Cycles and clique-minors in expanders

Benny Sudakov
UCLA and Princeton University

Expanders

Definition:

The vertex boundary of a subset X of a graph G:

$$\partial X = \{\text{all vertices in } G \backslash X \text{ with at least one neighbor in } X\}.$$

Definition:

- Graph G on n vertices is an expander if $\forall X \subset G$ of order at most $n/2$, $\frac{|\partial X|}{|X|}$ is "large" (at least constant).
- G is locally expander if $\frac{|\partial X|}{|X|}$ is "large" $\forall X \subset G$ up to a certain size.

Applications:

Communication networks, Derandomization, Metric embeddings, Computational complexity, Coding theory, Markov chains, ...

H-free graphs

Definition:

Graph G is called H-free if it contains no subgraph (not necessarily induced) isomorphic to H.

Key observation:

Let H be a fixed bipartite graph and let G be an H-free graph with minimum degree d. Then G is locally expanding.

Examples of H:

- C_{2k} $=$ cycle of length $2k$.
- $K_{s,t}$ $=$ complete bipartite graph with parts of size $s \leq t$.

Theorem: (*Bondy-Simonovits, Kövári-Sós-Turán*)

- For every C_{2k}-free graph G on n vertices, $e(G) \le c\, n^{1+1/k}$.
- For every $K_{s,t}$-free graph G on n vertices, $e(G) \le c\, n^{2-1/s}$.

Corollary:

- If G is a C_{2k}-free graph with minimum degree d, then
$$|\partial X| > 2|X| \text{ for all subsets } X, |X| \le O(d^k).$$
- If G is a $K_{s,t}$-free graph with minimum degree d, then
$$|\partial X| > 2|X| \text{ for all subsets } X, |X| \le O(d^{s/(s-1)}).$$

Proof. If $|\partial X| \le 2|X|$ and $|X|$ is "small", then $Y = X \cup \partial X$ has size at most $3|X|$ and at least $d|X|/2$ edges. This contradicts the theorem. $\square$

Cycle lengths

Definition:

- $cir(G)$ = circumference of graph G is the length of the longest cycle in G.
- $\mathcal{C}(G)$ = set of cycle lengths in G, i.e., all ℓ such that $C_\ell \subset G$.

Typical questions:

- How large is the circumference of G?
- How many different cycle lengths are in G?
- What are the arithmetic properties of $\mathcal{C}(G)$, e.g., is there a cycle in G whose length is a power of 2?

Toy example:

If G has minimum degree d then $cir(G) \geq d$ and $|\mathcal{C}(G)| \geq d - 1$.

Graphs of large girth

Definition:

Girth of G is the length of the shortest cycle in G.

Question: (*Ore 1967*)

How large is $cir(G)$ in graphs with min. degree d and girth $2k+1$?

Known results:

If G has minimum degree d and girth $2k+1$, then

- $cir(G) \geq \Omega(kd)$ - *Ore 1967*
- Improvements by *Zhang, Zhao, Voss*
- $cir(G) \geq \Omega(d^{\lfloor \frac{k+1}{2} \rfloor})$ - *Ellingham and Menser 2000.*

Many cycle lengths

Conjecture: (*Erdős 1992*)

If graph G has average degree d and girth $2k+1$, then

$$|\mathcal{C}(G)| \geq \Omega(d^k).$$

Remarks:

- Tight for $k = 2, 3, 5$, and "probably" for all k. It is believed that for every fixed k there are graphs of order $O(d^k)$ with minimum degree d and girth $2k+1$.
- True for $k = 2$ by *Erdős, Faudree, Rousseau, and Schelp*.

Problem: (*Erdős*)

If G has large (but constant) average degree, does it contain a cycle of length 2^i for some $i > 0$?

New results

Theorem: (*S.- Verstraëte*)

Let G be a graph with average degree d.

- If G is C_{2k}-free then $\left|\mathcal{C}(G)\right| \geq \Omega(d^k)$.
- If G is $K_{s,t}$-free then $\left|\mathcal{C}(G)\right| \geq \Omega\left(d^{s/(s-1)}\right)$.

Moreover, $\mathcal{C}(G)$ contains an interval of consecutive even integers of this length.

Theorem: (*S.- Verstraëte*)

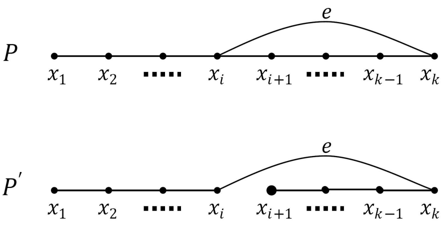
If graph G on n vertices has average degree at least $e^{c \log^* n}$ then G contains a cycle of length 2^i for some $i > 0$.

