Recent developments in Extremal Combinatorics: Ramsey and Turán type problems

Benny Sudakov
UCLA

Extremal combinatorics

Typical problem:

Determine or estimate the maximum or minimum possible size of a collection of finite objects (e.g., *graphs*, *sets*, *vectors*, *numbers*) satisfying certain restrictions.

TYPICAL PROBLEM:

Determine or estimate the maximum or minimum possible size of a collection of finite objects (e.g., *graphs*, *sets*, *vectors*, *numbers*) satisfying certain restrictions.

Modern tools:

- Combinatorial techniques (e.g. Regularity lemma)
- Probabilistic arguments
- Algebraic tools
- Harmonic Analysis
- Topological methods

General philosophy:

Every large system contains a large well organized subsystem.

T. Motzkin: *"Complete disorder is impossible!"*

General philosophy:

Every large system contains a large well organized subsystem.

T. Motzkin: *"Complete disorder is impossible!"*

Examples and applications:

- Combinatorics and graph theory
- Functional analysis
- Number theory
- Computer Science
- Geometry
- Information Theory
- Logic

Ramsey numbers

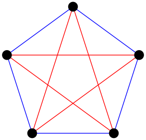
Definition:

The Ramsey number $r_k(s, n)$ is the minimum N such that every red-blue coloring of the k-tuples of an N-element set contains a red set of size s or a blue set of size n.

Definition:

The Ramsey number $r_k(s, n)$ is the minimum N such that every red-blue coloring of the k-tuples of an N-element set contains a red set of size s or a blue set of size n.

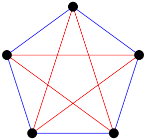
Example: $r_2(3, 3) = 6$



Definition:

The Ramsey number $r_k(s,n)$ is the minimum N such that every red-blue coloring of the k-tuples of an N-element set contains a red set of size s or a blue set of size n.

Example: $r_2(3,3) = 6$



Theorem: (Ramsey 1930)

For all k, s, n, the Ramsey number $r_k(s,n)$ is finite.

Diagonal Case: $s = n$

THEOREM: (ERDŐS 1947, ERDŐS-SZEKERES 1935)

$$2^{n/2} \le r_2(n, n) \le 2^{2n}.$$

Diagonal Case: $s = n$

Theorem: (Erdős 1947, Erdős-Szekeres 1935)

$$2^{n/2} \leq r_2(n, n) \leq 2^{2n}.$$

Upper bound: Induction: $r_2(s, n) \leq r_2(s-1, n) + r_2(s, n-1)$.
Every vertex has less than $r_2(s-1, n)$ red neighbors and less than $r_2(s, n-1)$ blue neighbors.

Graphs ($k = 2$)

Diagonal Case: $s = n$

Theorem: (Erdős 1947, Erdős-Szekeres 1935)

$$2^{n/2} \leq r_2(n, n) \leq 2^{2n}.$$

Upper bound: Induction: $r_2(s, n) \leq r_2(s-1, n) + r_2(s, n-1)$. Every vertex has less than $r_2(s-1, n)$ red neighbors and less than $r_2(s, n-1)$ blue neighbors.

Lower bound: Color every edge randomly. Probability that a given set of n vertices forms a monochromatic clique is $2 \cdot 2^{-\binom{n}{2}}$. Use the union bound.

Graphs $(k = 2)$

Diagonal Case: $s = n$

Theorem: (Erdős 1947, Erdős-Szekeres 1935)

$$2^{n/2} \leq r_2(n,n) \leq 2^{2n}.$$

Off-Diagonal Case:

Theorem: (Ajtai-Komlós-Szemerédi 80, Spencer 77, Kim 95)

- $r_2(3,n) = \Theta\left(\frac{n^2}{\log n}\right).$
- For $s \geq 4$, $\tilde{\Omega}\left(n^{\frac{s+1}{2}}\right) \leq r_2(s,n) \leq \tilde{O}\left(n^{s-1}\right).$

Hypergraphs $(k \geq 3)$, diagonal case

Theorem: (Erdős-Rado 1952, Erdős-Hajnal 1960s)

$$2^{cn^2} \leq r_3(n, n) \leq 2^{2^{c'n}}.$$

Hypergraphs ($k \geq 3$), diagonal case

Theorem: (Erdős-Rado 1952, Erdős-Hajnal 1960s)

$$2^{cn^2} \leq r_3(n,n) \leq 2^{2^{c'n}}.$$

Remarks:

- There is a similar gap of one exponential between the upper and the lower bound for $r_k(n,n)$ for $k > 3$. These bounds are towers of exponentials of height k and $k-1$ respectively.
- The $k = 3$ case is crucial. Determining the behavior of $r_3(n,n)$ will close the gap for all k as well.

