presentation.

The rest of the paper is organized as follows. In Section 2 we present the (simple) proof of Theorem 1.1. In Section 3 we prove an extension of a lemma established by Shearer in [21]. This lemma shows that graphs in which every edge lies in a relatively small number of triangles have large cuts. Combining this lemma with some additional ideas we prove Theorems 1.2, 1.3 and 1.4 in Section 4. The final Section 5 contains some concluding remarks and open problems.

2. Trees

In this section we prove Theorem 1.1. We need the following simple lemma proved in [14].

Lemma 2.1. *Let G be a graph with m edges and chromatic number at most h. If h is even, then $f(G) \geqslant \frac{m}{2} + \frac{m}{2h-2}$, and if h is odd, then $f(G) \geqslant \frac{m}{2} + \frac{m}{2h}$.*

The proof is simple: split the set of vertices of G into h independent sets, partition these randomly into a part consisting of $\lfloor h/2 \rfloor$ of these sets and its complement, and compute the expected number of edges in the cut obtained, to get the desired result.

Proof of Theorem 1.1. Let G be an H-free graph with m edges, where H is a tree with h edges. It is not difficult to see that G is $(h-1)$-degenerate, that is, any subgraph of it contains a vertex of degree at most $h-1$. Indeed, otherwise there is a subgraph G' of G in which all degrees are at least h. It is easy and well known that each such subgraph contains a copy of each tree with h edges, contradicting H-freeness. Therefore, G is indeed $(h-1)$-degenerate, and hence h-colourable. By Lemma 2.1, it follows that if h is even then $f(G) \geqslant \frac{m}{2} + \frac{m}{2h-2}$, and if h is odd, then $f(G) \geqslant \frac{m}{2} + \frac{m}{2h}$. This proves Theorem 1.1(i),(ii). The fact that if $\binom{h}{2}$ divides m then the result is tight follows by letting G be the vertex-disjoint union of $m/\binom{h}{2}$ cliques, each of size h. Then G is H-free, has m edges, and

$$f(G) = \left\lfloor \frac{h^2}{4} \right\rfloor \frac{m}{\binom{h}{2}},$$

which is $\frac{m}{2} + \frac{m}{2h-2}$ for even h and $\frac{m}{2} + \frac{m}{2h}$ for odd h. □

3. Cuts, degrees, and codegrees

In [21] Shearer proved that if $G = (V, E)$ is a triangle-free graph with n vertices and m edges, and if $d_1, d_2, \ldots, d_n$ are the degrees of its vertices, then

$$f(G) \geqslant \frac{m}{2} + \frac{1}{8\sqrt{2}} \sum_{i=1}^{n} \sqrt{d_i}. \tag{3.1}$$

Here we extend his argument and prove a similar result for graphs in which most edges do not lie in too many triangles.

Let $G = (V, E)$ be a graph, and consider the following randomized procedure for obtaining a cut of G. Let $h : V \mapsto \{0, 1\}$ be a random function obtained by picking, for each $v \in V$ randomly and independently, the value of $h(v) \in \{0, 1\}$, where both choices are equally likely. Call a vertex $v \in V$ *stable* if it has more neighbours u satisfying $h(u) \neq h(v)$ than neighbours w satisfying $h(w) = h(v)$, otherwise call it *active*. Let $h' : V \mapsto \{0, 1\}$ be the random function obtained from h as follows. For each $u \in V$, if u is active, then choose randomly $h'(u) \in \{0, 1\}$, where both choices are equally likely and all choices are independent. Otherwise, define $h'(u) = h(u)$. Finally, define $V_0 = (h')^{-1}(0)$ and $V_1 = (h')^{-1}(1)$.

This produces a cut (V_0, V_1) consisting of all edges of G that connect a vertex of V_0 and a vertex of V_1. Although in some cases (for example, when G is a complete graph), the expected number of edges in this cut is far smaller than $m/2$, it turns out that when the graph is relatively sparse, the expected size of the cut is large. This is shown by considering the probability that for a given edge uv, $h'(u) \neq h'(v)$. Clearly,

$$\Pr[h'(u) \neq h'(v)] = \frac{1}{2}\Pr[h'(u) \neq h'(v)|\ h(u) = h(v)] + \frac{1}{2}\Pr[h'(u) \neq h'(v)|\ h(u) \neq h(v)]. \tag{3.2}$$

In order to estimate the conditional probability $\Pr[h'(u) \neq h'(v)|h(u) = h(v)]$ note that if at least one of the vertices u or v is active, the probability that $h'(u) \neq h'(v)$ is precisely $1/2$, whereas if they are both stable then $h'(u)$ cannot differ from $h'(v)$. Thus

$$\Pr[h'(u) \neq h'(v)|\ h(u) = h(v)] = \frac{1}{2} - \frac{1}{2}\Pr[u \text{ and } v \text{ are stable}|\ h(u) = h(v)].$$

Similarly, if $h(u) \neq h(v)$ and both these vertices are stable, then certainly $h'(u) \neq h'(v)$, whereas if at least one of them is active, then the probability that $h'(u) \neq h'(v)$ is $1/2$. Thus

$$\begin{aligned}
&\Pr[h'(u) \neq h'(v)|\ h(u) \neq h(v)] \\
&\quad = \Pr[u \text{ and } v \text{ are stable}|\ h(u) \neq h(v)] + \frac{1}{2}\big(1 - \Pr[u \text{ and } v \text{ are stable}|\ h(u) \neq h(v)]\big) \\
&\quad = \frac{1}{2} + \frac{1}{2}\Pr[u \text{ and } v \text{ are stable}|\ h(u) \neq h(v)].
\end{aligned}$$

Substituting in (3.2) we conclude that

$$\begin{aligned}
&\Pr[h'(u) \neq h'(v)] \\
&= \frac{1}{2} + \frac{1}{4}(\Pr[u \text{ and } v \text{ are stable}|\ h(u) \neq h(v)] - \Pr[u \text{ and } v \text{ are stable}|\ h(u) = h(v)]).
\end{aligned} \tag{3.3}$$

The main remaining task is thus to estimate the difference between these last two conditional probabilities. Intuitively, if u and v do not have too many common neighbours, then the conditional probability of u and v to be stable is smaller when $h(u) = h(v)$ than when $h(u) \neq h(v)$, since in the first case each of them already has at least one vertex with the same h value as itself. In what follows we show that this is indeed the case.

