Induced Ramsey problems for trees and graphs with bounded treewidth

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Abstract

The induced q-color size-Ramsey number $\hat{r}_{\text{ind}}(H;q)$ of a graph H is the minimal number of edges a host graph G can have so that every q-edge-coloring of G contains a monochromatic copy of H which is an induced subgraph of G. A natural question, which in the non-induced case has a very long history, asks which families of graphs H have induced Ramsey numbers that are linear in $|H|$. We prove that for every k, w, q, if H is an n-vertex graph with maximum degree k and treewidth at most w, then $\hat{r}_{\text{ind}}(H;q) = O_{k,w,q}(n)$. This extends several old and recent results in Ramsey theory. Our proof is quite simple and relies upon a novel reduction argument.

1 Introduction

A celebrated theorem of Ramsey says that for every graph H, and every integer q, there exists N such that every q-edge-coloring of the complete graph K_N, contains a monochromatic copy of H. We write $r(H;q)$ to denote the smallest integer N with this property, this is the q-color Ramsey number of H. When $q = 2$, we often suppress this, writing $r(H)$ instead of $r(H;2)$. Determining or estimating Ramsey numbers is a central question in Combinatorics, which was extensively studied in the last seventy years. We refer the reader to the book [21] or the survey [7] for further literature.

More generally, one can consider a host graph G different from the complete graph and say that $G \to (H)_q$ if for every q-coloring of the edges of G, we can find a monochromatic copy of H. The Ramsey numbers ask for the minimal number of vertices a host graph G can have, in which case G should clearly be complete. But one can also consider having "sparser" host graphs G, which minimize other parameters. This led Erdős, Faudree, Rousseau, and Schelp in 1978 to the natural question of the minimum number of edges [13] that a Ramsey graph can have. They define $\hat{r}(H;q) := \min\{|E(G)| : G \to (H)_q\}$ to be the q-color size-Ramsey number (of H).

One of the extensively studied questions, which was asked by Erdős [16], is to understand which classes of graphs have size-Ramsey numbers that are linear in their number of edges. A classic result of Beck [2] is that the n-vertex path P_n has linear size-Ramsey numbers. Later on, Friedman and Pippenger proved that n-vertex trees T with maximum degree k also satisfies $\hat{r}(T;q) = O_{q,k}(n)$ [19]. Building upon this, Haxell and Kohayakawa proved the bound $\hat{r}(T;q) = O_q(kn)$ [23]. This is tight in general, since as observed by Beck [3] it is easy to construct trees T where $\hat{r}(T) \geq \Omega(kn)$.

Paths and trees are very special cases of graphs with bounded treewidth. Recall that a graph is G is *chordal* if every induced cycle is a triangle, and that H has treewidth at most w if $H \subset G$ for some chordal graph G without a clique of size $w + 2$. It was recently proved by Berger, Kohayakawa, Maeseka, Martins, Mendonça, Mota, and Parczyk that if H is an n-vertex

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graph with maximum degree at most k and treewidth at most w, then $\hat{r}(H;q) = O_{q,k,w}(n)$ [4] (see also [26], which handles the case of $q = 2$). Furthermore Draganić, Kaufmann, Munhá Correia, Petrova, and Steiner [12] has obtained the nearly linear bound $\hat{r}(H) \leq O_k(nw \log n)$ on the size-Ramsey numbers of H with maximum degree k and treewidth w growing with n.

In this paper, we will focus on the induced analogues of the above quantities. We say $G \to_{\text{ind}} (H)_q$ if for every q-coloring of $E(G)$, there is some monochromatic copy of H which is an induced subgraph of G. We define $r_{\text{ind}}(H;q)$ and $\hat{r}_{\text{ind}}(H;q)$ in the same way as the non-induced versions, being the minimum number of vertices/edges the host graph G must have to satisfy $G \to_{\text{ind}} (H)_q$. The existence of these numbers is non-trivial and is an important extension of Ramsey's theorem. It was originally proven independently by Deuber [10], Erdős, Hajnal, and Posa [14], and Rödl [29]. Naturally, it is much harder to get upper bounds for these induced variants. For example, a short and simple argument shows that $r(H) \leq 4^n$ for any n-vertex graph H. On the other hand, the best upper bound in the induced setting is only $r_{\text{ind}}(H) \leq n^{O(n)}$ (see [6]), and it is a famous open question of Erdős whether there is an exponential bound $2^{O(n)}$ for this problem.

Another natural question (see e.g., [5]), that had a gap between bounds for induced and non-induced settings, was bounded degree trees. Although, as we mentioned above, it was well known that the size-Ramsey number of such trees are linear, the induced counterpart was not known. Very recently, Girão and Hurley [20] proved that for any tree T with maximum degree k, $\hat{r}_{\text{ind}}(T;q) = O_q(k^2 n)$. This is a very nice result that in particular shows how to find induced trees in sparse expander graphs. The main contribution of this note is a shorter, alternative proof of a more general result (albeit with worse quantitative bounds).

