Journal of Combinatorial Theory, Series B **81**, 209–223 (2001)
doi:10.1006/jctb.2000.2007, available online at http://www.idealibrary.com on IDEAL®

Nowhere-Zero Flows in Random Graphs

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Received February 25, 2000

A nowhere-zero 3-flow in a graph G is an assignment of a direction and a value of 1 or 2 to each edge of G such that, for each vertex v in G, the sum of the values of the edges with tail v equals the sum of the values of the edges with head v. Motivated by results about the region coloring of planar graphs, Tutte conjectured in 1966 that every 4-edge-connected graph has a nowhere-zero 3-flow. This remains open. In this paper we study nowhere-zero flows in random graphs and prove that almost surely as soon as the random graph $G(n, p)$ has minimum degree two it has a nowhere-zero 3-flow. This result is clearly best possible.

1. INTRODUCTION

Let $G=(V, E)$ be a digraph. For a subset $X \subseteq V$ let $\delta^-(X)$ be the set of all edges entering X from $V-X$ and let $\delta^+(X)=\delta^-(V-X)$. A function ϕ: $E \to \mathbb{R}$ is called a *circulation* if for every vertex $v \in V$ it satisfies the following conservation rule:

$$\sum_{e \in \delta^-(v)} \phi(e) = \sum_{e \in \delta^+(v)} \phi(e).$$

For an integer $k \geqslant 1$ and a graph G, a *nowhere-zero k-flow* in G is an orientation of its edges and a circulation ϕ such that for every edge e, $|\phi(e)|$ is equal to one of $1, 2, ..., k-1$. It is easy to see that if an undirected graph has a nowhere-zero k-flow for some orientation of the edges, then it has one for every orientation (just replace $\phi(e)$ by $-\phi(e)$ if the direction of e changed).

The notion of a nowhere-zero flow was introduce by Tutte, and it provides an interesting way to generalize theorems about region coloring of planar graphs to general graphs. Indeed, given a coloring of regions of the planar graph G by colors $0, 1, ..., k-1$ one can construct a nowhere-zero k-flow in G by giving G an arbitrary orientation and assigning each edge

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e the difference between the values of the colors of the regions bordering e to its left and right respectively. The converse is also true; it is possible to construct a region coloring of a planar graph from a nowhere-zero flow. Motivated by this connection, Tutte [13] raised the general problem of determining for which value of k a graph has a nowhere-zero k-flow. In particular, by a theorem of Grötzsch [6] it is known that every planar graph without cycles of length $\leqslant 3$ is 3-vertex-colorable. This implies, by duality, that every 4-edge-connected planar graph has a nowhere-zero 3-flow. Tutte [14] asked in 1966 whether in this statement planarity is needed and conjectured that it is not.

Conjecture 1.1. Every 4-edge-connected graph has a nowhere-zero 3-flow.

This conjecture is still open. It is not even known if there is any fixed s such that every s-edge-connected graph has a nowhere-zero 3-flow. The best result so far was obtained by Jaeger [8], who proved that every 4-edge-connected graph has a nowhere-zero 4-flow. The following related conjecture was made by P. Seymour [11].

Conjecture 1.2. For any integer $k \geqslant 1$ there exist $s(k)$ such that every s-edge-connected graph has a flow with values 1 and $(k+1)/k$.

Motivated by these two conjectures, we study here nowhere-zero flows in the random graph $G(n, p)$, as well as in some other models of random graphs. Formally, $G(n, p)$ denotes the probability space whose points are graphs on a fixed set of n labeled vertices, where each pair of vertices forms an edge, randomly and independently, with probability p. The term ‘the random graph $G(n, p)$” means, in this context, a random point chosen in this probability space. Similarly, we define $G(n, M)$ to be a random point in the probability space of all graphs on n vertices with M edges, where all such graphs are equiprobable. Each graph property A (that is, a family of graphs closed under graph isomorphism) is an event in this probability space, and one may study its probability $\Pr[A]$, that is, the probability that the random graph $G(n, p)$ ($G(n, M)$) lies in this family. In particular, we say that A holds *almost surely* (or a.s., for short), if the probability that $G(n, p)$ ($G(n, M)$) satisfies A tends to 1 as n tends to infinity.