Lemma 3.1. *There are two absolute positive constants c_1, c_2 such that the following holds. Let $G = (V, E)$ be a graph, and let $h, h' : V \mapsto \{0, 1\}$ be the two random functions defined by the randomized procedure described above. Let u, v be two adjacent vertices of G, and suppose they have k common neighbours. Let the degree of u in G be $a + k + 1$ and let the degree of v in G be $b + k + 1$. Suppose, further, that $a \geqslant k$ and $b \geqslant k$. Then the probability* $\Pr[h'(u) \neq h'(v)]$ *that the edge uv lies in the random cut produced by the procedure satisfies*

$$\Pr[h'(u) \neq h'(v)] \geqslant \frac{1}{2} + \frac{c_1}{\sqrt{a+k+1}} + \frac{c_1}{\sqrt{b+k+1}} - \frac{c_2 k}{\sqrt{ab}}.$$

Proof. For convenience we assume that a, b, k are all even. The computation for the other possible cases is similar. For any two integers $m \geqslant r \geqslant 0$ let

$$B(m, r) = \frac{1}{2^m} \sum_{i=0}^{r} \binom{m}{i}$$

denote the probability that at most r of m random coin flips are heads. For $\epsilon_1, \epsilon_2 \in \{0, 1\}$, put

$$P_{\epsilon_1,\epsilon_2} = \Pr[u \text{ and } v \text{ are stable} |\ h(u) = \epsilon_1, h(v) = \epsilon_2].$$

By (3.3) it suffices to show that

$$P_{0,1} + P_{1,0} - P_{0,0} - P_{1,1} \geqslant \frac{4c_1}{\sqrt{a+k+1}} + \frac{4c_1}{\sqrt{b+k+1}} - \frac{4c_2 k}{\sqrt{ab}}.$$

Let K denote the set of common neighbours of u and v, let A denote the set of neighbours of u which are not adjacent to v, and let B denote the set of neighbours of v which are not adjacent to u. By the assumptions of the lemma $|K| = k$, $|A| = a$, $|B| = b$. Note that the stability of u and v is determined by the h-values of the vertices in $\{u, v\} \cup K \cup A \cup B$. Thus, for example, to compute $P_{0,1}$ note that the probability that the h-value of exactly $k/2 + \delta$ of the members of K is 0 is given by

$$\frac{1}{2^k} \binom{k}{k/2 + \delta}.$$

In this case, and assuming $h(u) = 0$ and $h(v) = 1$, both u and v will be stable if and only if the h-value of at most $a/2 - \delta$ members of A is 0, and the h-value of at most $b/2 + \delta$ members of B is one. Hence

$$P_{0,1} = \frac{1}{2^k} \sum_{-k/2 \leqslant \delta \leqslant k/2} \binom{k}{k/2 + \delta} B\left(a, \frac{a}{2} - \delta\right) B\left(b, \frac{b}{2} + \delta\right).$$

The computation of the other terms $P_{\epsilon_1,\epsilon_2}$ is similar. Thus we have

$$\begin{aligned}
&P_{0,1}+P_{1,0}-P_{0,0}-P_{1,1}\\
&\quad=\frac{1}{2^k}\sum_{-k/2\leqslant\delta\leqslant k/2}\binom{k}{k/2+\delta}\Big[B\Big(a,\frac{a}{2}-\delta\Big)B\Big(b,\frac{b}{2}+\delta\Big)+B\Big(a,\frac{a}{2}+\delta\Big)B\Big(b,\frac{b}{2}-\delta\Big)\\
&\qquad-B\Big(a,\frac{a}{2}-\delta-1\Big)B\Big(b,\frac{b}{2}-\delta-1\Big)-B\Big(a,\frac{a}{2}+\delta-1\Big)B\Big(b,\frac{b}{2}+\delta-1\Big)\Big]\\
&\quad=\frac{1}{2^k}\sum_{-k/2\leqslant\delta\leqslant k/2}\binom{k}{k/2+\delta}[\Delta_1(\delta)+\Delta_2(\delta)],
\end{aligned}$$

where

$$\begin{aligned}
\Delta_1=\Delta_1(\delta)=&\frac{1}{2^a}\binom{a}{a/2-\delta}B\Big(b,\frac{b}{2}+\delta\Big)+B\Big(a,\frac{a}{2}-\delta-1\Big)\frac{1}{2^b}\binom{b}{b/2+\delta}\\
&+\frac{1}{2^a}\binom{a}{a/2+\delta}B\Big(b,\frac{b}{2}-\delta\Big)+B\Big(a,\frac{a}{2}+\delta-1\Big)\frac{1}{2^b}\binom{b}{b/2-\delta},
\end{aligned}$$

and

$$\begin{aligned}
\Delta_2=\Delta_2(\delta)=&B\Big(a,\frac{a}{2}-\delta-1\Big)B\Big(b,\frac{b}{2}+\delta-1\Big)+B\Big(a,\frac{a}{2}+\delta-1\Big)B\Big(b,\frac{b}{2}-\delta-1\Big)\\
&-B\Big(a,\frac{a}{2}-\delta-1\Big)B\Big(b,\frac{b}{2}-\delta-1\Big)-B\Big(a,\frac{a}{2}+\delta-1\Big)B\Big(b,\frac{b}{2}+\delta-1\Big).
\end{aligned}$$