Theorem 1. *Let H be an n-vertex graph with maximum degree k and treewidth at most w. Then*

$$\hat{r}_{ind}(H;q) = O_{k,w,q}(n).$$

In particular, this theorem extends a celebrated result of Haxell, Kohayakawa, and Luczak [24] that the induced size-Ramsey numbers of cycles are linear (since cycles have treewidth 2).

While [20] works by extending Friedman-Pippenger-type techniques to the induced setting, our approach is quite different. We use a novel reduction (Theorem 3, discussed in the next subsection) which converts constructions for the size-Ramsey numbers $\hat{r}(H;q)$ into constructions for $r_{\text{ind}}(H;q)$ and $\hat{r}_{\text{ind}}(H;q)$. Then, we obtain Theorem 1 by applying the result of [4] as a blackbox, which implicitly builds on the standard *non-induced* Friedman-Pippenger embedding method.

Due to the usage of (weak) regularity-based arguments, the absolute constants from Theorem 1 are quite large. Therefore for trees, we shall also present a more efficient version of our argument, which we believe may be of independent interest. We first need a key definition, that will be used throughout the paper.

Definition. Given a graph G, we say G' is an *s-blowup* of G, if there is a homomorphism $\phi : V(G') \to V(G)$ so that $uv \in E(G')$ implies $\phi(u)\phi(v) \in E(G)$ and furthermore $|\phi^{-1}(v)| \leq s$ for each vertex $v \in V(G)$. In particular, we do not have edges inside the subsets $\phi^{-1}(v)$.

Theorem 2. *Let T be an n-vertex tree with maximum degree k. Then*

$$r_{ind}(T;q) \leq \hat{r}_{ind}(T;q) \leq (kq)^{Cq^4k^3} n$$

for some absolute constant C. Furthermore, if H is a w-blowup of T, then

$$\hat{r}_{ind}(H;q) \leq (kq)^{Cq^4k^3w} n.$$

The latter bound gives the correct dependence on w. Indeed, the complete w-blowup H of P_n contains $\Omega(n)$ vertex disjoint copies of $K_{w,w}$. Meanwhile (for $w \geq 11$), using the local lemma it is easy to show that any graph with maximum degree at most $2^{w/2}$ has a 2-edge coloring without any monochromatic copy of $K_{w,w}$ (see Lemma 2.3). Thus if $G \to H$, G must have $\Omega(n)$ vertices of degree $\geq 2^{w/2}$, and so $e(G) \geq \Omega(2^{w/2}n)$. We note that in the non-induced setting an upper bound of $\hat{r}(H;q) \leq (kq)^{q^{O(qk)}w}n$ was obtained by Jiang, Milans, and West [25, Theorem 5.3].

1.1 Idea of proof

A large amount of work on the size-Ramsey numbers and induced Ramsey numbers (including [2, 17, 11, 20]) have considered the host graph G to be a random graph $G(N,p)$ for appropriately chosen N, p and used the generic pseudorandom properties it satisfies. However, we shall instead argue by using a more carefully constructed host graph (taking a "gadget-based" approach).

Our construction will start with a non-induced Ramsey host graph G of H, where $\Delta(G)$ is bounded, and then consider an appropriate "pseudorandom blowup" of G. More formally, to prove Theorem 1, we rely on the following.

Theorem 3. *Fix k, Δ, q, there exists some $s = s(k, \Delta, q)$ so that the following holds. Let H, G be graphs with $G \to (H)_q$. Suppose $\Delta(H) \leq k$ and $\Delta(G) \leq \Delta$. Then, there is an s-blowup of G, G', so that $G' \to_{ind} (H)_q$.*

To prove Theorem 2, we will use a bipartite-analogue of this result (Theorem 5), which has stronger bounds (and adds a new parameter w to get the second part of Theorem 2). Theorems 3 and 5 are the two main results of our work.

Next, we describe our construction of G' in a bit more detail. We start with some bounded degree host graph G satisfying $G \to (H)_q$. Then, we take some "pseudorandom bipartite gadget Γ" with vertex sets of size s on both sides. We define our new host graph G' by replacing each $v \in V(G)$ by a set X_v of s vertices and for each $uv \in E(G)$ adding a copy of Γ between X_u and X_v. In this paper our gadgets are properly chosen random bipartite graphs, but we can also choose Γ to be a dense spectral expander, for which there are various explicit constructions (e.g., Example 5 from the survey [27, Section 3]). To show that G' is induced-Ramsey for H, we employ the following high-level strategy.

1. First, fix some q-coloring C' of $E(G')$.

2. We then apply a "cleaning procedure", producing a q-coloring C of $E(G)$, and subsets $(X^*_v)_{v \in V(G)}$, so that for any edge $e = uv \in E(G)$, the $C(e)$-monochromatic subgraph of $G'[X^*_u, X^*_v]$ is appropriately "robust" (for Theorem 3, we shall require said graph is lower-regular).