(True for any exponentially growing sequence of even numbers)

Expansion and long path

Lemma: (*Pósa 1976*)

If every subset $X \subset G$ of size $|X| \leq m$ has $|\partial X| > 2|X|$ then G contains a path of length $3m$.



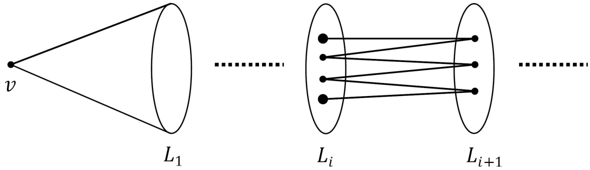
Rotation using the edge (x_i, x_k) , x_{i+1} is a new endpoint.

Corollary:

If G has average degree d and no cycle of length $2k$ then it contains a path of length $\Omega(d^k)$.

Let G be a connected, bipartite graph with average degree d and no cycle of length $2k$. Fix any $v \in G$ and let L_i be all the vertices in G within distance i from v. Note that

$$\sum e(L_i, L_{i+1}) = e(G) = \frac{d}{2}|G| = \frac{d}{2}\sum |L_i| \geq \frac{d}{4}\sum \left(|L_i| + |L_{i+1}|\right).$$

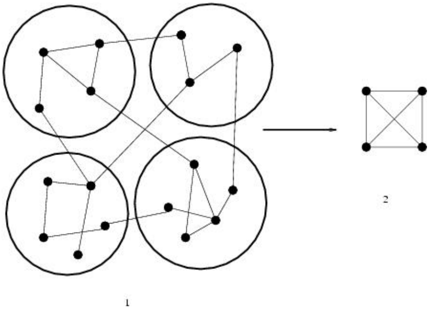


Therefore one of the induced subgraphs $G[L_i, L_{i+1}]$ has average degree at least $d/2$. It has a path of length $\Omega(d^k)$ between L_i and L_{i+1}, which together with v gives a long cycle. $\square$

Definition:

Graph Γ is a minor of graph G if for every vertex $u \in \Gamma$ there is a connected subgraph $G_u \subset G$ such that

- G_u and $G_{u'}$ are vertex disjoint for all $u \neq u'$.
- For every edge (u, u') of Γ there is an edge from G_u to $G_{u'}$.



QUESTION:

Find sufficient condition for graph G to contain given Γ as a minor.

(e.g., when Γ is a clique)

CONJECTURE: (*Hadwiger 1943*)

Every graph with chromatic number at least k contains a clique on k vertices as a minor.

Remark: Every graph with chromatic number k contains a subgraph with minimum degree $k-1$.

THEOREM: (*Kostochka, Thomason 80's*)

Every graph with average degree d contains a clique-minor of order

$$c\frac{d}{\sqrt{\log d}}.$$

Minors in graphs of large girth

Theorem: (*Thomassen, Diestel-Rompel, Kühn-Osthus*)

If G has girth $2k+1$ and average degree d, then it contains a minor with average degree $\Omega\left(d^{\frac{k+1}{2}}\right)$.

Observation:

Any minor of graph G has at most $e(G)$ edges and therefore has average degree at most $\sqrt{2e(G)}$.

Remark: Graphs of order $O(d^k)$, with minimum degree d and girth $2k+1$, have $O(d^{k+1})$ edges, so they cannot have minors with average degree $\gg d^{\frac{k+1}{2}}$.

Question:

Do we need the girth assumption? Is it enough to forbid cycle C_{2k}?

Minors in H-free graphs

Theorem: (*Kühn-Osthus*)

Every $K_{s,t}$-free graph G with average degree d contains a minor with average degree $\Omega\left(\frac{d^{1+\frac{1}{2(s-1)}}}{\text{polylog } d}\right)$.

Remarks:

- This implies that H-free graphs (for bipartite H) satisfy Hadwiger's conjecture.
- There is a $K_{s,t}$-free graph on $O\left(d^{s/(s-1)}\right)$ vertices, with average degree d. It has $O\left(d^{2+1/(s-1)}\right)$ edges and cannot contain a minor with average degree $\gg d^{1+\frac{1}{2(s-1)}}$.

Conjecture: (*Kühn-Osthus*)

Every $K_{s,t}$-free graph with average degree d contains a minor with average degree $\Omega\left(d^{1+\frac{1}{2(s-1)}}\right)$.