However, since

$$B\Big(b,\frac{b}{2}+\delta\Big)+B\Big(b,\frac{b}{2}-\delta\Big)=1+\frac{\binom{b}{b/2+\delta}}{2^b},$$

and

$$B\Big(a,\frac{a}{2}-\delta-1\Big)+B\Big(a,\frac{a}{2}+\delta-1\Big)=1-\frac{\binom{a}{a/2-\delta}}{2^a},$$

it follows that

$$\Delta_1=\frac{\binom{a}{a/2-\delta}}{2^a}+\frac{\binom{a}{a/2-\delta}}{2^a}\cdot\frac{\binom{b}{b/2+\delta}}{2^b}+\frac{\binom{b}{b/2+\delta}}{2^b}-\frac{\binom{b}{b/2+\delta}}{2^b}\cdot\frac{\binom{a}{a/2-\delta}}{2^a}=\frac{\binom{a}{a/2-\delta}}{2^a}+\frac{\binom{b}{b/2+\delta}}{2^b}.$$

Therefore

$$\frac{1}{2^k}\sum_{-k/2\leqslant\delta\leqslant k/2}\binom{k}{k/2+\delta}\Delta_1=\frac{\binom{k+a}{(k+a)/2}}{2^{k+a}}+\frac{\binom{k+b}{(k+b)/2}}{2^{k+b}}=\Theta\Big(\frac{1}{\sqrt{k+a+1}}+\frac{1}{\sqrt{k+b+1}}\Big).$$

In addition, for $\delta=0$, $\Delta_2=0$, whereas for $\delta>0$,

$$\begin{aligned}
\Delta_2&=B\Big(a,\frac{a}{2}-\delta-1\Big)\frac{1}{2^b}\sum_{j=b/2-\delta}^{b/2+\delta-1}\binom{b}{j}-B\Big(a,\frac{a}{2}+\delta-1\Big)\frac{1}{2^b}\sum_{j=b/2-\delta}^{b/2+\delta-1}\binom{b}{j}\\
&=-\frac{1}{2^a}\sum_{i=a/2-\delta}^{a/2+\delta-1}\binom{a}{i}\cdot\frac{1}{2^b}\sum_{j=b/2-\delta}^{b/2+\delta-1}\binom{b}{j}.
\end{aligned}$$

For $\delta<0$ the result is obtained from the one above by replacing each δ by $-\delta$.

For each fixed i, $\frac{1}{2^a}\binom{a}{i} \leqslant O(1/\sqrt{a})$ holds uniformly in i, a, and a similar estimate holds for $\frac{1}{2^b}\binom{b}{j}$. Also note that

$$\frac{1}{2^k} \sum_{-k/2 \leqslant \delta \leqslant k/2} \binom{k}{k/2+\delta} \delta^2 = \frac{k}{4},$$

since the expression on the left-hand side is the variance of the binomially distributed random variable with parameters k and $1/2$. Therefore

$$\frac{1}{2^k} \sum_{-k/2 \leqslant \delta \leqslant k/2} \binom{k}{k/2+\delta} \Delta_2 \leqslant O\left(\frac{1}{\sqrt{ab}}\right) \left(\frac{1}{2^k} \sum_{-k/2 \leqslant \delta \leqslant k/2} \binom{k}{k/2+\delta} \delta^2\right) = O\left(\frac{k}{\sqrt{ab}}\right).$$

This completes the proof. □

In order to apply the last lemma, we need the following simple facts, which appear, in some versions, already in [12], [13], and [2].

Lemma 3.2.

(i) *There is a positive constant c such that for every graph $G = (V, E)$ with n vertices, m edges, and positive minimum degree, $f(G) \geqslant \frac{m}{2} + cn$.*

(ii) *Let $G = (V, E)$ be a graph with m edges, suppose $U \subset V$ and let G' be the induced subgraph of G on U. If G' has m' edges, then $f(G) \geqslant f(G') + \frac{m-m'}{2}$.*

Proof. A simple proof of part (i) is to take a random linear order $v_1, v_2, \dots, v_n$ on the vertices of G, and to define a cut (A, B) by starting with $A = B = \emptyset$ and placing each vertex v_i, in its turn, either in A or in B, trying to maximize the number of edges in the bipartite graph spanned by the vertex classes A and B. In each such addition, the number of edges added to the bipartite graph is clearly at least half the number of edges connecting v_i to the previous vertices $\{v_1, \dots, v_{i-1}\}$, and hence at the end of the process we have at least $\frac{m}{2}$ edges with ends at A and B. Moreover, if v_i has an odd number of neighbours among the vertices $\{v_1, \dots, v_{i-1}\}$, then the number of edges added to the cut while inserting v_i exceeds half the number of edges connecting it to the previous vertices by at least $1/2$. As the probability that a given vertex has an odd number of neighbours preceding it in the randomly chosen order is bounded away from zero, the assertion of part (i) follows.

The proof of part (ii) is even simpler. Given a partition $U = A \cup B$, $A \cap B = \emptyset$, of U such that the number of edges of G between A and B is $f(G')$, add the vertices in $V - U$ one by one, where each vertex in its turn is added to A or to B, trying to maximize the number of edges in the bipartite graph spanned by these vertex classes. This clearly produces a cut in G that contains all the $f(G')$ edges of the initial cut of G', and at least half of all other edges, implying the assertion of (ii). □

We can now prove the following lemma, which will be one of the main tools used in the next section.

Lemma 3.3. *There exists an absolute positive constant ϵ such that for every positive constant C there is a $\delta = \delta(C) > 0$ with the following property. Let $G = (V, E)$ be a graph with n vertices (with positive degrees), m edges, and degree sequence $d_1, d_2, \dots, d_n$. Suppose, further, that the induced subgraph on any set of $d \geqslant C$ vertices, all of which have a common neighbour, contains at most $\epsilon d^{3/2}$ edges. Then*

$$f(G) \geqslant \frac{m}{2} + \delta \sum_{i=1}^{n} \sqrt{d_i}.$$

Proof. As long as there is a vertex of degree smaller than C in G, delete it. If during this process we delete more than $\frac{1}{4C} \sum_{i=1}^{n} \sqrt{d_i}$ vertices, then the desired result, with $\delta = \frac{c}{4C}$, where c is the constant from Lemma 3.2(i), follows from the assertion of that lemma. Therefore we may assume that the process terminates after at most $\frac{1}{4C} \sum_{i=1}^{n} \sqrt{d_i}$ such deletion steps. It thus terminates with an induced subgraph G' on n' vertices in which all degrees are at least C. Let $d'_1, d'_2, \dots, d'_{n'}$ be the degree sequence of G', and let m' denote its total number of edges. Note that G' is obtained from G by deleting fewer than $C \frac{1}{4C} \sum_{i=1}^{n} \sqrt{d_i} = \frac{1}{4} \sum_{i=1}^{n} \sqrt{d_i}$ edges. Since each deletion of an edge decreases the sum of roots of the degrees by at most 2, we conclude that

$$\sum_{i=1}^{n'} \sqrt{d'_i} \geqslant \frac{1}{2} \sum_{i=1}^{n} \sqrt{d_i} \tag{3.4}$$