The subject of random graphs was introduced by Erdős and Rényi [5], who also made a central observation that many natural graph-theoretic properties become true for $G(n, p)$ in a very narrow range of p. They made the following key definition. A function $r(n)$ is called a *threshold function* for a graph theoretic property A if $p(n)/r(n) \to 0$, as $n \to \infty$ implies that $G(n, p)$ a.s. does not have A, while $p(n)/r(n) \to \infty$, as $n \to \infty$ implies that $G(n, p)$ a.s. has this property. A graph property A is called *increasing* (*decreasing*) if the fact that G satisfies A and $E(G) \subseteq E(H)$ ($E(H) \subseteq E(G)$, respectively) implies that H has also property A. A property which is either increasing or decreasing is called *monotone*.

It was proved by Bollobás and Thomason [4] that every nontrivial monotone property has a threshold function. Unfortunately, the property of having a nowhere-zero k-flow is not monotone. To see this, consider any loopless graph in which every vertex has even degree. This graph has a nowhere-zero 2-flow, but the addition or deletion of any edge destroys this property. Nevertheless we can prove the following result.

THEOREM 1.3. *Let $k \geqslant 1$ be a fixed integer and let $\omega(n)$ be any function tending to infinity together with n, then*:

(i) *If $p = (\ln n + (2k-1) \ln \ln n + \omega(n))/n$ then a.s. $G(n, p)$ has a flow with values* 1 *and* $(k+1)/k$.

(ii) *If $p = (\ln n + (2k-1) \ln \ln n - \omega(n))/n$ then a.s. $G(n, p)$ is either empty or does not have such a flow.*

In particular, for $k = 1$ this shows that if we discard the case when p is very small (since for $p = cn^{-2}$ the random graph can be empty and then obviously will have any flow), we obtain that the property of having a nowhere-zero 3-flow has a threshold. In addition we show that as soon as the minimum degree of the random graph is $2k$, it has a flow with values 1 and $(k+1)/k$. This is clearly best possible, since a graph containing a vertex of degree $2k-1$ does not have such a flow. In order to formulate this result precisely we introduce the notion of a random graph process.

A *random graph process* $(G_t)_{t=0}^{\binom{n}{2}}$ is a sequence of graphs on $V = \{1, 2, ..., n\}$ such that G_t has exactly t edges and G_{t+1} arises from G_t by adding to it a single edge chosen uniformly at random from the $\binom{n}{2} - t$ remaining edges. Clearly, the probability space of all graphs obtained at time M, $0 \leqslant M \leqslant \binom{n}{2}$, can be identified with $G(n, M)$. Denote by τ_k the smallest time t such that the minimal degree of G_t is at least $2k$. Then we obtain the following result.

THEOREM 1.4. *Almost surely a random graph process $(G_t)_{t=0}^{\binom{n}{2}}$ is such that, for every $t \geqslant \tau_k$, G_t has a flow with values* 1 *and* $(k+1)/k$.

The rest of this paper is organized as follows. In Section 2 we prove our main theorems. Section 3 deals with nowhere-zero flows in random regular graphs. The final section contains some concluding remarks and open problems. Throughout this paper all logarithms are in base $e = 2.71828...$ and we assume whenever this is needed that n is sufficiently large.

2. FLOWS IN $G(n, p)$ AND $G(n, M)$

In this section we prove our main theorems, Theorems 1.3 and 1.4. We start with the statement of some preliminary results about flows in graphs. Next we present some lemmas dealing with the properties of random graphs and we conclude this section with the proofs of Theorems 1.3 and 1.4.

2.1. *Preliminaries*

To show the existence of the flow with values 1 and $(k+1)/k$ in a random graph, we use the following generalization of the max flow–min cut theorem due to Hoffman [7].

PROPOSITION 2.1. *Let G be a digraph and let $f, g: E(G) \to \mathbb{R}$ be two functions such that $f \leqslant g$. Then there exists a circulation $\phi: E(G) \to \mathbb{R}$ satisfying $f \leqslant \phi \leqslant g$ if and only if for every subset $X \subseteq V(G)$,*

$$\sum_{e \in \delta^-(X)} f(e) \leqslant \sum_{e \in \delta^+(X)} g(e).$$

If in addition f and g are integer-valued, then there is an integer-valued circulation satisfying $f \leqslant \phi \leqslant g$.

As an easy corollary of this result, we obtain the necessary and sufficient conditions for a graph to have a flow with values 1 and $(k+1)/k$. This is one of the main ingredients in our proofs.