3. By assumption, since $G \to (H)_q$, we can find a monochromatic copy of H inside G.

4. Finally, using this monochromatic copy of H, we run an embedding procedure to find a monochromatic induced copy of H inside G'. Here we use the pseudorandomness assumptions about G' together with the fact that G having bounded degree, to get all our desired non-edges. Meanwhile, the robustness properties ensured by Step 2 are used so that we get our desired monochromatic edges.

In the above, it will be important that s is sufficiently large. Indeed, to run our "induced embedding procedure" (Step 4), we will need a certain lower bound on the sizes of our sets X^*_v produced by (Step 2). Meanwhile, in the "cleaning procedure" (Step 2), our sets will be forced to shrink by some constant factor. Both the lower bound and this constant factor will end up depending on k, Δ, q.

Given the outlined framework, a proof of Theorem 1 can be phrased quite concisely. First, we take an appropriate pseudorandom blowup (as described above) of the Ramsey graph G, that is provided to us by the work of [4]. For the cleaning procedure needed in Step 2, we can use Proposition 4.1 from [8]. Hence, the only new ingredient we require is a simple embedding result for Step 4, which we prove in Proposition 3.1. Putting these three pieces together immediately gives Theorem 1.

2 Preliminaries

2.1 Notation

The graph theoretic notation is mostly standard, although we recall a few specific concepts here. We denote an edge e between two vertices u and v by uv. Given a graph G, we write $N_G(x)$ to denote the neighborhood of x. For a vertex $x \in V(G)$ and $S \subset V(G)$, we write $d_S(x) := |N_G(x) \cap S|$ to count the number of neighbors x has in S. Given sets $A, B \subset V(G)$, we write $G[A]$ to denote the subgraph induced by A, and $G[A, B]$ to denote the bipartite induced between A and B.

2.2 Lemmata

In this subsection we collect some technical lemmas which we will need in our proofs. We start by describing pseudorandom bipartite graphs, which we will use for blowups.

Definition. We say a bipartite graph $G = (X, Y, E)$ is (L, p)-*regular* if: for any $X' \subset X$ of size $|X'| \geq L$, there are at most L vertices $y \in Y$ such that $d_{X'}(y) < \frac{1}{2}p|X'|$ or $d_{X'}(y) > 2p|X'|$, and similarly for any $Y' \subset Y$ with $|Y'| \geq L$ there are at most L vertices $x \in X$ satisfying $d_{Y'}(x) < \frac{1}{2}p|Y'|$ or $d_{Y'}(x) > 2p|Y'|$.

We also say that $G = (X, Y, E)$ is (L, p)-lower-regular if for any $X' \subset X$ of size $|X'| \geq L$, we have $|\{y \in Y : d_{X'}(y) < \frac{1}{2}p|X'|\}| < L$ (and the similar inequality holds for $Y' \subset Y$ with $|Y'| \geq L$).

A fully standard application of the probabalistic method yields the following claim, which we will use to construct gadgets.

Proposition 2.1. *Consider $p \in (0, 1)$ and a, b. There exists a bipartite graph $\Gamma = (A, B, E)$ which is $(\frac{48}{p} \ln(a + b), p)$-regular.*

To prove this proposition we use the following lemma.

Lemma 2.2. *Consider $p, \epsilon \in (0, 1)$ and $t \geq 1$. Assume $n \leq \exp\left(\frac{t\epsilon^2 p}{6}\right)$. Then there exists an n-vertex graph G, so that for any two disjoint sets $X', Y' \subset V(G)$ of size t,*

$$|e_G(X', Y') - pt^2| \leq \epsilon pt^2.$$

Proof. We sample $G \sim G(n, p)$. Let $\mathbf{Z}$ count the number of unordered "bad" pairs of disjoint $X', Y' \in \binom{V(G)}{t}$ with

$$|e_G(X', Y') - pt^2| > \epsilon pt^2.$$

For any fixed choice of X', Y', we estimate the probability that they are bad using a standard Chernoff bound (see [1, Corollary A.1.14]). It says that for any $\delta \in (0, 1)$ and binomial random variable $X \sim \mathrm{Bin}(n, p)$,

$$\mathbb{P}(|X - np| > \delta np) \leq 2e^{-\delta^2 np/3}. \tag{2.1}$$

Since the number of edges between X', Y' is distributed like $\mathrm{Bin}(t^2, p)$, we can bound the probability that they are bad by $2e^{-\epsilon^2 pt^2/3}$. So, considering all possible unordered pairs, we have that

$$\mathbb{E}[\mathbf{Z}] \leq \frac{1}{2}\binom{n}{t}\binom{n-t}{t} \cdot 2e^{-\epsilon^2 pt^2/3} < n^{2t}e^{-\epsilon^2 pt^2/3}.$$

Using our assumptions on n, it is easy to check that $\mathbb{E}[\mathbf{Z}] < 1$, whence there must be some outcome of G with the desired properties. □

Proof of Proposition 2.1. Write $n := a + b, \epsilon := 1/2, p := p, t := \frac{6}{\epsilon^2 p}\ln n = \frac{24}{p}\ln n$. We can apply Lemma 2.2 with these parameters to get some appropriate n-vertex graph G. We will take $\Gamma = G[A, B]$ where A, B are any two disjoint sets of size a, b respectively, and argue this is $(2t, p)$-regular (as desired).