Let V' be the set of vertices of G', and consider the randomized procedure for obtaining a cut of G' by choosing the random functions $h, h' : V' \mapsto \{0, 1\}$ as described in the beginning of the section. Lemma 3.1 enables us to estimate the probability that an edge uv of G' for which the degrees of u and v in G' are $d(u), d(v)$, respectively, and

$$\text{the number of common neighbours of } u \text{ and } v \text{ is at most } \min\left(\frac{d(u)}{2}, \frac{d(v)}{2}\right) \tag{3.5}$$

lies in the cut produced in this way. We first show that the number of edges uv for which the condition (3.5) is violated is not too large. Indeed, assign each edge violating this condition to its end with the smaller degree (or to either of them in case of equality). Let u be a vertex of degree d in G', and consider the set of all edges uv assigned to u. For each such edge, the degree of v in the subgraph induced on the neighbours of u in G' is at least $d/2$. Since $d \geqslant C$, there are at most $\epsilon d^{3/2}$ edges in this subgraph, and hence the number of edges uv assigned to u is at most $4\epsilon\sqrt{d}$. Summing over all vertices we conclude that the number of edges of G' that violate (3.5) is at most

$$4\epsilon \sum_{i=1}^{n'} \sqrt{d'_i}. \tag{3.6}$$

Every other edge uv of G' satisfies the assumptions of Lemma 3.1. Therefore, the probability that it lies in our cut is at least

$$\frac{1}{2} + \frac{c_1}{\sqrt{d(u)}} + \frac{c_1}{\sqrt{d(v)}} - \frac{2c_2 k}{\sqrt{d(u)d(v)}},$$

where $d(u), d(v)$ are the degrees of u and v in G', and k is the number of their common neighbours in G'. By linearity of expectation, and assuming, for a moment, that all edges of G' satisfy (3.5), we conclude, from Lemma 3.1, that the expected size of the cut is at least

$$\frac{m'}{2} + c_1 \sum_{i=1}^{n'} \sqrt{d'_i} - \Delta,$$

where Δ is the sum of contributions of the last term (the term $\frac{2c_2k}{\sqrt{d(u)d(v)}}$) over all edges uv of G'. In reality, however, not all edges of G' satisfy (3.5); for each edge that does not satisfy it we do not add the corresponding term from the lemma. Thus we lose, for each such edge, at most 1, and as the number of these edges is bounded by (3.6), and we may assume, for example, that $\epsilon < \frac{c_1}{8}$, we still conclude that the expected size of our cut is at least

$$\frac{m'}{2} + \frac{c_1}{2} \sum_{i=1}^{n'} \sqrt{d'_i} - \Delta \tag{3.7}$$

where Δ is now the sum of contributions of the $\frac{2c_2k}{\sqrt{d(u)d(v)}}$-term over all edges uv of G' that satisfy (3.5). To bound Δ, as before, assign each edge uv, as above, to its end with the smaller degree (or to any of them in case of equality). Let u be a vertex of degree d in G', and consider the set of all edges uv assigned in this way to u. For each such edge, the degree of v in the subgraph induced on the neighbours of u in G' is k. In addition $\frac{1}{\sqrt{d(u)d(v)}} \leqslant \frac{1}{d(u)}$. Since $d(u) \geqslant C$, there are at most $\epsilon d(u)^{3/2}$ edges in this induced subgraph, and hence the total contribution of the $\frac{2c_2k}{\sqrt{d(u)d(v)}}$-terms assigned to u is at most $4c_2\epsilon\sqrt{d(u)}$. Summing over all vertices we conclude that

$$\Delta \leqslant 4c_2\epsilon \sum_{i=1}^{n'} \sqrt{d'_i}.$$

This, together with (3.7), implies that

$$f(G') \geqslant \frac{m'}{2} + \left(\frac{c_1}{2} - 4c_2\epsilon\right) \sum_{i=1}^{n'} \sqrt{d'_i}.$$

By the last inequality, (3.4) and part (ii) of Lemma 3.2, and by choosing $\epsilon < \frac{c_1}{16c_2}$ the desired result follows. □

In what follows, it will be convenient to use the following variant of the last lemma as well.

Lemma 3.4. *There exist two absolute positive constants ϵ and δ such that the following holds. Let $G = (V, E)$ be a graph with n vertices, m edges, and degree sequence $d_1, d_2, \ldots, d_n$. Suppose, further, that for each i the induced subgraph on all the d_i neighbours of vertex* *number i contains at most* $\epsilon d^{3/2}$ *edges. Then*

$$f(G) \geqslant \frac{m}{2} + \delta \sum_{i=1}^{n} \sqrt{d_i}.$$

This lemma differs from Lemma 3.3 by the fact that here δ is an absolute constant, but more crucially, by the fact that we assume that the induced subgraph on the full neighbourhood of each vertex is sparse (and not that the induced subgraph on any large subset of this neighbourhood is sparse). Note that since ϵ is small this means that if G satisfies the assumptions, then the induced subgraph on the neighbourhood of each low-degree vertex of G (if there are any) contains no edges at all. The proof of this lemma is essentially identical to the proof of Lemma 3.3, without the extra complication of producing the graph G'. Here we apply our randomized procedure to get a cut of the original graph G, and estimate its expected size as before using the fact that each neighbourhood of size d spans at most $\epsilon d^{3/2}$ edges. By choosing, for example,

$$\epsilon < \min\left(\frac{c_1}{8}, \frac{c_1}{16c_2}\right) \quad \text{and} \quad \delta = \frac{c_1}{4},$$

where c_1, c_2 are the constants from Lemma 3.1, and by repeating the arguments in the previous proof, we obtain the desired result.