COROLLARY 2.2. *For any integer $k \geqslant 1$, a graph $G = (V, E)$ has a flow with values 1 and $(k+1)/k$ if and only if there exists an orientation of its edges such that for every subset X of V*

$$|\delta^-(X)| \leqslant \frac{k+1}{k} |\delta^+(X)|.$$

Proof. Suppose $G = (V, E)$ has an orientation such that $|\delta^-(X)| \leqslant \frac{k+1}{k} |\delta^+(X)|$ for every $X \subseteq V$. Define the f and g to be the constant integer-valued functions such that for any $e \in E$, $f(e) = k$ and $g(e) = k+1$. Then an easy computation shows that $\forall X \subseteq V$,

$$\sum_{e \in \delta^-(X)} f(e) = k\, |\delta^-(X)| \leqslant k \frac{k+1}{k} |\delta^+(X)| = (k+1)\, |\delta^+(X)| = \sum_{e \in \delta^+(X)} g(e).$$

Therefore, by Proposition 2.1, there exists an integer-valued circulation ϕ such that $k \leqslant \phi \leqslant k+1$. Since for every edge $\phi(e)$ is an integer, it must be equal to k or $k+1$. Let ϕ' be a circulation such that $\phi'(e)=\phi(e)/k$ for every e. Clearly it is a flow in G with values 1 and $(k+1)/k$. This completes the proof of the first part of the statement. The opposite direction follows immediately from Proposition 2.1 by substituting $f=1$ and $g=(k+1)/k$. ∎

2.2. *Some Properties of Random Graphs*

In this section we present some properties of random graphs which we will use in the proof of the main results. We start with the statement of known results about the behavior of the degree sequence of $G(n, p)$. These results can be found in the book of Bollobás [2].

LEMMA 2.3. (a) *If $p \leqslant n^{-3/2}$, then almost surely the random graph $G(n, p)$ is either empty or has a vertex of degree one.*

(b) *For any fixed integer $k \geqslant 0$ and $n^{-3/2} \leqslant p \leqslant 1-n^{-3/2}$ let $\lambda_k(n) = n\binom{n}{k} p^k (1-p)^{n-k-1}$. Then almost surely the following assertion holds.*

(i) *If $\lim_{n\to\infty} \lambda_k = 0$ then $G(n, p)$ has no vertices of degree k.*

(ii) *If $\lim_{n\to\infty} \lambda_k = \infty$ then $G(n, p)$ has a vertex of degree k.*

We also need some additional properties of random graphs, which we summarize in the following simple though somewhat technical lemma.

LEMMA 2.4. (a) *If $\ln n/n \leqslant p \leqslant 6 \ln n/n$, then with probability at least $1-n^{-3/4}$ the random graph $G(n, p)$ has the following properties:*

(i) *Every $i \leqslant n^{4/5}$ vertices of G span fewer than $3i$ edges.*

(ii) *The distance between any pair of vertices of G with degrees at most $\sqrt{\ln n}$ is at least three.*

(iii) *For any subset $U \subset V(G)$ of size $n^{4/5} \leqslant u \leqslant n/2$, the number of edges in the cut between U and $V(G)-U$ is at least $u\sqrt{\ln n}$.*

(b) *If $p \geqslant 6 \ln n/n$, then with probability at least $1-o(n^{-1})$ any cut $(U, V-U)$, with $|U| = u \leqslant n/2$ in the random graph $G(n, p)$, contains at least $u\sqrt{\ln n}$ edges.*

Proof. Throughout the proof of the lemma we will use the inequality $\binom{n}{k} \leqslant \left(\frac{en}{k}\right)^k$.

(a) (i) The probability of the existence of a subset of size i violating the assertion (i) of the lemma is at most

$$\sum_{i=7}^{n^{4/5}} \binom{n}{i} \binom{\binom{i}{2}}{3i} p^{3i} \leqslant \sum_{i=7}^{n^{4/5}} \left[\frac{en}{i} \left(\frac{ei}{6}\right)^3 p^3 \right]^i$$
$$= \sum_{i=7}^{n^{4/5}} \left[\frac{e^4 n i^2 p^3}{6^3} \right]^i \leqslant \sum_{i=7}^{n^{4/5}} \left[\frac{e^4 \ln^3 n}{n^{2/5}} \right]^i = o(n^{-3/4}).$$