Hypergraphs ($k \geq 3$), diagonal case

Theorem: (Erdős-Rado 1952, Erdős-Hajnal 1960s)

$$2^{cn^2} \leq r_3(n,n) \leq 2^{2^{c'n}}.$$

Remarks:

- There is a similar gap of one exponential between the upper and the lower bound for $r_k(n,n)$ for $k > 3$. These bounds are towers of exponentials of height k and $k-1$ respectively.
- The $k = 3$ case is crucial. Determining the behavior of $r_3(n,n)$ will close the gap for all k as well.

Conjecture: (Erdős)

The Ramsey number $r_3(n,n) \geq 2^{2^{cn}}$, for some constant $c > 0$.

Many colors

Erdős-Hajnal: *For 4 colors, $r_3(n, n, n, n)$ is doubly-exponential.*

Many colors

Erdős-Hajnal: *For 4 colors, $r_3(n, n, n, n)$ is doubly-exponential.*

Question:

What about the 3-color Ramsey number $r_3(n, n, n)$?

Many colors

Erdős-Hajnal: *For 4 colors, $r_3(n, n, n, n)$ is doubly-exponential.*

Question:

What about the 3-color Ramsey number $r_3(n, n, n)$?

Theorem 1 (Conlon-Fox-S. 2010)

$$r_3(n, n, n) \geq 2^{n^{c \log n}}.$$

Game:

- Two players, *builder* and *painter*.
- At step i a new vertex v_i is added. For every existing vertex $v_j, j < i$, *builder* decides whether to draw the edge (v_j, v_i).
- If the edge (v_j, v_i) was exposed, *painter* has to color it red or blue immediately.

On-line Ramsey game

Game:

- Two players, *builder* and *painter*.
- At step i a new vertex v_i is added. For every existing vertex $v_j, j < i$, *builder* decides whether to draw the edge (v_j, v_i).
- If the edge (v_j, v_i) was exposed, *painter* has to color it red or blue immediately.

Definition:

The vertex on-line Ramsey number $\tilde{r}(k, \ell)$ is the minimum number of edges that *builder* has to draw in order to force *painter* to create a red k-clique or a blue ℓ-clique.

On-line Ramsey game

Game:

- Two players, *builder* and *painter*.
- At step i a new vertex v_i is added. For every existing vertex $v_j, j < i$, *builder* decides whether to draw the edge (v_j, v_i).
- If the edge (v_j, v_i) was exposed, *painter* has to color it red or blue immediately.

Definition:

The vertex on-line Ramsey number $\tilde{r}(k, \ell)$ is the minimum number of edges that *builder* has to draw in order to force *painter* to create a red k-clique or a blue ℓ-clique.

Theorem: (Conlon-Fox-S. 2010)

$$r_3(s, n) \leq 2^{O(\tilde{r}(s-1, n-1))}.$$

Off-diagonal case $(k = 3)$

Theorem: (Conlon-Fox-S. 2010)

For small s and large n

$$2^{c' sn \log n} \leq r_3(s, n) \leq 2^{cn^{s-2} \log n}.$$

Off-diagonal case $(k = 3)$

Theorem: (Conlon-Fox-S. 2010)

For small s and large n

$$2^{c' sn \log n} \leq r_3(s,n) \leq 2^{cn^{s-2} \log n}.$$

Remarks:

- The upper bound uses the online Ramsey game and improves by a factor of roughly n^{s-2} the exponent of the previous best estimate of Erdős-Rado from 1952.
- The lower bound construction combines probabilistic reasoning with some combinatorial ideas, and answers an open question of Erdős-Hajnal from 1972.