4. *H*-free graphs

In this section we present the proofs of Theorems 1.2, 1.3 and 1.4. We need several known results. The first one is a simple, well-known upper bound for the size of the maximum cut in a regular graph in terms of its eigenvalues. See, *e.g.*, [2] for a proof.

Lemma 4.1. *Let* $G = (V, E)$ *be a d-regular graph with n vertices and* $m = nd/2$ *edges, and let* $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ *be the eigenvalues of (the adjacency matrix of) G. Then*

$$f(G) \leqslant (d - \lambda_n)n/4 = \frac{m}{2} - \lambda_n n/4.$$

(Note that since the trace of the adjacency matrix of G is zero, $\lambda_n \leqslant 0$.)

The second result is due to Bondy and Simonovits [10].

Lemma 4.2. *Let* $k \geqslant 2$ *be an integer and let G be a graph on n vertices. If G contains no cycle of length 2k, then the number of edges of G is at most* $100kn^{1+1/k}$.

We also need the following result of Kővári, T. Sós and Turán [17].

Lemma 4.3. *Let* $K_{t,s}$ *denote, as in Theorem 1.4, the complete bipartite graph with* $t + s$ *vertices and ts edges. For every* $s \geqslant t$, *every graph with n vertices that contains no copy of* $K_{t,s}$ *has at most*

$$\frac{1}{2}(s-1)^{1/t} n^{2-1/t} + \frac{1}{2}(t-1)n$$

edges.

4.1. Even cycles

Proof of Theorem 1.2. Let $r-1=2k \geqslant 4$ be an even integer, and let $G=(V,E)$ be a C_{2k}-free graphs with n vertices and m edges.

Define $D = bm^{2/(r+1)}$, where $b=b(r)>1$ will be chosen later. We consider two possible cases depending on the existence of dense subgraphs in G.

Case 1. G is $(D-1)$-degenerate, that is, it contains no subgraph with minimum degree at least D. In this case, as is well known, there exists a labelling $v_1,\dots,v_n$ of the vertices of G so that for every i, the number of neighbours v_j of v_i with $j<i$ is strictly smaller than D. (To see this, let u be a vertex of minimum degree in G, define $v_n=u$, delete it from G and repeat the process.) Let $d^+(v_i)$ denote the number of neighbours v_j of v_i with $j<i$ and let $d(v_i)$ denote the total degree of v_i. Then

$$\sum_{i=1}^{n}\sqrt{d(v_i)} \geqslant \sum_{i=1}^{n}\sqrt{d^+(v_i)} > \frac{\sum_{i=1}^{n} d^+(v_i)}{\sqrt{D}} = \frac{m}{\sqrt{D}} \geqslant \frac{m^{r/(r+1)}}{\sqrt{b}}.$$

Note that the neighbourhood of a vertex of G cannot contain a path of length $2k-2$, and hence the number of edges induced on any subset of cardinality d in it is smaller than kd, which is smaller than $\epsilon d^{3/2}$ for all $d>(k/\epsilon)^2 = (\frac{r-1}{2\epsilon})^2$. Therefore, by Lemma 3.3,

$$f(G) \geqslant \frac{m}{2} + \delta\sum_{i=1}^{n}\sqrt{d(v_i)} \geqslant \frac{m}{2} + \delta\frac{m^{r/(r+1)}}{\sqrt{b}},$$

where $\delta=\delta(r)$, as needed.

Case 2. G contains a subgraph G' with minimum degree at least D. But in this case, the number of vertices of G' is $N \leqslant 2m^{(r-1)/(r+1)}/b$ and as the minimum degree is at least $bm^{2/(r+1)} \geqslant \frac{b^{(r+1)/(r-1)}}{2^{2/(r-1)}}N^{2/(r-1)}$ this is impossible, in view of Lemma 4.2, for a sufficiently large chosen value of $b=b(r)$.

This completes the proof of the required lower bound for $f(m, C_{r-1})$ for all odd $r>3$. The fact that the error term is tight, up to the value of $c(r)$, for $r\in\{5,7,11\}$ is proved using Lemma 4.1. An (n,d,λ)-*graph* is a d-regular graph $G=(V,E)$ on n vertices, such that the absolute value of every eigenvalue of the adjacency matrix of G, besides the largest one, is at most λ. The properties of such graphs in which λ is much smaller than d have been studied extensively. It is known that in this case the graph has some strong pseudo-random properties; see, *e.g.*, [8], Chapter 9.2 or [18] and their references. The extremal graphs we use here are, indeed, appropriate (n,d,λ)-graphs.

The Erdős–Rényi graph G, constructed in [15], is the polarity graph of a finite projective plane of order p. This graph is a C_4-free (n,d,λ)-graph, where $n=p^2+p+1$, $d=p+1$ and $\lambda=\sqrt{p}$, and it exists for every prime power p. It has, in fact, $p+1$ loops, which we omit. See, *e.g.*, [3] for the proof of these facts. Let m denote the number of edges of this graph. By Lemma 4.1 its maximum cut is of size at most

$$\frac{m}{2} + \frac{n\sqrt{p}}{4} \leqslant \frac{m}{2} + O(m^{5/6}).$$

Therefore, the error term in Theorem 1.2 is tight for $r-1=4$.

For every q which is an odd power of 2, the incidence graph of the generalized 4-gon has a polarity. The corresponding polarity graph is a $(q+1)$-regular graph with

$n = q^3 + q^2 + q + 1$ vertices. See [11], [19] for more details. This graph contains no cycle of length 6 and it is not difficult to compute its eigenvalues (they can be derived, for example, from the eigenvalues of the corresponding incidence graph, given in [23]; see also [6]). Indeed, all the eigenvalues, besides the trivial one (which is $q + 1$) are either 0 or $\sqrt{2q}$ or $-\sqrt{2q}$. Let m denote the number of edges of this graph. By Lemma 4.1 we conclude that

$$f(G) \leqslant \frac{m}{2} + \frac{n\sqrt{2q}}{4} \leqslant \frac{m}{2} + O(m^{7/8}),$$

showing that the assertion of the theorem is tight for $r - 1 = 6$ as well.