New results

Theorem: (*Krivelevich-S.*)

Let G be a graph with average degree d.

- If G is C_{2k}-free then it contains a minor with average degree
$$\Omega\left(d^{\frac{k+1}{2}}\right).$$
- If G is $K_{s,t}$-free then it contains a minor with average degree
$$\Omega\left(d^{1+\frac{1}{2(s-1)}}\right).$$

Remark:

The same approach is used to prove both statements. One of its key ingredients is the expansion property of H-free graphs.

DEFINITION:

A *separator* of a graph G of order n is a set of vertices whose removal separates G into connected components of size $\leq 2n/3$.

THEOREM: (*Plotkin-Rao-Smith, improving on Alon-Seymour-Thomas*)

If an n-vertex graph G has no clique-minor on h vertices, then it has a separator of order $O\left(h\sqrt{n \log n}\right)$.

COROLLARY: (*Plotkin-Rao-Smith, Kleinberg-Rubinfeld*)

Let G be a graph on n vertices and let $t > 0$ be a constant. If all subsets $X \subset G$ of size at most $n/2$ have $|\partial X| \geq t|X|$ then G contains a clique-minor of order $\Omega\left(\sqrt{\frac{n}{\log n}}\right)$.

Question:

What size clique-minors can one find in a graph G on n vertices, whose expansion factor $t \to \infty$ together with n?

Remarks:

- Proof based on separators can't give bound better than $\sqrt{\frac{n}{\log n}}$ since every graph of order n has separator of size $n/3$.
- Random graph $G(n, t/n)$ has almost surely expansion factor t and clique-minor of order $\Omega\left(\sqrt{\frac{nt}{\log n}}\right)$.

  (*Bollobás-Erdős-Catlin*, improved by *Fountoulakis-Kühn-Osthus*)

Question:

What if edges of G are distributed like in the random graph?

Expansion of C_4-free graphs revisited

Proposition:

Let G be a C_4-free graph with minimum degree d, and let X be a subset of G of size $|X| \le d$. Then $|\partial X| \ge \frac{d}{3}|X|$.

Proof. Suppose $|\partial X| < \frac{d}{3}|X|$. Since $e(X, \partial X) = (1 + o(1))d|X|$,

$$\sum_{y \in \partial X} \binom{d_X(y)}{2} \ge |\partial X| \binom{\frac{e(X,\partial X)}{|\partial X|}}{2} \approx d|X| \ge \binom{|X|}{2}.$$

This gives a 4-cycle, contradiction. $\square$

Remarks:

- If G is a C_4-free graph with $n = O(d^2)$ vertices and minimum degree d, then it has expansion factor $t = \frac{d}{3} = \Omega(\sqrt{n})$.
- There are similar results for all H-free graphs (H bipartite).

New result: expanders

Theorem: (*Krivelevich-S.*)

If all subsets $X \subset G$ of size $|X| \le O(n/t)$ have $|\partial X| \ge t|X|$ then G contains a minor with average degree

$$\Omega\left(\sqrt{\frac{nt\log t}{\log n}}\right).$$

Remarks:

- For $t = n^{\epsilon}$ this gives a minor with average degree $\Omega(\sqrt{nt})$ and thus also a clique-minor of order $\Omega(\sqrt{nt/\log n})$.
- The random graph $G(n, t/n)$ shows that this bound is tight.
- If G is a C_4-free graph of order $n = O(d^2)$, then for $t = d/3$ every subset of G of size $O(n/t)$ expands by factor of t. Thus G has a minor with average degree $\Omega(\sqrt{nt}) = \Omega(d^{3/2})$

New result: pseudo-random graphs

Definition:

A graph G is (p, β)-jumbled if for every subset of vertices X

$$\left| e(X) - p|X|^2/2 \right| \leq \beta|X|.$$

Theorem: *(Krivelevich-S.)*

Every (p, β)-jumbled graph G on n vertices with $\beta = o(np)$ contains a minor with average degree $\Omega(n\sqrt{p})$.

Remarks:

- This is tight since such a graph G has $O(n^2 p)$ edges. It extends results of Thomason and Drier-Linial.
- Expansion factor of (p, β)-jumbled G can be only $\frac{np}{\beta} \ll np$.
- If G is an (n, d, λ)-graph then it is $(d/n, \lambda)$-jumbled. Hence if $\lambda = o(d)$ it has a minor with average degree $\Omega(\sqrt{nd})$.