Question: (Erdős-Hajnal 1989)

Given a red-blue coloring of triples of an N-element set, how large of an "almost monochromatic" subset (i.e., subset with density $1-\epsilon$ in one color) must it contain?

QUESTION: (ERDŐS-HAJNAL 1989)

Given a red-blue coloring of triples of an N-element set, how large of an "almost monochromatic" subset (i.e., subset with density $1-\epsilon$ in one color) must it contain?

Remark: The largest "almost monochromatic" subset in the random red-blue coloring of the edges of the complete graph on N vertices still has size $O(\log N)$.

Large almost monochromatic sets

Theorem : (Conlon-Fox-S. 2010+)

For any ϵ there exists a constant $c = c(\epsilon)$ such that every red-blue coloring of the triples of an N-element set contains a set S of size $n = c\sqrt{\log N}$ such that at least $(1-\epsilon)\binom{n}{3}$ triples of S have the same color.

Theorem : (Conlon-Fox-S. 2010+)

For any ϵ there exists a constant $c = c(\epsilon)$ such that every red-blue coloring of the triples of an N-element set contains a set S of size $n = c\sqrt{\log N}$ such that at least $(1-\epsilon)\binom{n}{3}$ triples of S have the same color.

Remarks:

- Random coloring shows that this result is tight.

Large almost monochromatic sets

Theorem : (Conlon-Fox-S. 2010+)

For any ϵ there exists a constant $c = c(\epsilon)$ such that every red-blue coloring of the triples of an N-element set contains a set S of size $n = c\sqrt{\log N}$ such that at least $(1-\epsilon)\binom{n}{3}$ triples of S have the same color.

Remarks:

- Random coloring shows that this result is tight.
- For coloring triples in $\ell > 2$ colors we can still find a subset of size $c\sqrt{\log N}$ with density $1-\epsilon$ in a single color.
- For hypergraphs $(k \geq 3)$

  "Discrepancy $\neq$ Ramsey"!

Definition:

$r(G)$ is the minimum N such that every 2-edge coloring of the complete graph K_N contains a monochromatic copy of graph G.

Sparse graphs

Definition:

$r(G)$ is the minimum N such that every 2-edge coloring of the complete graph K_N contains a monochromatic copy of graph G.

Question: *How large is $r(G)$ for a "sparse" graph on n vertices?*

Sparse graphs

Definition:

$r(G)$ is the minimum N such that every 2-edge coloring of the complete graph K_N contains a monochromatic copy of graph G.

Question: *How large is $r(G)$ for a "sparse" graph on n vertices?*

Definition:

A graph is *d-degenerate* if each of its subgraphs has a vertex of degree at most d.

Sparse graphs

Definition:

$r(G)$ is the minimum N such that every 2-edge coloring of the complete graph K_N contains a monochromatic copy of graph G.

Question: *How large is $r(G)$ for a "sparse" graph on n vertices?*

Definition:

A graph is *d-degenerate* if each of its subgraphs has a vertex of degree at most d.

Remarks:

- Every s vertices of such a graph span at most $d \cdot s$ edges.
- Graphs with maximum degree d are d-degenerate.
- Degenerate graphs include planar graphs, sparse random graphs and might have vertices of very large degree.

Conjecture: (Burr-Erdős 1975)

For every d there exists a constant c_d such that

$$r(G) \leq c_d n$$

for every d-degenerate graph on n vertices.

Conjecture: (Burr-Erdős 1975)

For every d there exists a constant c_d such that

$$r(G) \leq c_d n$$

for every d-degenerate graph on n vertices.

Known for: bounded degree graphs [CRST '83], planar graphs and graphs drawn on bounded genus surfaces [CS '93], graphs with a fixed forbidden minor [RT 97] and sparse random graphs [FS '09].

Conjecture: (Burr-Erdős 1975)

For every d there exists a constant c_d such that

$$r(G) \leq c_d n$$

for every d-degenerate graph on n vertices.

Known for: bounded degree graphs [CRST '83], planar graphs and graphs drawn on bounded genus surfaces [CS '93], graphs with a fixed forbidden minor [RT 97] and sparse random graphs [FS '09].