For every q which is an odd power of 3, the incidence graph of the generalized 6-gon has a polarity. The corresponding polarity graph is a $(q + 1)$-regular graph with $q^5 + q^4 + \cdots + q + 1$ vertices. See [11], [19] for more details. This graph contains no cycle of length 10 and its eigenvalues can be easily derived from the the eigenvalues of the corresponding incidence graph, given in [23]; see also [6]. All the eigenvalues, besides the trivial one, are either $\sqrt{3q}$ or $-\sqrt{3q}$ or $\sqrt{q}$ or $-\sqrt{q}$. Using Lemma 4.1 this shows that the assertion of Theorem 1.2 is tight for $r - 1 = 10$ as well. □

Remark. Using similar reasoning, we can show that for any graph H which is the union of an arbitrary number of cycles of length 4, all having a single common point,

$$f(m, H) \geqslant \frac{m}{2} + c(H)m^{5/6},$$

and this is tight, up to the value of $c(H)$. To do so we have to combine the previous proof with the fact that the number of edges in any H-free graph on n vertices is at most $c(H)n^{3/2}$, a result that follows from the main result of [16] (see also [5]). A similar result holds for certain other graphs H; we omit the details.

4.2. Forests with a common neighbour

Proof of Theorem 1.3. The proof is similar to one of the proofs in [2] (see also [3]). Let H be the graph obtained from a forest F with at least one edge, by adding an additional vertex and by connecting it to all vertices of F. Let $G = (V, E)$ be an H-free graph with n vertices and m edges. Define $D = m^{2/5}$ and proceed as before, by considering two possible cases.

Case 1. G contains no subgraph with minimum degree at least D. In this case, we proceed as in the previous proof. Here, too, the induced subgraph of G on any set of common neighbours of a vertex can span only a linear number of edges, as it contains no copy of F. Thus we can apply, again, Lemma 3.3 and conclude, as in the proof of Theorem 1.2, that in this case

$$f(G) \geqslant \frac{m}{2} + \Omega\left(\frac{m}{\sqrt{D}}\right) \geqslant \frac{m}{2} + \Omega(m^{4/5}),$$

as needed.

Case 2. There exists a subset W of x vertices of G such that the induced subgraph G' of G on W has minimum degree at least D. We first prove that in this case G' (and hence G) contains an induced subgraph G'' on a set W' of vertices of G, with at least $xD/4$

edges, which is r-colourable for $r = O(\frac{x}{D})$. To see this, let R be a random subset of at most $2x/D$ vertices of G' obtained by picking, with repetitions, $2x/D$ vertices of G', each chosen randomly with uniform probability. Let u be a fixed vertex of G'. The probability that u does not have a neighbour in R is

$$\left(1 - \frac{d_{G'}(u)}{x}\right)^{2x/D} < \exp\{-(D/x)(2x/D)\} < 1/4,$$

where $d_{G'}(u)$ denotes the degree of u in G'. It follows that for every fixed edge uv of G', the probability that both u and v have neighbours in R is at least $1/2$. Let W' be the set of all vertices of W that have a neighbour in R, and let G'' be the induced subgraph of G on W'. By linearity of expectation, the expected number of edges of G'' is at least half the number of edges of G', that is, at least $xD/4$. Hence there exists a set R of at most $2x/D$ vertices of G' so that the corresponding graph G'' has at least $xD/4$ edges. Fix such an R and note that as the neighbourhood of each vertex of G contains no copy of the forest F, the induced subgraph on it is colourable by at most $|F|$ colours, implying that indeed the induced subgraph G'' is $r = O(x/D)$-colourable.

By Lemma 2.1 it follows that $f(G'')$ exceeds half the number of edges of this subgraph by at least

$$\Omega\left(\frac{xD}{4} \cdot \frac{1}{2r}\right) = \Omega(D^2) = \Omega(m^{4/5}).$$

This implies, by Lemma 3.2(ii), that

$$f(G) \geqslant \frac{m}{2} + \Omega(m^{4/5}),$$

as needed. The fact that this is tight up to the constant in the Ω-notation follows from the spectral properties of the graph constructed in [1]. This is a triangle-free (n, d, λ)-graph with $d = \Theta(n^{2/3})$ and $\lambda = \Theta(n^{1/3})$. As it is triangle-free, it contains no copy of H, and its spectral properties imply, by Lemma 4.1, that if m is the number of its edges then

$$f(G) \leqslant \frac{m}{2} + O(n^{4/3}) = \frac{m}{2} + O(m^{4/5}). \qquad \square$$

4.3. Complete bipartite graphs

Proof of Theorem 1.4(i) (sketch). The proof is similar to the previous ones. Let G be a $K_{2,s}$-free graph with m edges, where $s \geqslant 2$. If G is D-degenerate, for $D = bm^{1/3}$ where $b = b(s) > 0$ will be chosen later, then the desired result follows. Indeed, as before, Lemma 3.3 implies that in this case

$$f(G) \geqslant \frac{m}{2} + \Omega\left(\frac{m}{\sqrt{D}}\right) = \frac{m}{2} + \Omega(m^{5/6}).$$

Otherwise we get, for the right choice of $b = b(s)$, a contradiction to Lemma 4.3. The Erdős–Rényi graph shows that the result is tight. $\square$

Proof of Theorem 1.4(ii) (sketch). The proof of this case is similar to the previous ones as well, but contains two extra twists. Let $G = (V, E)$ be a $K_{3,s}$-free graph with m edges, and n vertices (of positive degrees), where $s \geqslant 3$. We may and will assume that m is sufficiently large. If $n \geqslant \frac{m^{4/5}}{2}$, the desired result follows from Lemma 3.2(i). Thus we may assume that $n < \frac{m^{4/5}}{2}$. As long as there is a vertex of degree smaller than $m^{1/5}$ in G, omit it. This process terminates after deleting fewer than $m^{1/5}n < \frac{m}{2}$ edges, and thus we obtain an induced subgraph G' of G with at least $m/2$ edges and minimum degree at least $m^{1/5}$. Let V' denote the set of vertices of G'. Note that the induced subgraph on the neighbourhood of any vertex of degree d of G' contains no copy of $K_{2,s}$, and hence contains fewer than $sd^{3/2}$ edges, by Lemma 4.3.

