Disjoint Systems

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ABSTRACT

A *disjoint system of type* $(\forall, \exists, k, n)$ is a collection $\mathscr{C} = \{\mathscr{A}_1, \ldots, \mathscr{A}_m\}$ of pairwise disjoint families of k-subsets of an n-element set satisfying the following condition. For every ordered pair $\mathscr{A}_i$ and $\mathscr{A}_j$ of distinct members of $\mathscr{C}$ and for every $A \in \mathscr{A}_i$ there exists a $B \in \mathscr{A}_j$ that does not intersect A. Let $D_n(\forall, \exists, k)$ denote the maximum possible cardinality of a disjoint system of type $(\forall, \exists, k, n)$. It is shown that for every fixed $k \geq 2$,

$$\lim_{n\to\infty} D_n(\forall, \exists, k)\binom{n}{k}^{-1} = \frac{1}{2}.$$

This settles a problem of Ahlswede, Cai, and Zhang. Several related problems are considered as well.

1. INTRODUCTION

In extremal finite set theory one is usually interested in determining or estimating the maximum or minimum possible cardinality of a family of subsets of an n element set that satisfies certain properties. See [5], [7], and [8] for a comprehensive study of problems of this type. In several recent papers (see [1–3]), Ahlswede, Cai, and Zhang considered various extremal problems that study the maximum or minimum possible cardinality of a collection of families of subsets of an n-set that satisfies certain properties. They observed that many of the classical extremal problems dealing with families of sets suggest numerous intriguing

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questions when one replaces the notion of a family of sets by the more complicated one of a collection of families of sets.

In the present note we consider several problems of this type that deal with disjoint systems. Let $N = \{1, 2, \ldots, n\}$ be an n element set, and let $\mathscr{C} = \{\mathscr{A}_1, \ldots, \mathscr{A}_m\}$ be a collection of pairwise disjoint families of k-subsets of N. $\mathscr{C}$ is a *disjoint system of type* $(\exists, \forall, k, n)$ if for every ordered pair $\mathscr{A}_i$ and $\mathscr{A}_j$ of distinct members of $\mathscr{C}$ there exists an $A \in \mathscr{A}_i$ that does not intersect any member of $\mathscr{A}_j$. Similarly, $\mathscr{C}$ is a *disjoint system of type* $(\forall, \exists, k, n)$ if for every ordered pair $\mathscr{A}_i$ and $\mathscr{A}_j$ of distinct members of $\mathscr{C}$ and for every $A \in A_i$ there exists a $B \in \mathscr{A}_j$ that does not intersect A. Finally, $\mathscr{C}$ is a *disjoint system of type* $(\exists, \exists, k, n)$ if for every ordered pair $\mathscr{A}_i$ and $\mathscr{A}_j$ of distinct members of $\mathscr{C}$ there exists an $A \in \mathscr{A}_i$ and a $B \in \mathscr{A}_j$ that does not intersect A.

Let $D_n(\exists, \forall, k)$ denote the maximum possible cardinality of a disjoint system of type $(\exists, \forall, k, n)$. Let $D_n(\forall, \exists, k)$ denote the maximum possible cardinality of a disjoint system of type $(\forall, \exists, k, n)$, and let $D_n(\exists, \exists, k)$ denote the maximum possible cardinality of a disjoint system of type $(\exists, \exists, k, n)$. Trivially, for every n,

$$D_n(\exists, \forall, 1) = D_n(\forall, \exists, 1) = D_n(\exists, \exists, 1) = n .$$

It is easy to see that every disjoint system of type $(\exists, \forall, k, n)$ is also a system of type $(\forall, \exists, k, n)$, and every system of type $(\forall, \exists, k, n)$ is also of type $(\exists, \exists, k, n)$. Therefore, for every $n \geq k$

$$D_n(\exists, \forall, k) \leq D_n(\forall, \exists, k) \leq D_n(\exists, \exists, k) .$$

In this note we determine the asymptotic behavior of these three functions for every fixed k, as n tends to infinity.