(ii) The probability that $G(n, p)$ contains a pair of adjacent vertices with degrees s and t respectively is at most $n^2 n^{s-1} n^{t-1} p^{s+t-1} (1-p)^{2n-s-t-2}$. Similarly, the probability of the existence of a path of length two connecting vertices with degrees s and t is at most $n^3 n^{s-1} n^{t-1} p^{s+t} (1-p)^{2n-s-t-2}$. Therefore the probability that there exists a pair of vertices violating the assertion (ii) of the lemma is at most

$$\sum_{s=0}^{\sqrt{\ln n}} \sum_{t=0}^{\sqrt{\ln n}} (n^{s+t} p^{s+t-1} (1-p)^{2n-s-t-2} + n^{s+t+1} p^{s+t} (1-p)^{2n-s-t-2})$$
$$\leqslant 2 \ln n \, n^{2\sqrt{\ln n}+1} p^{2\sqrt{\ln n}} (1-p)^{2n-2\sqrt{\ln n}-2}$$
$$\leqslant (1+o(1)) \frac{(6 \ln n)^{2\sqrt{\ln n}+1}}{n} = o(n^{-3/4}).$$

(iii) Let U be a subset of vertices of size $n^{4/5} \leqslant u \leqslant n/2$. The number of edges $e(U, V-U)$ in the cut between U and $V-U$ is a binomially distributed random variable with parameters $u(n-u)$ and p. Therefore, it follows, by the standard large deviation inequality of Chernoff (see, e.g., [1, Appendix A]), that

$$\Pr(e(U, V-U) \leqslant u\sqrt{\ln n}) = \Pr(e(U, V-U) - pu(n-u)$$
$$\leqslant -(pu(n-u) - u\sqrt{\ln n}))$$
$$\leqslant e^{-[(pu(n-u)-u\sqrt{\ln n})^2]/[2pu(n-u)]}$$
$$\leqslant e^{-[pu(n-u)]/2 + u\sqrt{\ln n}} \leqslant e^{-(u \ln n)/4 + u\sqrt{\ln n}}.$$

Thus the probability that there exists such a cut in $G(n, p)$ is at most

$$\sum_{u=n^{4/5}}^{n/2} \binom{n}{u} e^{-(u \ln n)/4 + u\sqrt{\ln n}} \leqslant \sum_{u=n^{4/5}}^{n/2} \left(\frac{en}{u} \frac{e^{\sqrt{\ln n}}}{n^{1/4}} \right)^u$$
$$\leqslant \sum_{u=n^{4/5}}^{n/2} \left(\frac{e^{\sqrt{\ln n}+1}}{n^{1/20}} \right)^u = o(n^{-3/4}).$$

(b) Similar to the proof of (iii), the probability that $G(n, p)$ contains a cut with fewer than $u\sqrt{\ln n}$ edges is at most

$$\sum_{u=1}^{n/2} \binom{n}{u} e^{-pu(n-u)/2+u\sqrt{\ln n}} \leqslant \sum_{u=1}^{n/2} \binom{n}{u} e^{-(3u(n-u)\ln n)/n+u\sqrt{\ln n}}$$

$$\leqslant \sum_{u=1}^{n/2} \left(\frac{en}{u} e^{\sqrt{\ln n}} n^{-3+3u/n}\right)^u$$

$$= \sum_{u=1}^{n/2} \left(\frac{e^{\sqrt{\ln n}+1}}{n^2} \frac{n^{3u/n}}{u}\right)^u$$

$$\leqslant \sum_{u=1}^{n/2} \left(\frac{2e^{\sqrt{\ln n}+1}}{n^{3/2}}\right)^u = o(n^{-1}).$$

This completes the proof of the lemma. ▮

2.3. *The Proof of Theorem* 1.3

First we need the following result.

LEMMA 2.5. *Every graph G has an orientation of its edges such that the indegree* $(|\delta^-(v)|)$ *and outdegree* $(|\delta^+(v)|)$ *of every vertex* v *in G differ by at most one.*

Proof. Consider a graph $G = (V, E)$. Since the sum of the degrees of G is even (it is equal $2|E|$), it contains an even number of vertices of odd degree. Denote by G' a multigraph (it may have parallel edges) obtained from G by adding to it a perfect matching on the set all vertices of odd degree. Such matching exists since this set has an even size. Clearly, by definition, G' has only vertices of even degree. Therefore there exists an *eulerian orientation* of G' (see, e.g., [9, problem 5.13]) such that every vertex has the same outdegree as indegree. Since in every vertex G differs from G' by at most one additional edge we can conclude that the restriction of the same orientation to G satisfies the assertion of the lemma. This completes the proof. ▮

Next we show that if a graph is oriented such that the indegree and outdegree of every vertex differ by at most one, then every sufficiently large cut in this graph is almost balanced. This is formulated more precisely in the following easy lemma.