Let $\eta > 0$ be a small fixed real, to be chosen later, and consider a random subset V'' of V' obtained by picking each vertex of V' randomly and independently, with probability η. Let G'' be the induced subgraph of G' (and hence of G) on V''. If a vertex has degree d in G', then its expected degree in G'' (assuming it is a vertex of G'') is ηd. The expected number of edges in its neighbourhood is at most $\eta^2 sd^{3/2}$. Since all degrees are large, it follows that with high probability, every vertex of degree d in G' that lies in V'' has degree at least, say, $\eta d/2$ in G''. Similarly, for every pair of vertices with codegree exceeding, say, $\eta\sqrt{d}$, the number of their common neighbours in V'' is highly concentrated around its expectation. This implies that with high probability the number of edges in every neighbourhood of G'' is at most, say, $2\eta^2 sd^{3/2}$. If η is sufficiently small as a function of s and ϵ, where ϵ is the number from Lemma 3.4, we can ensure that the assumptions of Lemma 3.4 hold for G''. Moreover, this graph has m' edges, where $m' \geqslant \eta^2 m/4 = \Omega(m)$, and it is, of course, $K_{3,s}$-free.

We can now proceed as in the previous proofs. If G'' is $bm^{2/5}$-degenerate, then by Lemma 3.4 $f(G'')$ exceeds half the number of its edges by at least $\Omega(m^{4/5})$ and the desired result follows, in this case, from Lemma 3.2(ii). Otherwise, it contains a subgraph with minimum degree $bm^{2/5}$, and hence at most $2m^{3/5}/b$ vertices, but this is impossible, for a sufficiently large $b = b(s)$, in view of Lemma 4.3 and the fact that the graph contains no copy of $K_{3,s}$. This completes the proof of the lower bound for $f(m, K_{3,s})$.

The fact that the result is tight follows from Lemma 4.1 and the spectral properties of the projective norm graphs constructed in [7] (see [6] or [22] for a computation of their eigenvalues). For any prime p and for appropriate choice of the parameters, this construction gives a $K_{3,3}$-free (n, d, λ)-graph with $n = p^3 - p^2, d = p^2 - 1$ and $\lambda = p$. □

5. Concluding remarks and open problems

(1) Closely related to the MaxCut problem is the so-called *judicious partition problem*, where the task is to find a partition $V = V_1 \cup V_2$ such that both parts V_1 and V_2 span the smallest possible number of edges. Formally, for a graph $G = (V, E)$ we define

$$g(G) = \min_{V=V_1\cup V_2} \max\{e(V_1), e(V_2)\},$$

where as usual $e(V_i)$ denotes the number of edges of G spanned by V_i. Bounding $g(G)$ from above immediately supplies a lower bound for $f(G)$: $f(G) \geqslant m - 2g(G)$. In the other direction, in our joint paper with Bollobás [3] we obtained a general result, connecting the size of an optimal bipartite cut with the best value of a judicious partition in it. We proved that if a graph $G = (V, E)$ with m edges has a bipartite cut of size $\frac{m}{2} + \delta$, then there exists a partition $V = V_1 \cup V_2$ such that both parts V_1, V_2 span at most $\frac{m}{4} - (1 - o(1))\frac{\delta}{2} + O(\sqrt{m})$ edges for the case $\delta = o(m)$, and at most $\left(\frac{1}{4} - \Omega(1)\right)m$ edges for $\delta = \Omega(m)$. This result immediately enables us to extend Theorems 1.1, 1.2, 1.3 and 1.4 to the corresponding ones for judicious partitioning.

(2) The main technical part of Section 3 is given by Lemma 3.1 and Lemma 3.3, extending the result of Shearer in [21]. We can give an alternative proof of a variant of Lemma 3.3, which gives essentially the same result, but is (possibly) somewhat more natural. Here is the argument for the triangle-free case. The same reasoning can be extended to the more general case considered in Section 3 as well.

Theorem 5.1. *There exists an absolute constant $c_1 > 0$ such that every triangle-free graph G with vertex set $V(G) = [n]$ and degree sequence $(d_1, \dots, d_n)$ has a cut with at least $dn/4 + c_1 \sum_{i=1}^{n} \sqrt{d_i}$ edges, where $d = \frac{1}{n}\sum_{i=1}^{n} d_i$ is the average degree of G.*

Proof. The theorem is an easy consequence of the following lemma.

Lemma 5.2. *Let G be as above. Then there exist disjoint subsets $A, B \subset [n]$ such that $e(A,B) - e(A) - e(B) \geqslant c_2 \sum_{i=1}^{n} \sqrt{d_i}$, where $c_2 > 0$ is an absolute constant.*

To derive Theorem 5.1, set $c_1 = c_2/2$, apply Lemma 5.2, and then add vertices from $V(G) \setminus (A \cup B)$ vertex by vertex to A or to B, each time adding a vertex to the set where it has fewer neighbours, breaking ties arbitrarily. The obtained cut clearly has at least $dn/4 + c_1 \sum_{i=1}^{n} \sqrt{d_i}$ edges. (See also Lemma 3.2(ii).)

Proof of Lemma 5.2. Let

$$\psi(d) = \max\left\{k : \sum_{i=k}^{d} \binom{d}{i} \left(\frac{1}{4}\right)^{i} \left(\frac{3}{4}\right)^{d-i} \geqslant \frac{1}{3}\right\}. \tag{5.1}$$

By de Moivre–Laplace $\psi(d) \geqslant d/4 + c\sqrt{d}$, for some absolute constant $c > 0$. Now, for $1 \leqslant i \leqslant n$, set

$$p_i = \sum_{j=\psi(d_i)}^{d_i} \binom{d_i}{j} \left(\frac{1}{4}\right)^{j} \left(\frac{3}{4}\right)^{d_i - j}, \tag{5.2}$$

and observe that $p_i \geqslant 1/3$ by the definition (5.1) of $\psi(d)$.