Theorem 1.1. *For every* $k \geq 2$

$$\lim_{n\to\infty} D_n(\exists, \forall, k)\binom{n}{k}^{-1} = \frac{1}{k+1} .$$

Theorem 1.2. *For every* $k \geq 2$

$$\lim_{n\to\infty} D_n(\forall, \exists, k)\binom{n}{k}^{-1} = \frac{1}{2} .$$

Corollary 1.3. *For every* $k \geq 2$

$$\lim_{n\to\infty} D_n(\exists, \exists, k)\binom{n}{k}^{-1} = \frac{1}{2} .$$

Theorem 1.1 settles a conjecture of Ahlswede, Cai, and Zhang [3], who proved it for $k = 2$ [2]. The main tool in its proof, presented in Section 2, is a result of Frankl and Füredi [9]. The proof of Theorem 1.2, which settles another question raised in [3] and proved for $k = 2$ in [2], is more complicated and combines combinatorial and probabilistic arguments. This proof and the simple derivation of Corollary 1.3 from its assertion are presented in Section 3.

2. HYPERGRAPH DECOMPOSITION AND DISJOINT SYSTEMS

In this section we prove Theorem 1.1. A *k-graph* is a hypergraph in which every edge contains precisely k vertices. We need the following result of Frankl and Füredi.

Lemma 2.1 ([9]). *Let $H=(U,\mathcal{F})$ be a fixed k-graph with $|\mathcal{F}|=f$ edges. Then one can place*

$$(1-o(1))\binom{n}{k}\Big/f$$

copies $H_1=(U_1,\mathcal{F}_1)$, $H_2=(U_2,\mathcal{F}_2),\ldots$ of H into a complete k-graph on n vertices such that $|U_i\cap U_j|\le k$ for all $i\ne j$, and if $|U_i\cap U_j|=k$ and $U_i\cap U_j=B$, then $B\not\in\mathcal{F}_i$, $B\not\in\mathcal{F}_j$. Here the $o(1)$ term tends to zero as n tends to infinity. □

Proof of Theorem 1.1. The lower bound for $D_n(\exists,\forall,k)$ is a direct corollary of Lemma 2.1. Let $(U,\mathcal{A})$ be the k-graph consisting of $k+1$ pairwise disjoint edges. By the lemma we can place $(1-o(1))\binom{n}{k}/(k+1)$ edge disjoint copies of this graph into a complete k-graph on n vertices, so that any two copies will have at most k common vertices. Therefore, if we take the edges of each copy as a family, we get $(1-o(1))\binom{n}{k}/(k+1)$ pairwise disjoint families, which form a disjoint system. Let $H_1=(U_1,\mathcal{A}_1)$ and $H_2=(U_2,\mathcal{A}_2)$ be two such families. Since $|U_1\cap U_2|\le k$ and the family H_1 consists of $k+1$ pairwise disjoint sets, there is a set $A\in\mathcal{A}_1$ which does not contain any point of $U_1\cap U_2$. This A does not intersect any set of the family H_2. Therefore, our disjoint system is of type $(\exists,\forall,k,n)$.

We next establish an upper bound for $D_n(\exists,\forall,k)$. Let $\mathcal{C}=\{\mathcal{A}_1,\mathcal{A}_2,\ldots\}$ be a disjoint system of type $(\exists,\forall,k,n)$. We denote by N_1, $|N_1|=n_1$, the set of all families of $\mathcal{C}$ containing one element, by N_2, $|N_2|=n_2$, the set of those containing from two up to k elements, and by N_3, $|N_3|=n_3$, the set of those containing more than k elements. Since sets in any two one-element families are disjoint, we have $n_1\le n/k$. Let $\mathcal{A}_i=\{A_1,\ldots,A_t\}$ be a family with $2\le t\le k$ elements. By the definition of a system of type $(\exists,\forall,k,n)$, we conclude that any set B with the properties

$$|B|=k,\quad B\subset\bigcup_{j=1}^{t}A_j,\quad B\cap A_j\ne\emptyset\quad\forall j \tag{1}$$

cannot be used as an element of any other family. We next bound the number of such sets B from below. Choose not necessarily distinct $a_j\in A_j$ for $2\le j\le t$ such that $a_j\not\in A_1$. Put $L=\{a_2,\ldots,a_t\}$, then $|L|\le k-1$. Let $\mathcal{L}$ be the family of all sets of the form $L_r=L\cup Y_r$ where Y_r ranges over all $(k-|L|)$ element subsets of A_1. Clearly each such L_r satisfies the properties (1). Moreover, $L_r\ne A_1$ and $|\mathcal{L}|\ge k$.