LEMMA 2.6. *Let $G=(V, E)$ be a digraph in which the indegree and outdegree of every vertex differ by at most one, and let X be a subset of vertices in G such that the number of edges crossing the cut $(X, V-X)$ is at least $(2k+1)\,|X|$, $k \geqslant 0$. Then the inequality*

$$\frac{k}{k+1}\,|\delta^+(X)| \leqslant |\delta^-(X)| \leqslant \frac{k+1}{k}\,|\delta^+(X)|$$

holds.

Proof. We prove only the first part of the inequality; the second part can be shown similarly. By definition, the number of edges in the cut is equal to $|\delta^+(X)|+|\delta^-(X)| \geqslant (2k+1)\,|X|$. Since the indegree and outdegree of every vertex differ by at most one, then an easy computation shows that

$$|\delta^+(X)|-|\delta^-(X)| = \sum_{v \in X} (|\delta^+(v)|-|\delta^-(v)|) \leqslant |X| \leqslant \frac{1}{2k+1}\,(|\delta^+(X)|+|\delta^-(X)|).$$

This inequality implies that $(2k/(2k+1))\,|\delta^+(X)| \leqslant ((2k+2)/(2k+1))\,|\delta^-(X)|$. Multiplying both sides by $(2k+1)/(2k+2)$ we conclude that $k/(k+1)\,|\delta^+(X)| \leqslant |\delta^-(X)|$. ∎

Having finished all necessary preparations, we are now ready to complete the proof of our first theorem.

Proof of Theorem 1.3. (i) Let $G=(V, E)$, $|V|=n$, be a random graph with edge probability, $p=(\ln n+(2k-1)\ln\ln n+\omega(n))/n$, where $\omega(n)$ tends to infinity arbitrarily slowly. By Lemma 2.5 there exists an orientation of G such that the indegree and outdegree of each vertex differ by at most one. This also implies that for vertices of even degree these two quantities are actually equal. We claim that almost surely this orientation satisfies $|\delta^-(X)| \leqslant ((k+1)/k)\,|\delta^+(X)|$ for every $X \subseteq V$. Thus by Corollary 2.2 the random graph $G(n, p)$ a.s. has flow with values 1 and $(k+1)/k$. Since by definition $\delta^-(X)=\delta^+(V-X)$, it is clearly enough to show that for any subset X of size at most $n/2$ the following inequality hold almost surely

$$\frac{k}{k+1}\,|\delta^+(X)| \leqslant |\delta^-(X)| \leqslant \frac{k+1}{k}\,|\delta^+(X)|. \tag{1}$$

Note also that in order to prove this inequality we can restrict our attention only to the $X \subseteq V$, for which the subgraph of G induced by X is connected.

First suppose that $p \geqslant 6 \ln n/n$ or the size of set X is at least $n^{4/5}$. Then by Lemma 2.4 the number of edges between X and $V-X$ is at least $\sqrt{\ln n}\,|X| \geqslant (2k+1)\,|X|$. Thus the inequality (1) follows from Lemma 2.6.

Next we consider the case $p = (\ln n + (2k-1) \ln \ln n + \omega(n))/n \leqslant 6 \ln n/n$ and $|X| \leqslant n^{4/5}$. By substituting the value of probability p in Lemma 2.3 we obtain that almost surely the minimum degree of G is at least $2k$. If the set X consists of a single vertex v of even degree, then the indegree of v is equal to the outdegree. This implies that $|\delta^+(X)| = |\delta^-(X)|$. If the degree of v is odd, it should be at least $2k+1$. In that case the inequality (1) follows from Lemma 2.6. Thus we can assume that X contains at least two vertices.

Let X_1 be the set of vertices from X with degree less than $\sqrt{\ln n}$ and let $X_2 = X - X_1$. Note that by Lemma 2.4 the distance between any two vertices in X_1 is a.s. at least three. Since the subgraph of G induced by X is connected, it follows that each vertex in X_1 has at least one neighbor in X_2 and any vertex in X_2 is adjacent to at most one vertex in X_1. This implies that $|X_2| \geqslant |X_1|$ and has a size at least $|X|/2$. Denote by e the number of edges spanned by the set X. Then from Lemma 2.4 we obtain that almost surely this quantity is at most $3\,|X|$. Therefore an easy computation shows that the number of edges between X and $V-X$ is equal to

$$\sum_{v \in X} d(v) - 2e \geqslant |X_2|\,\sqrt{\ln n} - 6\,|X| \geqslant \sqrt{\ln n}\,|X|/2 - 6\,|X|$$

$$= (\sqrt{\ln n}/2 - 6)\,|X| \geqslant (2k+1)\,|X|.$$

Then again the inequality (1) follows from Lemma 2.6. This completes the proof that a.s. $G(n, p)$ has a flow with values 1 and $(k+1)/k$.