Indeed, suppose there was some set $X' \subset X$ of size at least $2t$ so that there are at least $2t$ vertices $y \in Y$ satisfying $d_{X'}(y) < \frac{1}{2}p|X'|$ or $2p|X'| < d_{X'}(y)$. By pigeonhole principle, we can find some Y' (disjoint from X') of size t so that either $d_{X'}(y) > 2p|X'| > (1+\epsilon)p|X'|$ for all $y \in Y'$ or $d_{X'}(y) < (1-\epsilon)p|X'|$ for all $y \in Y'$. Taking a random $X'' \subset X'$ of size exactly t, we either have $\mathbb{E}[e(G[X'', Y'])] > (1+\epsilon)pt^2$ or $\mathbb{E}[e(G[X'', Y'])] < (1-\epsilon)pt^2$, contradicting in either case the properties of G. Arguing symmetrically for sets $Y' \subset Y$ establishes that Γ is $(2t, p)$-regular, as desired. □

Next, we prove the bound from the introduction which we use to show the tightness of Theorem 2 for blow-ups of trees

Lemma 2.3. *Fix $w \geq 11$. Let G be a graph with maximum degree at most $2^{w/2}$. Then G has a 2-edge coloring with no monochromatic $K_{w,w}$.*

Proof. To prove this statement we use the Lovász Local Lemma (see, e.g., [1, Lemma 5.1.1]), which says that if $(E_i)_{i \in I}$ is a collection of events so that E_i is mutually independent from all but at most D other events E_j and

$$\mathbb{P}(E_i) \leq \frac{1}{e(D+1)}, \tag{2.2}$$

then with positive probability none of the events E_i occur.

Consider a uniformly random 2-edge coloring of G. For each $H \subset G$ isomorphic to $K_{w,w}$, let E_H be the event that H is monochromatic. Clearly, $\mathbb{P}(E_H) = p := 2^{1-w^2}$ for each H. Moreover, two events $E_H, E_{H'}$ can be dependent only if H, H' share an edge. We can now use that G has bounded degree to control dependencies. Indeed, for any $e = uv \in E(G)$, the number of copies H containing e is at most $\binom{d(u)}{w-1}\binom{d(v)}{w-1} \leq 2^{w^2-w} \leq \frac{1}{12w^2}2^{w^2}$ (assuming $w \geq 11$). Since every copy of $K_{w,w}$ has w^2 edges, each event is independent from all but at most $D \leq \frac{1}{12}2^{w^2}$ other events. As $e(D+1)p < 1$ we can use the local lemma to get a coloring without monochromatic $K_{w,w}$. □

Finally, we need a short lemma whose proof is based on *dependent random choice*, which is a powerful tool in extremal combinatorics (see [18] for a discussion of many applications). Given a sufficiently dense bipartite graph $\Gamma = (X, Y, E)$, this method allows one to pass to some subgraph $\Gamma' = (X, Y', E)$ so that any k vertices in Y' have many common neighbors in X.

However, for our applications, we need a slightly more technical statement. Specifically, given multiple subsets $Y_1, \ldots, Y_t \subset Y$, and appropriate assumptions on the subgraphs $\Gamma[X, Y_i]$, we want to find an outcome of Y' where $Y' \cap Y_i$ is simultaneously non-empty for each $i = 1, \ldots, t$. This forces us to be more careful and use a simple but slightly different variant of the usual dependent random choice argument.

Lemma 2.4. *Let $h \geq 1$, $p \in (0,1)$ and let $\Gamma = (X, Y, E)$ be a bipartite graph with at least $p|X||Y|$ edges. If we sample $x_1, \ldots, x_h \in X$ uniformly at random with repetitions, then*

$$\mathbb{P}\big(|\cap_{i=1}^h N(x_i)| \geq \frac{p^h}{2}|Y|\big) > \frac{p^h}{2}|Y|.$$

Proof. Consider the random variable $\mathbf{Z} := |\cap_{i=1}^h N(x_i)|$. Using Jensen's inequality, we have

$$\mathbb{E}[\mathbf{Z}] = \sum_{y \in Y} (d(y)/|X|)^h \geq |Y| \cdot \left(\frac{\sum_{y \in Y} d(y)}{|X||Y|}\right)^h \geq p^h|Y|.$$

Since $\mathbf{Z} \leq |Y|$ always holds, we have

$$p^h|Y| \leq \mathbb{E}[\mathbf{Z}] \leq |Y| \cdot \mathbb{P}(\mathbf{Z} \geq p^h|Y|/2) + p^h|Y|/2.$$

Rearranging gives $\mathbb{P}(|N(x_1) \cap \cdots \cap N(x_h)| \geq \frac{p^h}{2}|Y|) \geq p^h/2$, as required. □