We claim that no k-set can satisfy the properties (1) for two or more families from $\mathcal{C}$. Indeed assume this is false. Let B be a set which satisfies the properties (1) for two families $\mathcal{A}=\{A_1,\ldots,A_l\}$ and $\mathcal{F}=\{F_1,\ldots,F_m\}$. By the definition of a disjoint system of type $(\exists,\forall,k,n)$ there exists a set $A_i\in\mathcal{A}$, $1\le i\le l$ such that $A_i\cap F_j=\emptyset$ for all j. Since by (1) $B\subseteq\bigcup_{j=1}^{m}F_j$ we conclude that $B\cap A_i=\emptyset$, contradicting (1) and proving our claim.

Therefore, with each family $\mathcal{A}_i$ in N_2 we can associate $k+1$ sets (k-sets of the form L_r as above together with the set A_1) that cannot be associated with any other family and are not members of any other family. In addition each family in N_3 contains at least $k+1$ k-sets. This implies that

$$(k+1)n_2 + (k+1)n_3 \le \binom{n}{k} .$$

Therefore, $(n_2 + n_3) \le \binom{n}{k}/(k+1)$. Together with the fact that $n_1 \le n/k$, we conclude that $D_n(\exists, \forall, k, n) \le \frac{n}{k} + \binom{n}{k}/(k+1)$ completing the proof. □

3. RANDOM GRAPHS AND DISJOINT SYSTEMS

In this section we prove Theorem 1.2. We need the following two probabilistic lemmas.

Lemma 3.1 (Chernoff, see, e.g., [4], Appendix A). *Let X be a random variable with the binomial distribution $B(n, p)$. Then for every $a > 0$ we have*

$$\Pr(|x - np| > a) < 2e^{-2a^2/n} .$$ □

Let L be a graph-theoretic function. L satisfies the *Lipschitz condition* if for any two graphs H, H' on the same set of vertices that differ only in one edge we have $|L(H) - L(H')| \le 1$. Let $G(n, p)$ denote, as usual, the random graph on n labeled vertices in which every pair, randomly and independently, is chosen to be an edge with probability p. (See, e.g., [6].)

Lemma 3.2 ([4], Chap. 7). *Let L be a graph-theoretic function satisfying the Lipschitz condition and let $\mu = E[L(G)]$ be the expectation of $L(G)$, where $G = G(n, p)$. Then for any $\lambda > 0$*

$$\Pr[|L(G) - \mu| > \lambda\sqrt{m}] < 2e^{-\lambda^2/2}$$

where $m = \binom{n}{2}$. □

Proof of Theorem 1.2. Let n_1 be the number of families containing only one element. The same argument as in the proof of Theorem 1.1 shows that $n_1 \le n/k$. This settles the required upper bound for $D_n(\forall, \exists, k)$, since all other families contain at least two sets.

We prove the lower bound using probabilistic arguments. We show that for any $\epsilon > 0$ there are at least $\frac{1}{2}(1-\epsilon)\binom{n}{k}$ families which form a disjoint system of type $(\forall, \exists, k, n)$, provided that n is sufficiently large (as a function of ϵ and k). We first outline the main idea in the (probabilistic) construction and then describe the details. Let $G = G(n, p)$ be a random graph, where p is a constant, to be specified later, which is very close to 1. We use this graph to build another random graph G_1, whose vertices are all k-cliques in G. Two vertices of G_1 are adjacent if and only if the induced subgraph on the corresponding k-cliques in G is the union of two vertex disjoint k-cliques with no edges between them. Following the standard terminology in the study of random graphs, we say that an event holds *almost* *surely* if the probability it holds tends to 1 as n tends to infinity. We will prove that almost surely the following two events happen. First, the number of vertices in G_1 is greater than $(1-\epsilon/2)\binom{n}{k}$. Second, G_1 is almost regular, i.e., for every (small) $\delta > 0$ there exists a (large) number d such that the degree $d(x)$ of any vertex x of G_1 satisfies $(1-\delta)d < d(x) < (1+\delta)d$, provided that n is sufficiently large.