(ii) The second part of the theorem follows easily from Lemma 2.3. Indeed, if $p \leqslant \ln n/(2n)$ then by this lemma (Parts (a) and (b)) the random graph $G(n, p)$ is almost surely either empty or contains a vertex of degree one. In the latter case it obviously does not have a flow with nonzero values. So suppose that $\ln n/(2n) \leqslant p \leqslant (\ln n + (2k-1) \ln \ln n - \omega(n))/n$, where $\omega(n)$ tends to infinity arbitrarily slowly. Then an easy computation shows that the value of $\lambda_{2k-1}(n) = n\binom{n}{2k-1} p^{2k-1}(1-p)^{n-2k} \geqslant \Omega(e^{\omega(n)})$ tends to infinity. Therefore by Lemma 2.3 the random graph $G(n, p)$ will a.s. contain a vertex of degree $2k-1$. Clearly such a vertex prevents a graph from having a flow with values 1 and $(k+1)/k$. This completes the proof of the theorem. ∎

2.4. *The Proof of Theorem* 1.4

The proof of Theorem 1.4 which we present here is quite similar to that of Theorem 1.3. Therefore we omit most of the details and outline only additional ideas and techniques we use.

First we will establish a connection between the behavior of two models of random graph $G(n, p)$ and $G(n, M)$. The following result (see., e.g., [2]) shows that these models are practically interchangeable, provided M is close to $p\binom{n}{2}$.

PROPOSITION 2.7. *Let A be any graph property and let $0 < p = M/\binom{n}{2} < 1$. Denote by $P_M(A)$, $P_p(A)$ the probabilities that $G(n, M)$ and $G(n, p)$ respectively have a property A. Then*

$$P_M(A) \leqslant 3M^{1/2} P_p(A).$$

This proposition together with Lemma 2.4 immediately implies the following properties of $G(n, M)$.

LEMMA 2.8. (a) *Let $(n \ln n)/2 \leqslant M \leqslant 3n \ln n$, then almost surely $G(n, M)$ has the following properties*:

(i) *Every $i \leqslant n^{4/5}$ vertices of G span fewer than $3i$ edges.*

(ii) *The distance between any pair of vertices of G with degrees at most $\sqrt{\ln n}$ is at least three.*

(iii) *For any subset $U \subset V(G)$ of size $n^{4/5} \leqslant u \leqslant n/2$, the number of edges in the cut between U and $V(G) - U$ is at least $u\sqrt{\ln n}$.*

(b) *If $M \geqslant 3n \ln n$, then a.s. any cut $(U, V-U)$ with $|U| = u \leqslant n/2$ in the random graph $G(n, M)$ contains at least $u\sqrt{\ln n}$ edges.*

Plugging this lemma into the proof of Theorem 1.3, we obtain the following corollary about orienting $G(n, M)$.

COROLLARY 2.9. *For any integer $k \geqslant 1$ and $M \geqslant (n \ln n)/2$, the random graph $G(n, M)$ a.s. has an orientation such that the indegree and outdegree of every vertex differ by at most one, and for any set $X \subset V(G)$ of size $2 \leqslant |X| \leqslant n-2$ it holds that*

$$\frac{k}{k+1}\,|\delta^+(X)| \leqslant |\delta^-(X)| \leqslant \frac{k+1}{k}\,|\delta^+(X)|.$$

Now it remains to show how this corollary implies Theorem 1.4.

Proof of Theorem 1.4. Let $(G_t)_{t=0}^{\binom{n}{2}}$ be a random graph process and let τ_k be the smallest time t such that the minimum degree of G_t is equal to $2k$. It is well known (see, e.g., [2]) that for $M \leqslant (n \ln n)/2$ the random graph $G(n, M)$ a.s. has a vertex of degree at most one. Since the probability space of all graphs obtained in a random process at time M can be identified with $G(n, M)$ we conclude that almost surely $\tau_k \geqslant (n \ln n)/2$. Thus if $t \geqslant \tau_k$ then a.s. the graph G_t satisfies the assertion of Corollary 2.9 and has a minimal degree of at least $2k$. Then a proof, similar to that of Theorem 1.3, shows that the inequality $(k/(k+1))\,|\delta^+(X)| \leqslant |\delta^-(X)| \leqslant ((k+1)/k)\,|\delta^+(X)|$ holds also for any subset X of G_t consisting of a single vertex. By Corollary 2.2 this implies that G_t, $t \geqslant \tau_k$, almost surely has a flow with values 1 and $\frac{k+1}{k}$, completing the proof. ∎

3. RANDOM REGULAR GRAPHS

In this section we study nowhere-zero flows in random regular graphs. We use $G_{n,d}$ to denote the probability space of d regular graphs on n vertices (dn is even), where each such graph is picked uniformly at random. We consider d fixed and $n \to \infty$ and say that some event in this space occurs almost surely if the probability of this event tends to one when n tends to infinity.

