Density theorems for bipartite graphs and related Ramsey-type results

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Ramsey's theorem

Definition:

$r(G)$ is the minimum N such that every 2-edge-coloring of the complete graph K_N contains a monochromatic copy of graph G.

Theorem: (Ramsey-Erdős-Szekeres, Erdős)

$$2^{t/2} \leq r(K_t) \leq 2^{2t}.$$

Question: (*Burr-Erdős 1975*)

How large is $r(G)$ for a sparse graph G on n vertices?

CONJECTURE: (*Burr-Erdős 1975*)

For every d there exists a constant c_d such that if a graph G has n vertices and maximum degree d, then

$$r(G) \leq c_d n.$$

THEOREM:

1. (*Chvátal-Rödl-Szemerédi-Trotter 1983*) c_d exists.
2. (*Eaton 1998*) $c_d \leq 2^{2^{\alpha d}}$.
3. (*Graham-Rödl-Ruciński 2000*) $2^{\beta d} \leq c_d \leq 2^{\alpha d \log^2 d}$.

   Moreover, if G is bipartite, $r(G) \leq 2^{\alpha d \log d} n$.

Density theorem for bipartite graphs

Theorem: (*Fox-S.*)

Let G be a bipartite graph with n vertices and maximum degree d and let H be a bipartite graph with parts $|V_1| = |V_2| = N$ and εN^2 edges. If $N \geq 8d\varepsilon^{-d}n$, then H contains G.

Corollary:

For every bipartite graph G with n vertices and maximum degree d,

$$r(G) \leq d2^{d+4}n.$$

(D. Conlon independently proved that $r(G) \leq 2^{(2+o(1))d}n$.)

Proof: Take $\varepsilon = 1/2$ and H to be the graph of the majority color.

Ramsey numbers for cubes

Definition:

The *binary cube* Q_d has vertex set $\{0,1\}^d$ and x, y are adjacent if x and y differ in exactly one coordinate.

Conjecture: (*Burr-Erdős 1975*)

Cubes have linear Ramsey numbers, i.e., $r(Q_d) \leq \alpha 2^d$.

Theorem:

1. (*Beck 1983*) $r(Q_d) \leq 2^{\alpha d^2}$.
2. (*Graham-Rödl-Ruciński 2000*) $r(Q_d) \leq 2^{\alpha d \log d}$.
3. (*Shi 2001*) $r(Q_d) \leq 2^{2.618d}$.

New bound: (*Fox-S.*)

$$r(Q_d) \leq 2^{(2+o(1))d}.$$

Ramsey multiplicity

Conjecture: (*Erdős 1962, Burr-Rosta 1980*)

Let G be a graph with v vertices and m edges. Then every 2-edge-coloring of K_N contains

$$\gtrsim \quad 2^{1-m} N^v$$

labeled monochromatic copies of G.

Theorem:

1. *(Goodman 1959)* *True for* $G = K_3$.
2. *(Thomason 1989)* *False for* $G = K_4$.
3. *(Fox 2007)* *For some* G, *# of copies can be* $\leq m^{-\alpha m} N^v$.

Subgraph multiplicity

Conjecture: (*Sidorenko 1993, Simonovits 1984*)

Let G be a bipartite graph with v vertices and m edges and H be a graph with N vertices and $\varepsilon\binom{N}{2}$ edges. Then the number of labeled copies of G in H is $\gtrsim \varepsilon^m N^v$.

It is true for:

complete bipartite graphs, trees, even cycles, and binary cubes.

Theorem:

If G is bipartite with maximum degree d and $m = \Theta(dv)$ edges, then the number of labeled copies of G in H is at least $\varepsilon^{\Theta(m)} N^v$.

Topological subdivision

Definition:

A *topological copy* of a graph Γ is any graph formed by replacing edges of Γ by internally vertex disjoint paths.
It is called a *k-subdivision* if all paths have k internal vertices.

Conjecture: (*Mader 1967, Erdős-Hajnal 1969*)

Every graph with n vertices and at least cp^2n edges contains a topological copy of K_p.

(*Proved by Bollobás-Thomason and by Komlós-Szemerédi*)

Conjecture: (*Erdős 1979, proved by Alon-Krivelevich-S 2003*)

Every n-vertex graph H with at least c_1n^2 edges contains the 1-subdivision of K_m with $m = c_2\sqrt{n}$.

Subdivided graphs

Question:

Can one find a 1-subdivision of graphs other than cliques?