Theorem: *(Kostochka-S. 2003)*

The Ramsey number of any d-degenerate graph G on n vertices satisfies $r(G) \leq n^{1+\epsilon}$ for any fixed $\epsilon > 0$.

Maximizing the Ramsey number

Conjecture: (Erdős-Graham 1973)

Among all the graphs with $m = \binom{n}{2}$ edges and no isolated vertices, the n-vertex complete graph has the largest Ramsey number.

Maximizing the Ramsey number

Conjecture: (Erdős-Graham 1973)

Among all the graphs with $m = \binom{n}{2}$ edges and no isolated vertices, the n-vertex complete graph has the largest Ramsey number.

Remark: The complete graph with m edges has $O(\sqrt{m})$ vertices and a Ramsey number bounded by $2^{O(\sqrt{m})}$. In the early 80's Erdős asked to show that this bound holds for all graphs with m edges.

Maximizing the Ramsey number

Conjecture: (Erdős-Graham 1973)

Among all the graphs with $m = \binom{n}{2}$ edges and no isolated vertices, the n-vertex complete graph has the largest Ramsey number.

Remark: The complete graph with m edges has $O(\sqrt{m})$ vertices and a Ramsey number bounded by $2^{O(\sqrt{m})}$. In the early 80's Erdős asked to show that this bound holds for all graphs with m edges.

Theorem: *(S. 2010+)*

If G is a graph with m edges without isolated vertices, then

$$r(G) \leq 2^{250\sqrt{m}}.$$

Methods: Dependent random choice

Rough claim:

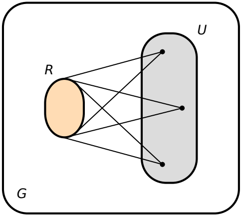
Every *sufficiently dense* graph G contains a *large* subset U in which every/almost all sets of d vertices have *many* common neighbors.

Rough claim:

Every *sufficiently dense* graph G contains a *large* subset U in which every/almost all sets of d vertices have *many* common neighbors.

Proof:

Let U be the set of vertices adjacent to every vertex in a random subset R of G of an *appropriate* size.

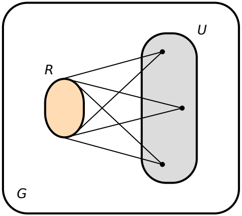


Rough claim:

Every *sufficiently dense* graph G contains a *large* subset U in which every/almost all sets of d vertices have *many* common neighbors.

Proof:

Let U be the set of vertices adjacent to every vertex in a random subset R of G of an *appropriate* size.



If some set of d vertices has only *few* common neighbors, it is unlikely that all the members of R will be chosen among these neighbors. Hence we do not expect U to contain any such d vertices. $\square$

Forbidden subgraph problem:

Given a fixed graph H, determine $ex(n, H)$, the maximum number of edges in a graph on n vertices that does not contain a copy of H.

Extremal Graph Theory

Forbidden subgraph problem:

Given a fixed graph H, determine $ex(n, H)$, the maximum number of edges in a graph on n vertices that does not contain a copy of H.

Appears in:

- Discrete geometry
- Additive number theory
- Probability
- Harmonic Analysis
- Computer Science
- Coding Theory

Extremal Graph Theory

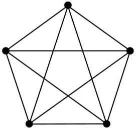
Forbidden subgraph problem:

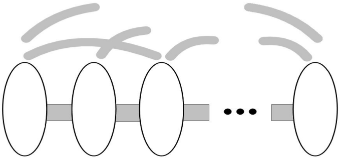
Given a fixed graph H, determine $ex(n, H)$, the maximum number of edges in a graph on n vertices that does not contain a copy of H.

Appears in:

- Discrete geometry
- Additive number theory
- Probability
- Harmonic Analysis
- Computer Science
- Coding Theory

Mantel 1907: *Every triangle-free graph on n vertices has at most $\lfloor n^2/4 \rfloor$ edges.*

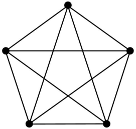




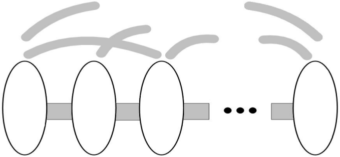
K_{r+1} = complete graph of order $r+1$

Turán graph $T_r(n)$: complete r-partite graph with equal parts.
$t_r(n) = e(T_r(n)) = \frac{r-1}{2r}n^2 + O(r)$

Turán's theorem



K_{r+1} = complete graph of order $r+1$



Turán graph $T_r(n)$: complete r-partite graph with equal parts.