To generate a random d-regular graph one can use the following model given in [2, pp. 48–52]. Let $W = \bigcup_{j=1}^{n} W_j$ be a fixed set of $2m = dn$ labeled vertices, where $|W_j| = d$ for each j. A *configuration* F is a partition of W into m pairs of vertices, called edges of F. Clearly the number of all possible configurations is $N(m) = 2m!/(m!2^m)$. Let $\mathscr{F}_{n,d}$ be a probability space where all configurations are equiprobable. For $F \in \mathscr{F}_{n,d}$, let $\phi(F)$ be the graph on the vertex set $\{1, 2, ..., n\}$ in which ij is an edge iff F has an edge joining W_i to W_j. Clearly $\phi(F)$ is a graph with maximal degree at most d. More importantly, $\phi(F)$ is a d-regular graph with probability bounded away from 0 as $n \to \infty$, and all d-regular graphs are obtained with the same probability. Thus instead of studying properties of random d-regular graphs we may consider the space of configurations.

If the degree d of a random regular graph is even then it has an orientation such that the indegree of every vertex is equal to its outdegree. By definition, this implies that for even d the random graph $G_{n,d}$ has a nowhere-zero 2-flow. This is clearly best possible. Now we consider the case when d is odd. It is well known (see [12]) that a 3-regular graph has a nowhere-zero 3-flow only if it is bipartite. Since the random graph $G_{n,3}$ almost surely has an odd cycle (see, [2]), we conclude that it does not have a nowhere-zero 3-flow. On the other hand, by a theorem of Robinson and Wormald [10], for any odd d the edge set of the random d-regular graph a.s. can be partitioned into d disjoint perfect matchings. This implies (see, [12]) that for all odd values of d almost surely $G_{n,d}$ has a nowhere-zero 4-flow. Therefore it remains to determine the odd values of $d \geqslant 5$, for which the random d-regular graph a.s. has a nowhere-zero 3-flow. We obtain the following partial result in that direction.

THEOREM 3.1. *For every odd $d \geqslant 11$ the random d-regular graph $G_{n,d}$ almost surely has a nowhere-zero 3-flow.*

Proof. First suppose that $d \geqslant 13$. Then it was proved by Bollobás [3] that a.s. for every subset X of $G_{n,d}$, with $|X| \leqslant n/2$ the number of edges in the cut $(X, V-X)$ is at least $3\,|X|$. This result combined with Lemmas 2.5 and 2.6 (for $k=1$) implies that almost surely there exists an orientation of $G_{n,d}$ such that the inequality

$$\tfrac{1}{2}\,|\delta^+(X)| \leqslant |\delta^-(X)| \leqslant 2\,|\delta^+(X)| \tag{2}$$

holds for every $X \subseteq V(G)$. Thus by Corollary 2.2 we obtain that $G_{n,d}$, $d \geqslant 13$, a.s. has a nowhere-zero 3-flow.

To deal with the case $d=11$, we first need to establish some properties of random 11-regular graphs.

LEMMA 3.2. *Almost surely $G_{n,11}$ has the following properties.*

(i) *For any subset $X \subset V(G)$, with $|X| \leqslant n/2$, the number of edges in the cut $(X, V-X)$ is at least $11\,|X|/4$.*

(ii) *For any subset $X \subset V(G)$ of size at most $n/3$ the number of edges in the cut $(X, V-X)$ is at least $3\,|X|$.*

(iii) *The number of subsets X such that the cut $(X, V-X)$ contains less than $3\,|X|$ edges is at most $e^{n/9}$.*

