Grid Ramsey problem and related questions

Benny Sudakov, ETH

joint with D. Conlon, J. Fox and C. Lee

Hales-Jewett Theorem

Definition:

Let $[m] = \{1, 2, \ldots, m\}$, $a \in [m]^n$ and let S be a non-empty set of coordinates.

A *combinatorial line* is $a_S(1), a_S(2), \ldots, a_S(m)$, where $a_S(t)$ is a vector $b \in [m]^n$ such that $b_i = t, i \in S$ and $b_i = a_i, i \notin S$.

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$m = 3, \quad n = 6$

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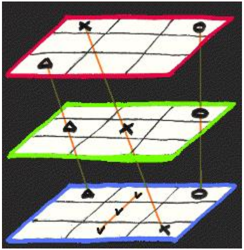

Informally, if the cells of a n-dimensional $m \times m \times \cdots \times m$ cube are colored with r colors, there must be one row, column, or certain diagonal all of whose cells are the same color, i.e., the multi-player tic-tac-toe game cannot end in a draw if the board has high dimesion.

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For every r, m and sufficiently large N, every r-coloring of $[N]$ contains a monochromatic arithmetic progression of length m.

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Proof. Consider mapping f from $[m]^n$ into $[N]$,

$$f(a_1, \ldots, a_n) = \sum_{i=1}^{n} a_i m^{i-1}.$$

Color every $a \in [m]^n$ by the color of $f(a)$. Then a monochromatic line in this coloring gives a monochromatic arithmetic progression of length m in the original coloring of $[N]$. $\square$

Easy corollary

Definition: A set $U \subset \mathbb{Z}^d$ is a *homothetic copy* of $V \subset \mathbb{Z}^d$ iff $U = u + \lambda V$ for some vector $u \in \mathbb{Z}^d$ and integer λ.

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For all r and $V \subset \mathbb{Z}^d$, every r-coloring of $\mathbb{Z}^d$ contains a monochromatic homothetic copy of V.

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Proof. Let $V = \{v_1, \ldots, v_m\}$. Map $[m]^n$ into $\mathbb{Z}^d$,

$$f(a_1, \ldots, a_n) = \sum_{i=1}^{n} v_{a_i}.$$

Color every $a \in [m]^n$ by the color of $f(a)$. Then a monochromatic line in this coloring gives a homothetic copy of V in the original coloring of $\mathbb{Z}^d$. $\square$

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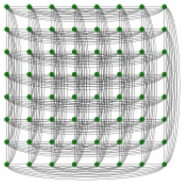
Remarks:

- Greatly improves the original Ackermann type bound.
- Main step in the proof is the "Grid-type lemma", which reduces the size of the alphabet from m to $m - 1$.

Grid graph

Definition:

A **Grid graph** $\Gamma_{m,n}$ is a graph on the set of vertices $[m] \times [n]$ such that (i, j) is adjacent to (i', j') iff $i = i'$ or $j = j'$.



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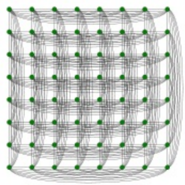

- We call the i^{th} *row* the set of vertices $\{i\} \times [n]$ and $[m] \times \{j\}$ is called the j^{th} *column*.
- Rows, columns of $\Gamma_{m,n}$ are complete graphs K_n, K_m respectively.

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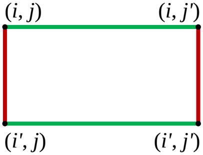

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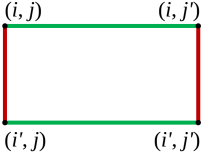

Grid Ramsey function:

$G(r)$ is the minimum integer n such that every r-edge coloring of $\Gamma_{n,n}$ contains an alternating rectangle.

Grid lemma

Lemma: *(Shelah 1988)*

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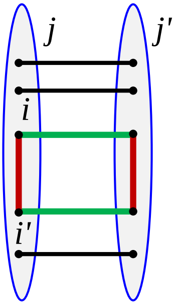

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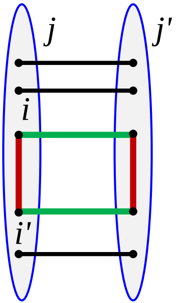

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Since $n > r^{\binom{r+1}{2}}$ there are two columns j, j', whose edges are identically colored. There are $r+1$ edges of the grid graph between vertices in these columns. Since there are only r colors, two of these edges (say in rows i and i') have the same color. Then $\{i, i'\} \times \{j, j'\}$ is an alternating rectangle. $\square$

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- A very similar (more general) grid-type lemma is a key step in Shelah's proof.
- This simple bound is difficult to improve. The only known improvement is by an additive lower-order term (Gyárfás).
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Remark: The second result gives some evidence why it is hard to improve the upper bound on $G(r)$.