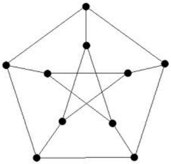
$t_r(n) = e(T_r(n)) = \frac{r-1}{2r}n^2 + O(r)$

Theorem: *(Turán 1941, Mantel 1907 for $r = 2$)*

For all $r \geq 2$, the unique largest K_{r+1}-free graph on n vertices is $T_r(n)$.

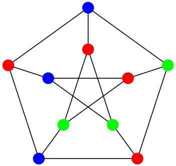
Question:

What is the Turán number $ex(n, H)$ for a general graph H?

E.g., $H =$ 

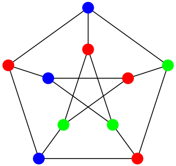
Definition:

The chromatic number $\chi(H)$ is the minimum number of colors needed to color $V(H)$, so that adjacent vertices have distinct colors.



Definition:

The chromatic number $\chi(H)$ is the minimum number of colors needed to color $V(H)$, so that adjacent vertices have distinct colors.



Theorem: (*Erdős-Stone 1946, Erdős-Simonovits 1966*)

Let H be a fixed graph with $\chi(H) = r + 1$. Then

$$ex(n, H) = t_r(n) + o(n^2) = (1 + o(1))\frac{r-1}{2r}n^2.$$

Remark: Determines the asymptotics of Turán numbers $ex(n, H)$ for all graphs with chromatic number at least 3.

Problem:

It is known [KST '54] that $ex(n, H) \leq O(n^{2-\epsilon_H})$ for some $\epsilon_H > 0$. What parameter of the bipartite graph H might determine the growth of $ex(n, H)$?

Problem:

It is known [KST '54] that $ex(n, H) \leq O(n^{2-\epsilon_H})$ for some $\epsilon_H > 0$. What parameter of the bipartite graph H might determine the growth of $ex(n, H)$?

Known:

- For complete bipartite graphs $K_{r,s}$ for $s > (r-1)!$.
- For cycles of even length C_{2k} for $k = 2, 3, 5$.

Problem:

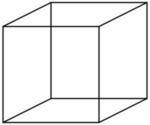
It is known [KST '54] that $ex(n, H) \leq O(n^{2-\epsilon_H})$ for some $\epsilon_H > 0$. What parameter of the bipartite graph H might determine the growth of $ex(n, H)$?

Known:

- For complete bipartite graphs $K_{r,s}$ for $s > (r-1)!$.
- For cycles of even length C_{2k} for $k = 2, 3, 5$.

Open:

- Complete bipartite graph with equal parts of size 4.
- Cycle of length 8.
- The 3-cube.



Recall: A graph is r-degenerate if each of its subgraphs has a vertex of degree at most r.

Recall: A graph is r-degenerate if each of its subgraphs has a vertex of degree at most r.

CONJECTURE: (*Erdős 1966*)

Every r-degenerate bipartite H satisfies $ex(n, H) \leq O(n^{2-1/r})$.

Remark: For all r this estimate is best possible.

Recall: A graph is r-degenerate if each of its subgraphs has a vertex of degree at most r.

Conjecture: (*Erdős 1966*)

Every r-degenerate bipartite H satisfies $ex(n, H) \leq O(n^{2-1/r})$.

Remark: For all r this estimate is best possible.

Theorem: (*Alon-Krivelevich-S. 2003*)

Conjecture holds for every H in which vertices of one part have degrees at most r. For general r-degenerate bipartite H

$$ex(n, H) \leq O(n^{2-\frac{1}{4r}}).$$

Question: *How many copies of a fixed bipartite graph H must exist in an n-vertex graph with m edges?*

Question: *How many copies of a fixed bipartite graph H must exist in an n-vertex graph with m edges?*

Definition:

- $h_H(G)$ = the number of homomorphisms from H to G.
- $t_H(G) = \frac{h_H(G)}{|G|^{|H|}}$ = fraction of mappings from H to G which are homomorphisms.