Suppose $G_1 = (V, E)$ satisfies these properties. By Vizing's theorem [10], the chromatic index $\chi'(G_1)$ of G_1 satisfies $\chi'(G_1) \le (1+\delta)d + 1$. Since for any $x \in G_1$, we have $d(x) \ge (1-\delta)d$, the number of edges $|E|$ of G_1 is at least $\frac{(1-\delta)d|V|}{2}$. Hence there exists a matching in G_1 which contains at least $\frac{(1-\delta)d|V|}{2}/\chi'(G_1) \sim \frac{(1-\delta)|V|}{2(1+\delta)}$ edges. This matching covers almost all vertices of G_1, as δ is small, providing a system of pairs of k-sets covering almost all the $\binom{n}{k}$ k-sets. Taking each pair as a family, we have a disjoint system of size at least $\frac{1}{2}(1-\epsilon)\binom{n}{k}$ and ϵ can be made arbitrarily small for all n sufficiently large.

We next show that the resulting system is a disjoint system of type $(\forall, \exists, k, n)$. Assume that this is false, and let $\mathcal{A} = \{A_1, A_2\}$ and $\mathcal{B} = \{B_1, B_2\}$ be two pairs where $A_1 \cap B_i \neq \emptyset$ for $i = 1, 2$. Choose $x_1 \in A_1 \cap B_1$ and $x_2 \in A_1 \cap B_2$. Since x_1 and x_2 belong to A_1, they are adjacent in $G = G(n, p)$. However, $x_1 \in B_1$, $x_2 \in B_2$, and this contradicts the fact that the subgraph of G induced on $B_1 \cup B_2$ has no edges between B_1 and B_2. Thus the system is indeed of type $(\forall, \exists, k, n)$ and

$$D_n(\forall, \exists, k) > \frac{1}{2}(1-\epsilon)\binom{n}{k}$$

for every $\epsilon > 0$, provided that $n > n_0(k, \epsilon_1)$, as needed.

The proof that indeed G_1 has the required properties almost surely will be deduced from the following two statements.

Fact 1. *$G = G(n, p)$ satisfies the following condition almost surely. For every set X of k vertices of G, the number of vertices which do not have any neighbor in X is*

$$(1 + o(1))(1-p)^k(n-k),$$

where here the $o(1)$ term tends to zero as n tends to infinity.

Fact 2. *For any $c > 0$, if n is sufficiently large, $G = G(n, p)$ satisfies the following condition almost surely. For every set Y of n_1 vertices of G, where $cn \le n_1 \le n$, the number of k-cliques of the induced subgraph of G on Y is close to its expectation, i.e., is*

$$(1 + o(1))\binom{n_1}{k}p^{\binom{k}{2}},$$

where the $o(1)$ term tends to zero as n tends to infinity.

The proof of Fact 1 is a standard application of Lemma 3.1 and is thus left to the reader.

Proof of Fact 2. Let $H(Y, p)$ denote the induced subgraph of $G = G(n, p)$ on a fixed set Y of vertices, where $|Y| = n_1$. Let L be the graph-theoretic function given by

$$L(H') = \frac{1}{\binom{n_1}{k-2}} N(H')\,,$$

where H' is a graph on Y and $N(H')$ denotes the number of k-cliques in H'.