Proposition 2.5. *Fix integers ℓ, h, r. Let $\Gamma = (X, Y, E)$ be a bipartite graph and let $Y_1, \ldots, Y_\ell$ be subsets of Y so that $|N(x) \cap Y_i| \geq p|Y_i|$ for each $x \in X$ and $i \in [\ell]$. Suppose that $p^{h\ell}/2 > |Y|^r|X|^{-h/2}$. Then there exists subsets $Y_i' \subset Y_i, i \in [\ell]$ such that $|Y_i'| \geq \frac{p^{h\ell}}{2}|Y_i|$ for $i \in [\ell]$ and for any $y_1, \ldots, y_r \in \bigcup_i Y_i'$, we have that $|N(y_1) \cap \ldots N(y_r)| \geq \sqrt{|X|}$.*

Proof. Sample $x_1, \ldots, x_h \in X$ uniformly at random with repetitions, and set $Y' := \bigcap_{i=1}^h N(x_i)$. Let $\mathcal{E}_{\text{good}}$ be the event $|Y_i \cap Y'| \geq \frac{p^{h\ell}}{2}|Y_i|$ for all $i \in [\ell]$, and $\mathcal{E}_{\text{bad}}$ be the event that $|\cap_{i=1}^r N(y_i)| < |X|^{1/2}$ for some choice of $y_1, \ldots, y_r \in Y'$. We prove that $\mathbb{P}(\mathcal{E}_{\text{good}}) > \mathbb{P}(\mathcal{E}_{\text{bad}})$, which implies the existence of an outcome of Y' satisfying only our good event. Taking $Y_i' := |Y_i \cap Y'|$ for $i \in [\ell]$ gives the desired result.

First, we note that $\mathbb{P}(\mathcal{E}_{\text{bad}}) \leq |Y|^r|X|^{-h/2}$. Indeed, for any fixed choice of $y_1, \ldots, y_r \in Y$, we have that $\mathbb{P}(\{y_1, \ldots, y_r\} \subset Y') = (|\cap_{i=1}^r N(y_i)|/|X|)^h$. In particular, when $|\cap_{i=1}^r N(y_i)| < |X|^{1/2}$, this probability is at most $|X|^{-h/2}$. Taking a union bound over all such tuples gives $\mathbb{P}(\mathcal{E}_{\text{bad}}) \leq |Y|^r|X|^{-h/2}$.

We now estimate the probability of the good event. Define an auxiliary bipartite graph $\tilde{\Gamma}$ with parts X and $\tilde{Y} := Y_1 \times \cdots \times Y_\ell$. For $x \in X$, we say $x \sim_{\tilde{\Gamma}} (y_1, \ldots, y_\ell)$ if $\{y_1, \ldots, y_\ell\} \subset N_\Gamma(x)$.

Let $\tilde{Y}' := N_{\tilde{\Gamma}}(x_1) \cap \cdots \cap N_{\tilde{\Gamma}}(x_h)$, where these x_i come from the same random tuple which determined Y'. Noting that $|\tilde{Y}'| = \prod_{i=1}^\ell |Y_i \cap Y'|$, we see that

$$\frac{|\tilde{Y}'|}{|\tilde{Y}|} = \prod_{i=1}^{\ell} \frac{|Y_i \cap Y'|}{|Y_i|} \leq \min_{i \in [\ell]} \left\{\frac{|Y_i \cap Y'|}{|Y_i|}\right\}.$$

Thus the probability $\mathcal{E}_{\text{good}}$ holds is at least $\mathbb{P}(|\tilde{Y}'| \geq \frac{p^{h\ell}}{2}|\tilde{Y}|)$.

By the minimum degree assumptions, $d_{\tilde{\Gamma}}(x) = \prod_{i=1}^\ell |N_\Gamma(x) \cap Y_i| \geq \prod_{i=1}^\ell p|Y_i| = p^\ell|\tilde{Y}|$ for every $x \in X$. It follows that $\tilde{\Gamma}$ has density at least p^ℓ. Applying Lemma 2.4, we have that $\mathbb{P}(|\tilde{Y}'| \geq \frac{p^{h\ell}}{2}|\tilde{Y}|) \geq \frac{p^{h\ell}}{2}$. Using this, together with the assumption $p^{h\ell}/2 > |Y|^r|X|^{-h/2}$, gives $\mathbb{P}(\mathcal{E}_{\text{good}}) > \mathbb{P}(\mathcal{E}_{\text{bad}})$, completing the proof. □

3 Induced embedding

In this section, we prove two "induced embedding results" for pseudorandom blowups of graphs. The idea behind the first result is to employ a greedy embedding procedure, using pseudorandomness (see Property 2 below) to control our non-edges.

Proposition 3.1 (Induced embedding with regularity). *Let $H \subset G$ be graphs, with $\Delta(H) \le k$ and $\Delta(G) \le \Delta$. Let s^*, L, L', p, ρ be constants so that*

$$s^*(\rho/2)^k(1-2p)^\Delta > \Delta L + kL'. \tag{3.1}$$

Now suppose there are graphs $H^* \subset G^*$, along with a homomorphism $\phi : G^* \to G$, so that writing $X_v := \phi^{-1}(v)$ for $v \in V(G)$, we have:

1. *$|X_v| \ge s^*$ for $v \in V(G)$;*
2. *$G^*[X_u, X_v]$ is (L,p)-regular for $uv \in E(G)$;*
3. *$H^*[X_u, X_v]$ is (L', ρ)-lower-regular for $uv \in H$.*

Then, we can find a set of vertices W so that $H^*[W] \cong H \cong G^*[W]$.