Sidorenko's conjecture

Question: *How many copies of a fixed bipartite graph H must exist in an n-vertex graph with m edges?*

Definition:

- $h_H(G)$ = the number of homomorphisms from H to G.
- $t_H(G) = \frac{h_H(G)}{|G|^{|H|}}$ = fraction of mappings from H to G which are homomorphisms.

Conjecture: (*Erdős-Simonovits 84, Sidorenko 93*)

For every bipartite H and every n-vertex G with $pn^2/2$ edges,

$$t_H(G) \geq p^{e(H)}.$$

Conjecture: (*Erdős-Simonovits 84, Sidorenko 93*)

For every bipartite H and every n-vertex G with $pn^2/2$ edges,

$$t_H(G) \geq p^{e(H)}.$$

Remarks:

- Random graphs with edge probability p achieve minimum.

Sidorenko's conjecture

Conjecture: (*Erdős-Simonovits 84, Sidorenko 93*)

For every bipartite H and every n-vertex G with $pn^2/2$ edges,

$$t_H(G) \geq p^{e(H)}.$$

Remarks:

- Random graphs with edge probability p achieve minimum.
- Has analytical form and connections to matrix theory [BR], Markov chains [BP], graph limits [L], and quasi-randomness.
- Known for *trees, even cycles, complete bipartite graphs, cubes.*

Sidorenko's conjecture

Conjecture: (*Erdős-Simonovits 84, Sidorenko 93*)

For every bipartite H and every n-vertex G with $pn^2/2$ edges,

$$t_H(G) \geq p^{e(H)}.$$

Remarks:

- Random graphs with edge probability p achieve minimum.
- Has analytical form and connections to matrix theory [BR], Markov chains [BP], graph limits [L], and quasi-randomness.
- Known for *trees, even cycles, complete bipartite graphs, cubes.*

Theorem: (*Conlon-Fox-S. 2010+*)

Conjecture holds for every bipartite H which has a vertex complete to all vertices in other part. This gives an asymptotic version of the conjecture for all graphs.

Observation:

The size of the maximum bipartite subgraph of a graph G	$\leq$	The size of the maximum triangle-free subgraph of a graph G

Observation:

The size of the maximum bipartite subgraph of a graph G	$\leq$	The size of the maximum triangle-free subgraph of a graph G

Turán's theorem: *Equality if G is a complete graph.*

Observation:

The size of the maximum bipartite subgraph of a graph G	$\leq$	The size of the maximum triangle-free subgraph of a graph G

TURÁN'S THEOREM: *Equality if G is a complete graph.*

PROBLEM: (*Erdős 1983*)

Find conditions on a graph G which imply that the largest K_{r+1}-free subgraph and the largest r-partite subgraph of G have the same number of edges.

Large minimum degree is enough

Theorem: (*Alon, Shapira, S. 2009*)

Let H be a fixed graph with chromatic number $r+1>3$. There exist constants $\gamma=\gamma(H)>0$ and $\mu=\mu(H)>0$ such that if G is a graph on n vertices with minimum degree at least $(1-\mu)n$ and Γ is the largest H-free subgraph of G, then Γ can be made r-partite by deleting $O\left(n^{2-\gamma}\right)$ edges.

Large minimum degree is enough

Theorem: (*Alon, Shapira, S. 2009*)

Let H be a fixed graph with chromatic number $r+1>3$. There exist constants $\gamma=\gamma(H)>0$ and $\mu=\mu(H)>0$ such that if G is a graph on n vertices with minimum degree at least $(1-\mu)n$ and Γ is the largest H-free subgraph of G, then Γ can be made r-partite by deleting $O\left(n^{2-\gamma}\right)$ edges.

Remarks:

- If H is a clique K_{r+1} then the largest H-free subgraph of such G is r-partite.
- Extends Turán's and Erdős-Stone-Simonovits theorems to all graphs with large minimum degree.

