Paul Erdős and Graph Ramsey Theory

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Ramsey theorem

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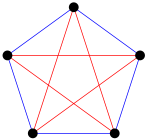

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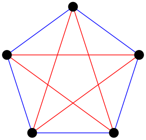

Theorem: (*Ramsey 1930*)

For all s, n, the Ramsey number $r(s, n)$ is finite.

Diagonal Ramsey numbers

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Lower bound: Color every edge randomly. Probability that a given set of n vertices forms a monochromatic clique is $2 \cdot 2^{-\binom{n}{2}}$.
Use the union bound.

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- Can we strengthen Ramsey's theorem to show that the monochromatic clique has some additional structure?
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Motivation: This will test the limits of current methods and may also lead to the development of new techniques that eventually would give better estimates for $r(n, n)$.

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Remark: A simple application of $r(n,n) \leq 2^{2n}$ only gives $w(S) \geq \frac{\log N}{2} \cdot \frac{1}{\log N} = \frac{1}{2}$.

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Theorem: (*Rödl 2003*)

There is a monochromatic clique with weight $\approx \log\log\log\log N$.

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Partition vertices $[2, N]$ into $t = \log\log N$ intervals $I_j = [2^{2^{j-1}}, 2^{2^j})$.

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Theorem: (*Conlon-Fox-S. 2013+*)

Maximum weight of monochromatic clique is $\Theta(\log\log\log N)$.

Conjecture: (*Burr-Erdős 1975*)

For every d there exists a constant c_d such that if a graph G has n vertices and maximum degree d, then

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Theorem: (*Graham-Rödl-Ruciński 2000*)

$$2^{\Omega(d)} \leq c_d \leq 2^{O(d \log^2 d)}.$$

Methods: Local density

Key idea: Suppose that a red-blue edge-coloring of the complete graph K_N has the following property:

For any two disjoint sets X, Y of linear size there are at least $\varepsilon|X||Y|$ red edges between them (i.e., density of red $d(X, Y) \geq \varepsilon$).

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Remark: If density of red edges is less than $\frac{1}{2d}$ then we can find blue copy of G greedily.

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The Ramsey number of n-vertex graph with maximum degree d is at most $c_d n$ with $c_d = 2^{O(d)}$.

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Theorem: *(Conlon-Fox-S. 2009 & 2012)*

Let G be an n-vertex graph with maximum degree d, then

$$r(G) \le 2^{O(d \log d)} \cdot n.$$

If G is bipartite, then

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Remark: The second result is tight and gives the best known bound for Ramsey numbers of *binary cubes*. Cube Q_d has vertex set $\{0,1\}^d$ and x, y are adjacent if x and y differ in exactly one coordinate.

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Conjecture: (*Burr-Erdős 1975*)

If G is a d-degenerate n-vertex graph, then $r(G)$ is linear in n.

Linear bound is known for:

- Bounded degree graphs (*Chvátal-Rödl-Szemerédi-Trotter 83*)
- Planar graphs and graphs drawn on bounded genus surfaces (*Chen-Schelp 93*)
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Remark: Best upper bound on $r(G)$ is $2^{c\sqrt{\log n}} \cdot n$ (*Fox-S 09*).

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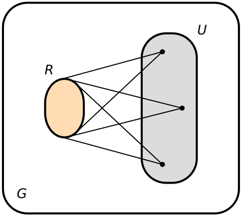

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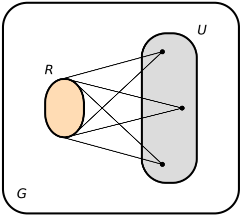

If some set of d vertices has only *few* common neighbors, it is unlikely that all the members of R will be chosen among these neighbors. Hence we do not expect U to contain any such d vertices. $\square$

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Remark: This coloring also contains two sets X and Y such that, all edges in X and from X to Y have the same color and

$$|X| = t\,, \quad |Y| \ge k^{-20t/k} N\,.$$

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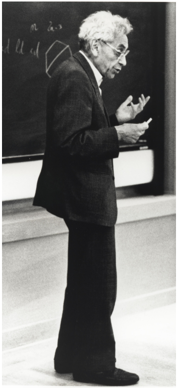
- Apply above process recursively with $k_1 \ll k_2 \ll \ldots$.



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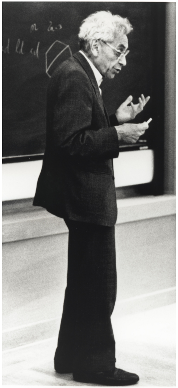

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These challenges, the fundamental nature of Graph Ramsey Theory and its tight connection with other areas will ensure that in the future this subject, started by Paul Erdős 40 years ago, will continue to play an essential role in the development of discrete mathematics.