Proof of Proposition 3.1. Write $h := |V(H)|$. Let $v_1, \dots, v_h$ be any ordering of $V(H)$. For $i \in [h]$, let $J_i := \{j < i : v_j \in N_H(v_i)\}, \overline{J}_i := \{j' < i : v_{j'} \in N_G(v_i) \setminus N_H(v_i)\}$. For $t = 0, \dots, h$, let $H^{(t)} := H[\{v_1, \dots, v_t\}]$ denote the subgraph of H induced by the first t vertices of our ordering.

We embed vertices v_i one by one using the following iterative procedure. For every $0 \le t \le h-1$, at the beginning of stage $t+1$ we have already chosen vertices $x_1, \dots, x_t$, and for $i > t$ have sets

$$X_i^{(t)} := X_{v_i} \cap \Big(\bigcap_{j \in [t] \cap J_i} N_{H^*}(x_j) \Big) \setminus \Big(\bigcup_{j' \in [t] \cap \overline{J}_i} N_{G^*}(x_{j'}) \Big)$$

which satisfy the following conditions.

- The map $x_i \mapsto v_i$ is an isomorphism into $H^{(t)}$ for both H^* and G^*,
- and $|X_i^{(t)}| \ge s^*(\rho/2)^{|[t] \cap J_i|}(1-2p)^{|[t] \cap \overline{J}_i|}$ for each $i > t$.

Note that for $t = 0$, this is vacuously satisfied, since $X_i^{(0)} = X_{v_i}$ and has size at least s^* by Property 1. Now, assuming that our assumptions hold at the end of stage $t-1$, we show how to pick $x_t \in X_t^{(t-1)}$ appropriately.

By definition, we have that every $x_t \in X_t^{(t-1)}$ satisfies $x_t \in N_{H^*}(x_j)$ for $j \in J_t$, and $x_t \notin N_{G^*}(x_{j'})$ for $j' \in \overline{J}_t$. We further have $x_t \notin N_{G^*}(x_{j''})$ for $j'' \in [t-1] \setminus (J_t \cup \overline{J}_t)$, since $v_{j''}$ is not adjacent to v_t in G and G^* is homomorphic to G. This shows that our first bullet holds for any choice of $x_t \in X_t^{(t-1)}$.

To ensure that there is a choice of x_t which satisfies our second bullet we use our regularity assumptions. First, by our hypotheses, we have for $i > t-1$ that

$$|X_i^{(t-1)}| \ge s^*(\rho/2)^{|[t] \cap J_i|}(1-2p)^{|[t] \cap \overline{J}_i|} > \Delta L + kL' \ge \max\{L, L'\}. \tag{3.2}$$

This uses the assumption given by Eq. 3.1 togther with $|[t-1] \cap J_i| \le d_H(v_i) \le k$, $|[t-1] \cap \overline{J}_i| \le d_G(v_i) \le \Delta$.

Next, let $I := \{i > t : t \in J_i\}, \overline{I} := \{i > t : t \in \overline{J}_i\}$. For $i \in I$, set B_i to be a set of vertices $x \in X_{v_t}$ which in the graph H^* have less than $\frac{\rho}{2}|X_i^{(t-1)}|$ neighbors in $X_i^{(t-1)}$. Similarly for $i' \in \overline{I}$ define

$$B_{i'} := \big\{ x \in X_{v_t} : |X_{i'}^{(t-1)} \setminus N_{G^*}(x)| < (1-2p)|X_{i'}^{(t-1)}| \big\}.$$

Recalling Eq. 3.2, we have $|X_i^{(t-1)}| \ge L$ for each $i \in I$. Whence, by Property 3 we get that $|B_i| \le L'$. Similarly, by Property 2 we get $|B_{i'}| \le L$ for all $i' \in \overline{I}$.

Using that $|I| \le d_H(v_t) \le k, |\overline{I}| \le d_G(v_t) \le \Delta$ together with Eq. 3.2, we deduce $|(\cup_{i\in I} B_i) \cup (\cup_{i'\in \overline{I}} B_{i'})| \le \Delta L + kL' < |X_t^{(t-1)}|$. Therefore, there exists a choice of $x_t \in X_t^{(t-1)}$ not belonging to any of the sets $B_i, B_{i'}$. Picking this x_t gives the desired lower bounds for each $|X_i^{(t)}|, i > t$ and allows us to continue the next embedding iteration. □

Next, we present another induced embedding result. Compared to Proposition 3.1, where we assumed that $H^*[X_u, X_v]$ was lower-regular between parts, we now have a weaker assumption (Property 3 below) that can be obtained using dependent random choice. Consequently, we can no longer greedily embed vertices one at a time in our analysis. Instead, we shall utilize the Lovász Local Lemma (see Eq. 2.2). Here we will be embedding w vertices into each blob of the host graph, to find induced copies of w-blowups H' of H. This does not pose any challenges beyond making our notation slightly more cumbersome. Nothing is lost in the proof by assuming $w = 1$ and replacing instances of '(a, j)', '(b, i)' respectively by 'a', 'b'.