Key observation

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There is an alternating-free r-edge coloring of $\Gamma_{m,n}$ iff there are r-edge colorings $c_1, \ldots, c_m$ of the complete graph K_n with

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This shows that there are no alternating rectangles. $\square$

- Choose a partition $E(K_n) = E_1 \cup \cdots \cup E_t$ such that any union of "*few*" parts has "*small*" chromatic number.
- Generate c_i by assigning to every part E_j randomly one of r colors.
- Any two such colorings will agree only on small number of parts, i.e., chromatic number of $\mathcal{G}_{c_i, c_{i'}}$ will be small.

An edge partition of K_n

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Indeed, every E_i is a bipartite graph with parts containing all $x, x_i = 0$ and all $y, y_i = 1$. Therefore $\chi(E_i) = 2$.

Since $\chi(H \cup H') \leq \chi(H) \cdot \chi(H')$ for any pair of graphs on the same vertex set, the claim follows.

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Thus, with high probability $\chi(\mathcal{G}_{c_i, c_{i'}}) \leq r$ for all $i \neq i'$, which gives alternating-free r-edge coloring of $\Gamma_{n,n}$. □

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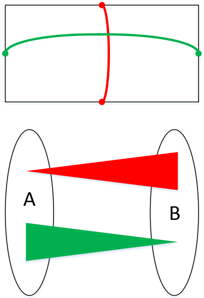

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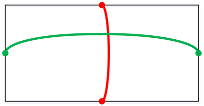

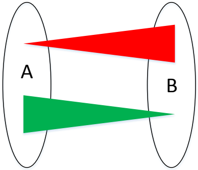

Every alternating rectangle in the grid gives a copy of $K_4^{(3)}$ with only two colors. □

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Claim:

Every $(4,3)$-coloring of $K_{2n}^{(3)}$ which uses r colors gives an alternating-free r-coloring of $\Gamma_{n,n}$.

Remarks: Converse statement is also true. By amplifying "*slightly*" the number of colors one can construct from an alternating-free coloring of grid a $(4,3)$-coloring of a complete 3-uniform hypergraph.

Ramsey-type problem

Definition:

$F(r,p,q) = \min n$ such that every r-edge-coloring of a complete graph K_n contains a K_p with at most $q-1$ colors on its edges.

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Question: *(Erdős-Gyárfás 90s)*

As q varies from 2 to $\binom{p}{2}$, when $F(r,p,q)$ becomes polynomial?

Polynomial growth

Proposition: *(Erdős-Gyárfás)*

$$F(r, p, p) = O\big(r^{p-2}\big).$$

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Hence $|X| = O(r^{p-3})$ and therefore $n = O(r|X|) = O(r^{p-2})$. □

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What is the minimum q such that $F(r, p, q)$ becomes *polynomial* in r? Is it $q = p$?

Erdős-Gyárfás conjecture

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Theorem: *(Conlon, Fox, Lee and S. 2014+)*

For all $p \geq 4$,

$$F(r,p,p-1) \geq r^{c \log^{\frac{1}{p-3}} r}.$$

Concluding remarks and open problem

Question: *(Conlon, Fox, Lee and S.)*

Is there an r-edge coloring of K_n such that the union of any q colors has chromatic number at most p?

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Question: What happens if we want the union of q colors to have chromatic number $\ll 2^q$?

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Proposition: *(Conlon, Fox, Lee and S.)*

There is an edge-coloring of K_n with $r = 2^{3\sqrt{\log n}}$ colors in which the union of any q colors has chromatic number at most $2^{3\sqrt{q \log q}}$.

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Question: *(Conlon, Fox, Lee and S.)*

What is the maximum $n = n(r)$ such that there is an r-edge coloring of K_n in which union of every 2 colors has chromatic number at most 3?

thank you
danke
謝謝
ngiyabonga
teşekkür ederim
gracias
рахмат
Баярлалаа
спасибо
faafetai lava
kiitos
dankie
dhanyavad
bedankt
hvala
maururu
köszönöm
dziękuję
enkosi
obrigado
sobodi
děkuji
mèsi
sagolun
sukriya
kop khun krap
chnorakaloutioun
gratias ago
grācies
sulpáy
taiku
go raibh maith agat
mochchakkeram
djiere dieuf
tapadh leat
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manana
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rahmet
ありがとう
terima kasih
najis tuke
kam sah hamnida
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감사합니다
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dhanyavadagalu
shukriya
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