Edge-deletion problems

Definition:

A graph property $\mathcal{P}$ is *monotone* if it is closed under deleting edges and vertices. It is *dense* if there are n-vertex graphs with $\Omega(n^2)$ edges satisfying it.

Edge-deletion problems

Definition:

A graph property $\mathcal{P}$ is *monotone* if it is closed under deleting edges and vertices. It is *dense* if there are n-vertex graphs with $\Omega(n^2)$ edges satisfying it.

Examples:

- $\mathcal{P} = \{G \text{ is 5-colorable}\}$.
- $\mathcal{P} = \{G \text{ is triangle-free}\}$.
- $\mathcal{P} = \{G \text{ has a 2-edge coloring with no monochromatic } K_6\}$.

Edge-deletion problems

Definition:

A graph property $\mathcal{P}$ is *monotone* if it is closed under deleting edges and vertices. It is *dense* if there are n-vertex graphs with $\Omega(n^2)$ edges satisfying it.

Examples:

- $\mathcal{P} = \{G \text{ is 5-colorable}\}$.
- $\mathcal{P} = \{G \text{ is triangle-free}\}$.
- $\mathcal{P} = \{G \text{ has a 2-edge coloring with no monochromatic } K_6\}$.

Definition:

Given a graph G and a monotone property $\mathcal{P}$, let

$$E_{\mathcal{P}}(G) = \text{smallest number of edge deletions needed to turn } G \text{ into a graph satisfying } \mathcal{P}.$$

Approximation and hardness

Theorem: (*Alon, Shapira, S. 2009*)

- For every monotone $\mathcal{P}$ and $\epsilon > 0$, there exists a linear-time deterministic algorithm that, given a graph G on n vertices, computes a number X such that $\left|X - E_{\mathcal{P}}(G)\right| \leq \epsilon n^2$.
- For every monotone dense $\mathcal{P}$ and $\delta > 0$, approximating $E_{\mathcal{P}}(G)$ within an additive error of $n^{2-\delta}$ is NP-hard.

Approximation and hardness

Theorem: (*Alon, Shapira, S. 2009*)

- For every monotone $\mathcal{P}$ and $\epsilon > 0$, there exists a linear-time deterministic algorithm that, given a graph G on n vertices, computes a number X such that $\left|X - E_{\mathcal{P}}(G)\right| \leq \epsilon n^2$.
- For every monotone dense $\mathcal{P}$ and $\delta > 0$, approximating $E_{\mathcal{P}}(G)$ within an additive error of $n^{2-\delta}$ is NP-hard.

Remarks:

- Answers in a strong form a question of Yannakakis from 1981. For many monotone dense $\mathcal{P}$ it even wasn't known before that computing $E_{\mathcal{P}}(G)$ *precisely* is NP-hard.

Approximation and hardness

Theorem: (*Alon, Shapira, S. 2009*)

- For every monotone $\mathcal{P}$ and $\epsilon > 0$, there exists a linear-time deterministic algorithm that, given a graph G on n vertices, computes a number X such that $\left|X - E_{\mathcal{P}}(G)\right| \leq \epsilon n^2$.
- For every monotone dense $\mathcal{P}$ and $\delta > 0$, approximating $E_{\mathcal{P}}(G)$ within an additive error of $n^{2-\delta}$ is NP-hard.

Remarks:

- Answers in a strong form a question of Yannakakis from 1981. For many monotone dense $\mathcal{P}$ it even wasn't known before that computing $E_{\mathcal{P}}(G)$ *precisely* is NP-hard.
- First result uses a strengthening of Szemerédi regularity lemma to approximate G by a fixed size weighted graph W.
- Second result uses generalizations of Turán and Erdős-Stone-Simonovits theorems together with spectral techniques.

Summary

The open problems which we mentioned, as well as many more additional ones which we skipped due to the lack of time, will provide interesting challenges for future research in extremal combinatorics.

Summary

The open problems which we mentioned, as well as many more additional ones which we skipped due to the lack of time, will provide interesting challenges for future research in extremal combinatorics.

These challenges, the fundamental nature of the area and its tight connection with other mathematical disciplines will ensure that in the future extremal combinatorics will continue to play an essential role in the development of mathematics.