The expected value of $L(H(Y, p))$ is easily seen to be $\mu(L) = \frac{1}{\binom{n_1}{k-2}}\binom{n_1}{k}p^{\binom{k}{2}}$, and the expected value of $N = N(H(Y, p))$ is $\mu(N) = \binom{n_1}{k}p^{\binom{k}{2}}$. By the definition of L, if H_1 and H_2 are two graphs on Y, which differ only in one edge than $|L(H_1) - L(H_2)| \le 1$, since the number of k-cliques of $H(Y, p)$ containing an edge is at most $\binom{n_1}{k-2}$. Thus, by Lemma 3.2

$$\Pr\left[|L(H(Y, p)) - \mu(L)| > n^{3/4}\sqrt{\binom{n_1}{2}}\right] < 2e^{-n^{3/2}/2}\,.$$

Consequently,

$$\Pr\left[|N(H(Y, p)) - \mu(N)| > n^{3/4}\binom{n_1}{k-2}\sqrt{\binom{n_1}{2}}\right] < 2e^{-n^{3/2}/2}\,.$$

Since k and p are constants and $n_1 \ge cn$, we conclude that

$$n^{3/4}\binom{n_1}{k-2}\sqrt{\binom{n_1}{2}} = \gamma\binom{n_1}{k}\,p^{\binom{k}{2}} = \gamma\mu(N)\,,$$

here $\gamma = \gamma(n, n_1, k, p)$ tends to 0 as n tends to infinity.

The total number of possible sets Y is clearly less than 2^n. Hence, the probability that, for some Y, $N(H(Y, p))$ deviates by more than $\gamma\mu(N(H(Y, p))$ from its expectation is less than $2^{n+1}e^{-n^{3/2}/2}$, which tends to zero as n tends to infinity. This completes the proof of Fact 2.

Returning to the proof of the theorem, consider a k-clique X in G. The degree d of X as a vertex of G_1 is the number of k-cliques in the induced subgraph of G on the set of all vertices which have no neighbors in X. By Facts 1 and 2 each such degree is almost surely

$$(1 + o(1))\binom{n_1}{k}p^{\binom{k}{2}}$$

where $n_1 = (1 + o(1))(1 - p)^k(n - k)$. Therefore, almost surely G_1 is almost regular and the degrees of its vertices tend to infinity with n.

In a similar manner, Fact 2 applied to the set Y of all vertices of G implies that the number of k-cliques in $G = G(n, p)$ (which is the number of vertices of G_1) is almost surely

$$(1 + o(1))\binom{n}{k}p^{\binom{k}{2}}\,.$$

Fix $p < 1$ so that $p^{\binom{k}{2}} > 1 - \epsilon/4$ for the required ϵ. With this p, almost surely the number of vertices in G_1 is more than

$$\left(1 - \frac{\epsilon}{2}\right)\binom{n}{k}\,,$$

as needed. Therefore, our procedure produces, with high probability, a disjoint system of the required type with at least $\frac{1}{2}(1-\epsilon)\binom{n}{k}$ pairs, completing the proof. □

Remark. By combining our method here with the technique of [9], we can prove the following extension of Lemma 2.1, which may be useful in further applications. Since the proof is similar to the last one, we omit the details.

Proposition 3.3. *Let $H=(U,\mathcal{F})$ be a fixed k-graph with $|\mathcal{F}|=f$ edges and let g denote the maximum cardinality of an intersection of two distinct edges of H. Then one can place*

$$(1-o(1))\binom{n}{k}\Big/f$$

pairwise edge-disjoint copies $H_1=(U_1,\mathcal{F}_1)$, $H_2=(U_2,\mathcal{F}_2),\ldots$ of H into a complete k-graph on n vertices such that $|U_i \cap U_j| \le k$ for all $i \ne j$, and such that if for some $i \ne j$ there is an $F_j \in \mathcal{F}_j$ so that $|F_j \cap U_i| \ge g+2$, then there is an $F_i \in \mathcal{F}_i$ so that $F_j \cap U_i \subset F_i$. Here the $o(1)$ term tends to zero as n tends to infinity.

Proof of Corollary 1.3. Let n_1 be the number of one element families in a disjoint system of type $(\exists,\exists,k,n)$. The trivial argument used in the proofs of Theorems 1.1 and 1.2 shows that $n_1 \le n/k$ and thus implies that

$$D_n(\exists,\exists,k) \le \frac{n}{k}+\frac{1}{2}\binom{n}{k}.$$

As observed in Section 1, $D_n(\forall,\exists,k) \le D_n(\exists,\exists,k)$ and hence, by Theorem 1.2, the desired result follows. □

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