Proposition 3.2. *Let $H \subset G$ be bipartite graphs with a common vertex bipartition (A, B); assume that $\Delta(H) = k, \Delta(G) = \Delta$. Finally consider some w-blowup H' of H.*

Now let $H^* \subset G^*$ be graphs, with a homomorphism $\phi : G^* \to G$, so that writing $X_a := \phi^{-1}(a)$ for $a \in A$ and $Y_b := \phi^{-1}(b)$ for $b \in B$, we have:

1. *$|Y_b| \ge ws^*$ for $b \in B$;*
2. *$G^*[X_a, Y_b]$ is (L, p)-regular for $ab \in E(G)$;*
3. *given any $a \in A$, $b_1, \dots, b_{wk} \in N_H(a)$ and $y_i \in Y_{b_i}$ for $i \in [wk]$, we have that*

$$(1 - 2p)^{\Delta w} \Big|X_a \cap \bigcap_{i=1}^{wk} N_{H^*}(y_i)\Big| \ge wL.$$

Then, assuming

$$w\Delta \frac{L}{s^*} \le \frac{1}{e(w\Delta^2 + 1)}, \tag{3.3}$$

we can find a set of vertices W so that $H^*[W] \cong H' \cong G^*[W]$.

Proof. We may write $V(H') = V(H) \times [w]$ so that $(v, i) \mapsto v$ is a homomorphism from H' to H. Next, for $b \in B$, we shall fix w disjoint sets $Y_{b,1} \dots, Y_{b,w} \subset Y_b$ each of size s^* (which is possible by Property 1).

For $(b, i) \in B \times [w]$, we shall pick a random vertex $y_{b,i} \in Y_{b,i}$ (uniformly at random, and independently). For $(a, j) \in A \times [w]$, we let $T_{a,j} \subset X_a$ be the set of "valid images" for (a, j), i.e.,

$$T_{a,j} := X_a \cap \left(\bigcap_{(b,i)\in N_{H'}(a,j)} N_{H^*}(y_{b,i}) \setminus \bigcup_{\substack{(b',i'):b'\in N_G(a),\\ (b',i')\notin N_{H'}(a,j)}} N_{G^*}(y_{b',i'}) \right).$$

We let $\mathcal{E}_{a,j}$ be the "bad" event that $|T_{a,j}| < w$. Note that if there is an outcome where none of the $\mathcal{E}_{a,j}$ hold, then we can greedily pick distinct vertices $x_{a,j}$ for $(a, j) \in A \times [w]$ so that $x_{a,j} \in T_{a,j}$. Indeed, we will pick w vertices $x_{a,1}, \dots, x_{a,w}$ inside X_a, and after embedding the first $t < w$ of them, we still have $w - t > 0$ choices for $x_{a,t+1}$ inside $T_{a,t+1} \setminus \{x_{a,1}, \dots, x_{a,t}\}$. The map $(a, j) \mapsto x_{a,j}, (b, i) \mapsto y_{b,i}$ shall then produce an induced copy of H. This follows from the definition of the sets $T_{a,j}$, and the fact that there are no edges between vertices in $\bigcup_{a\in A} X_a$ and $\bigcup_{b\in B} Y_b$. So it suffices to find such an outcome, which we do using the Lovász Local Lemma.

Observe that the event $\mathcal{E}_{a,j}$ is mutually independent from all events $\mathcal{E}_{a',j'}$ for which $N_G(a) \cap N_G(a') = \emptyset$ (in which case a, a' are adjacent in the square graph of G, which which has maximum degree at most Δ^2). It follows that our dependency graph has maximum degree at most $D := w\Delta^2$. So by Eq. 2.2, we are done if we can prove $\mathbb{P}(\mathcal{E}_{a,j}) \le \frac{1}{e(w\Delta^2+1)}$.

This will follow from Eq. 3.3 together with the following claim.

Claim 3.3. *For $(a,j) \in A \times [w]$, we have that $\mathbb{P}(\mathcal{E}_{a,j}) \le w\Delta\frac{L}{s^*}$.*

Proof. Write

$$T^{(0)} = T^{(0)}_{a,j} := X_a \cap \Big(\bigcap_{(b,i)\in N_{H'}(a,j)} N_{H^*}(y_{b,i})\Big).$$

Since $|N_{H'}(a,j)| \le w|N_H(a)| \le w\Delta(H) \le wk$ (by assumption, recalling H' is a w-blowup of H), we have by Property 3 that $(1-2p)^{w\Delta}|T^{(0)}| \ge wL$.