Known results: *(Alon-Duke-Lefmann-Rödl-Yuster, Alon)*

1. Every n-vertex H with at least $c_1 n^2$ edges contains the 3-subdivision of every graph Γ with $c_2 n$ edges.
2. If G is the 1-subdivision of a graph Γ with n edges, then $r(G) \leq cn$.

Theorem: *(Fox-S.)*

If H has N vertices, εN^2 edges, and $N > c\varepsilon^{-3} n$, then H contains the 1-subdivision of every graph Γ with n edges.

Erdős-Hajnal conjecture

Definition:

A graph on n vertices is *Ramsey* if both its largest clique and independent set have size at most $C \log n$.

Theorem: (*Erdős-Hajnal, Promel-Rödl*)

Every Ramsey graph on n vertices contains an induced copy of every graph G of constant size.

(Moreover, this is still true for G up to size $c \log n$.)

Conjecture (*Erdős-Hajnal 1989*)

Every graph H on n vertices without an induced copy of a fixed graph G contains a clique or independent set of size at least n^{ε}.

Erdős-Hajnal conjecture

A *bi-clique* is a complete bipartite graph with parts of equal size.

Known results: (*Erdős-Hajnal, Erdős-Hajnal-Pach*)

If H has n vertices and no induced copy of G, then

1. H contains a clique or independent set of size $e^{c\sqrt{\log n}}$.
2. H or its complement $\overline{H}$ has a bi-clique of size n^{ε}.

Theorem: (*Fox-S.*)

If H has n vertices and no induced copy of G of size k, then

1. H has a clique or independent set of size $ce^{c\sqrt{\frac{\log n}{k}}}\log n$.
2. H has a bi-clique or an independent set of size n^{ε}.

Hypergraph Ramsey numbers

A hypergraph is *k-uniform* if every edge has size k.

Definition:

For a k-uniform hypergraph G, let $r(G)$ be the minimum N such that every 2-edge-coloring of the complete k-uniform hypergraph $K_N^{(k)}$ contains a monochromatic copy of G.

Theorem: (*Erdős-Hajnal, Erdős-Rado*)

The Ramsey number of the complete k-uniform hypergraph $K_n^{(k)}$ satisfies

$$t_{k-1}(cn^2) \leq r(K_n^{(k)}) \leq t_k(n),$$

where the *tower function* $t_i(x)$ is defined by

$$t_1(x) = x,\ t_2(x) = 2^x,\ t_3(x) = 2^{2^x},\ \ldots,\ t_{i+1}(x) = 2^{t_i(x)},\ \ldots$$

Ramsey numbers for sparse hypergraphs

Conjecture: (*Hypergraph generalization of Burr-Erdős conjecture*)

For every d and k there exists $c_{d,k}$ such that if G is a k-uniform hypergraph with n vertices and maximum degree d, then

$$r(G) \leq c_{d,k} n.$$

1. (*Kostochka-Rödl* 2006) $\quad r(G) \leq n^{1+o(1)}$.
2. Proved for $k = 3$ by Cooley-Fountoulakis-Kühn-Osthus and Nagle-Olsen-Rödl-Schacht.
3. Proved for all k by Cooley-Fountoulakis-Kühn-Osthus and Ishigami.
4. These proofs give Ackermann-type bound on $c_{d,k}$.

Theorem: (*Conlon-Fox-S.*)

If G is a k-uniform hypergraph with n vertices and maximum degree d, then $\quad r(G) \leq c_{d,k} n \quad$ with $\quad c_{d,k} \leq t_k(cd)$.

Topological graphs

Definitions:

A *topological graph* G is a graph drawn in the plane with vertices as points and edges as curves connecting its endpoints such that any two edges have at most one point in common.

G is a *thrackle* if every pair of edges intersect.

Conjecture: (*Conway 1960s*)

Thrackle with n vertices has at most n edges.

In particular, every topological graph with more edges than vertices, contains a pair of disjoint edges.

Known: Every thrackle on n vertices has $O(n)$ edges.
(*Lovász-Pach-Szegedy, Cairns-Nikolayevsky*)

Disjoint edges in graph drawings

Question:

Do dense topological graphs contain large patterns of pairwise disjoint edges?

Theorem: (*Pach-Tóth*)

Every topological graph with n vertices and at least $n(c \log n)^{4k-8}$ edges has k pairwise disjoint edges.

Theorem: (*Fox-S.*)

Every topological graph with n vertices and $c_1 n^2$ edges has two edge subsets E', E'' of size $c_2 n^2$ such that every edge in E' is disjoint from every edge in E''.