Proof of Lemma 3.2. Denote by $P(x,s)$ the probability that a random configuration contains a set $X \subset V$ such that $|X| = x$ and there are exactly s edges with exactly one end vertex in $\bigcup_{j \in X} W_j$. By definition it is easy to see that

$$P(x,s) \leqslant \binom{n}{x}\binom{11x}{s}\binom{11(n-x)}{s}\, s!\, N\left(\frac{11x-s}{2}\right) N\left(\frac{11(n-x)-s}{2}\right) \Big/ N\left(\frac{11n}{2}\right)$$

$$= P_0(x,s).$$

In addition, note that for $0 \leqslant s < s' \leqslant 3x$ we have $P_0(x, s) \leqslant P_0(x, s')$. Our theorem follows if we show that

$$\text{(i)} \qquad \sum_{x \leqslant n/2,\, s \leqslant 11x/4} P_0(x, s) = o(1),$$

$$\text{(ii)} \qquad \sum_{x \leqslant n/3,\, s \leqslant 3x} P_0(x, s) = o(1),$$

$$\text{(iii)} \qquad \sum_{x \leqslant n/2,\, s \leqslant 3x} P_0(x, s) = o(e^{n/9}).$$

This can be done by straightforward calculations which are implicit in [3], therefore we omit them here. ∎

Now we are ready to complete the proof of Theorem 3.1. Let G be a random 11-regular graph and let M be a perfect matching in G which exists almost surely (see, e.g., [2]). Consider the following orientation on G. Every edge from the matching is oriented randomly and independently in one of two possible directions. The edges of $G - M$ are oriented such that the indegree of every vertex is equal to its outdegree (this is possible since $G - M$ is eulerian). We claim that this orientation a.s. satisfies the inequality (2) for every subset $|X| \leqslant n/2$ and therefore by Corollary 2.2 a.s. $G_{n,11}$ has a nowhere-zero 3-flow.

Note that by the definition of the orientation, the indegree and outdegree of every vertex differ by at most one. Therefore if the number of edges in the cut $(X, V - X)$ is at least $3|X|$ then the inequality (2) follows from Lemma 2.6. Thus it is enough to consider set X for which the size of the cut is smaller than $3|X|$. In that case, from Lemma 3.2 we conclude that $n/3 \leqslant |X| = x \leqslant n/2$ and the number of edges in the cut $(X, V - X)$ is at least $11x/4$. Now an easy computation shows that the set X violates the inequality (2) only if the number of edges from matching M with the head in X differs by at least $11x/12$ from the number of edges from M with the tail in X. But this difference z is a sum of at most x mutually independent random variables z_i such that $\Pr(z_i = 1) = \Pr(z_i = -1) = 1/2$. Therefore by the standard large deviation inequality the probability of this event is at most

$$\Pr(|z| \geqslant a = 11x/12) \leqslant 2e^{-a^2/2x} = 2e^{-(11/12)^2 x/2} \leqslant e^{-n/8}.$$

Hence, by Lemma 3.2, the probability that there exists a set X violating inequality (2) is at most $e^{n/9} e^{-n/8} = o(1)$. This completes the proof of the theorem. ∎

4. CONCLUDING REMARKS AND OPEN PROBLEMS

In the proof of Theorem 1.3 we show that almost surely as soon as the random graph $G(n, p)$, $\ln n/n \leqslant p$, has minimal degree $2k$, it has a flow with values 1 and $(k+1)/k$. Since the behavior of the degree sequence of $G(n, p)$ is very well understood (see [2]) we can obtain the following corollary.

COROLLARY 4.1. *Let c be a fixed real number and let $p = (\ln n + (2k-1)\ln\ln n + c)/n$, then the probability that $G(n, p)$ has a flow with values* 1 *and $(k+1)/k$ tends to $e^{-2e^{-c}/(2k-1)!}$ as n tends to infinity.*

Let us define the *isoperimetric number* of a graph $G = (V, E)$ to be $i(G) = \min_{X \subset V} e(X, V-X)/|X|$, where $e(X, V-X)$ is the number of edges in the cut $(X, V-X)$. It was proved by Bollobás [3] that when $d \to \infty$ then $i(G_{n,d})$ is almost surely at least $d/2 - O(\sqrt{d})$. Combining this with Lemmas 2.5 and 2.6 in a manner similar to that used in the proof of Theorem 1.3, we obtain the following proposition, which complies with Conjecture 1.2.

PROPOSITION 4.2. *For any integer $k \geqslant 1$, there exists a $d(k)$, such that the random d-regular graph $G_{n,d}$ with $d > d(k)$ a.s. has a flow with values* 1 *and* $(k+1)/k$.

The interesting question which remains open is to decide whether a 5-regular random graph a.s. has a nowhere-zero 3-flow. Since it is known that this graph is almost surely 5-edge-connected, Tutte's Conjecture 1.1 suggests that this question has an affirmative answer.

ACKNOWLEDGMENT

I thank M. DeVos for a stimulating discussion about nowhere-zero flows and for his comments on an early version of this paper. I also thank an anonymous referee for pointing out an error in the first version of the paper.

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