Form disjoint subsets A, B using the following procedure.

- Form A by taking each $i \in [n]$ into A independently and with probability $1/4$.
- Given A, define

$$B_0 = \{i \notin A : d(i, A) \geqslant \psi(d_i)\}.$$

- For each $i \in B_0$, include i in B independently and with probability $1/(3p_i)$.

Let X, Y, Z be random variables, counting the number of edges between A and B, inside A, and inside B, respectively. We estimate the means of X, Y, Z. Obviously, each $e \in E(G)$

is in A with probability 1/16, and therefore

$$\mathbb{E}[Y] = |E(G)|/16 = dn/32. \tag{5.3}$$

Let $i \in [n]$. Then

$$\Pr[i \in B_0] = \Pr[(i \notin A) \text{ and } (d(i,A) \geqslant \psi(d_i))] = \frac{3}{4} \cdot p_i,$$

and therefore $\Pr[i \in B] = \Pr[i \in B_0] \Pr[i \in B | i \in B_0] = 1/4$. Since the degree of each $i \in B$ to A is at least $\psi(d_i)$, we get

$$\mathbb{E}[X] \geqslant \sum_{i=1}^{n} \frac{\psi(d_i)}{4} \geqslant \sum_{i=1}^{n} \left(\frac{d_i}{16} + \frac{c\sqrt{d_i}}{4} \right) = \frac{dn}{16} + \frac{c}{4} \sum_{i=1}^{n} \sqrt{d_i}. \tag{5.4}$$

Now let $e = (i, j) \in E(G)$. Since G is triangle-free, i and j do not have common neighbours. It follows that

$$\Pr[i, j \in B_0] = \Pr[(i, j \notin A) \text{ and } (d(i,A) \geqslant \psi(d_i)) \text{ and } (d(j,A) \geqslant \psi(d_j))]$$

equals

$$\frac{9}{16} \left(\sum_{k=\psi(d_i)}^{d_i-1} \binom{d_i - 1}{k} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{d_i-1-k} \right) \left(\sum_{k=\psi(d_j)}^{d_j-1} \binom{d_j - 1}{k} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{d_j-1-k} \right) < \frac{9}{16} p_i p_j,$$

and thus $\Pr[(i, j) \in B] < 1/16$. Summing up, we obtain

$$\mathbb{E}[Z] < \frac{1}{16} |E(G)| = \frac{dn}{32}. \tag{5.5}$$

Using linearity of expectation and estimates (5.3), (5.4), (5.5), we derive

$$\mathbb{E}[X - Y - Z] \geqslant \frac{c}{4} \sum_{i=1}^{n} \sqrt{d_i}.$$

Set $c_2 = c/4$. Choose A, B for which $X - Y - Z \geqslant c_2 \sum_{i=1}^{n} \sqrt{d_i}$. The lemma and hence the theorem follow. □

(3) When some edges of the graph are contained in too many triangles, the assertion of Lemma 3.3 may cease to hold. As an example, consider the following graph $G = (V, E)$ constructed by Delsarte and Goethals, and by Turyn. Let q be a prime power and let V be the elements of the two-dimensional vector space over $GF(q)$. Thus G has $n = q^2$ vertices. Partition the $q + 1$ lines through the origin of the space into two sets P and N, where $|P| = k$. Two vertices x and y of the graph G are adjacent if and only if $x - y$ is parallel to a line in P. It is easy to check that G is $d = k(q - 1)$-regular, and hence its number of edges, m, is $kq^2(q - 1)/2 = \Theta(kq^3)$. Moreover, this is a strongly regular graph and its smallest eigenvalue is $-k$ (see, *e.g.*, [18]). It follows that $f(G)$ exceeds half the number of edges by at most $O(kq^2)$ (this can, in fact, be shown without using the spectral bound as well, since the graph is an edge-disjoint union of cliques of size q, and the maximum cut in such a clique exceeds half the number of its edges by only $O(q)$). The number $O(kq^2)$ is (much) smaller than $n\sqrt{d} = \Theta(q^{5/2}k^{1/2})$ whenever k is (much) smaller than q. Therefore, in each such case, the conclusion of Lemma 3.3 does not hold. Note that the number of common neighbours of each pair of adjacent vertices of G is $q-2+(k-1)(k-2)$. This is (of course) bigger than $\epsilon\sqrt{d}$.

(4) The authors of [3] conjecture that, for any fixed graph H, there is an $\epsilon(H)>0$ such that

$$f(m,H) \geqslant \frac{m}{2} + \Omega(m^{3/4+\epsilon}).$$

It clearly suffices to prove this conjecture for complete graphs H. A related plausible conjecture is that for every fixed graph H there is a constant $c(H)$ such that

$$f(m,H) = \frac{m}{2} + \Theta(m^{c(H)}).$$

It would be nice to prove this conjecture, and possibly even to determine the value of $c(H)$ for every graph H. This seems difficult.

(5) We conjecture that the assertion of Theorem 1.2 is tight, up to the value of $c(r)$, for all odd $r>3$. Even if this is true, however, a proof will not be easy, as it would imply that the maximum number of edges of a C_{r-1}-free graph on n vertices is $\Theta(n^{1+2/(r-1)})$ for all odd $r>3$. This is known only for $r-1\in\{4,6,10\}$, despite a considerable amount of effort to prove it for other values.

(6) It seems plausible to conjecture that the assertion of Theorem 1.4 can be extended for bigger values of t, as follows:

$$f(m,K_{t,s}) \geqslant \frac{m}{2} + c(s)m^{\frac{3t-1}{4t-2}},$$

for all $s\geqslant t\geqslant 2$. By the statement of the theorem this holds (and is tight) for $t\in\{2,3\}$. If true for larger values of t, this is tight at least for all $s\geqslant (t-1)!+1$, as shown by the projective norm graphs.

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