We now consider the set $\overline{N} := \{(b,i) \in B \times [w] : b \in N_G(a) \text{ but } (b,i) \notin N_{H'}(a,j)\}$ of potential non-neighbors. By assumption, $\ell := |\overline{N}| \le wd_G(a) \le w\Delta$. We fix some arbitrary ordering $(b_1,i_1),\dots,(b_\ell,i_\ell)$ of these vertices. For $t = 1,\dots,\ell$, let

$$T^{(i)} := T^{(i-1)} \setminus N_{G^*}(y_{b_t,i_t}).$$

We wish to show that $\mathbb{P}(|T^{(\ell)}| < w) \le w\Delta\frac{L}{s^*}$. The calculation is similar to Proposition 3.1.

As noted before, we deterministically have $|T^{(0)}| \ge (1-2p)^{-w\Delta}wL \ge (1-2p)^{-\ell}wL$. For $t = 1,\dots,\ell$, we now let $\mathcal{E}^{(t)}$ be the event that $|T^{(t-1)}| \ge L$, but $|T^{(t)}| < (1-2p)|T^{(t-1)}|$. So if none of the events $\mathcal{E}^{(t)}$ happens, we get $|T^{(\ell)}| \ge (1-2p)^\ell|T^{(0)}| \ge wL \ge w$, as desired.

To bound the probability that $\mathcal{E}^{(t)}$ holds for some $t \in [\ell]$, we show that for every t, $\mathbb{P}(\mathcal{E}^{(t)}) < \frac{L}{s^*}$. Indeed, conditioned on $|T^{(t-1)}| \ge L$, Property 2 tells us there are less than L "bad" vertices $y \in Y_{b_t,i_t}$ where $|N_{G^*}(y) \cap T^{(t-1)}| > 2p|T^{(t-1)}|$. Moreover, in order for the event $\mathcal{E}^{(t)}$ to hold we neeed that the random vertex y_{b_t,i_t}, which is chosen from the set Y_{b_t,i_t} of size s^*, is bad. This happens with probability $\mathbb{P}(\mathcal{E}^{(t)}) < \frac{L}{s^*}$. Recalling $\ell \le w\Delta$, a union bound gives $\mathbb{P}\big(\mathcal{E}^{(t)} \text{ holds for some } t \in [\ell]\big) \le \ell\frac{L}{s^*} \le w\Delta\frac{L}{s^*}$, as desired. □

□

4 Cleaning results

In this section, we prove our cleaning results which correspond to Step 2 of our strategy from Subsection 1.1. Recall that in Section 3, our embedding results had three kinds of assumptions; a largeness assumption (Property 1), a sparsity assumption (Property 2), and some type of local embedding assumption (Property 3). Our cleaning results are about how given an edge coloring C' of a pseudo-random blowup G' of G, we can shrink the vertex sets of G' slightly and define an auxiliary coloring C of $E(G)$ so that we have this local embedding assumption in appropriate monochromatic subgraphs of G'.

The first cleaning result is a quantitative version of a lemma proved by Conlon, Nenadov, and Trujic [8, Lemma 2.3].

Proposition 4.1. *Fix $q,\Delta \ge 1$ and $p,\eta > 0$. There exists a $\lambda = \lambda(q,\Delta,p,\eta) > 0$ so that the following holds. Let G be a graph with maximum degree Δ and let G' be a graph with a homomorphism $\phi : G' \to G$. Defining $X_v := \phi^{-1}(v)$ for each $v \in V(G)$, suppose that $|X_v| = s$ and $G'[X_u,X_v]$ is $(\lambda s,p)$-lower-regular for each $uv \in E(G)$. Then for any q-coloring C' of $E(G')$, we can find subsets $X^*_v \subset X_v$ for $v \in V(G)$ and a q-coloring C of $E(G)$, so that:*

1. *$|X^*_v| = s^* := \lambda|X_v|$ for $v \in V(G)$;*
2. *$G'_{C(uv)}[X^*_u,X^*_v]$ is $(\eta s^*,p/4q)$-lower-regular (where G'_i denotes the subgraph of edges in G' receiving color i).*

Moreover,

$$\lambda(q,\Delta,p,\eta) \geq (p/2q)^{14/(p/2q)^{14/(p/2q)^{\cdot^{\cdot^{\cdot^{14/\eta}}}}}},$$

i.e., has a tower-type dependence with $\Delta+1$ occurences of $p/2q$.

To get this quantitative bound, we recall a result of [28].

Theorem 4. *Consider a bipartite graph $G=(X,Y,E)$ with $|X|=|Y|:=n$ and density $p:=\frac{|E|}{|X||Y|}$, along with $\epsilon>0$. We can find $X'\subset X, Y'\subset Y$ of size $n':=\frac{1}{2}np^{12/\epsilon}$ so that $G[X',Y']$ is $(\epsilon n', p/2)$-lower-regular.*

This immediately gives the cleaning result for matchings, which we iterate.

Lemma 4.2. *We have $\lambda(q,1,p,\eta)\geq \frac{1}{2}(p/2q)^{12/\eta}\geq (p/2q)^{13/\eta}$.*

Proof. When $\Delta(G)=1$, its edges form a matching. For each edge $e=uv\in E(G)$, we can use lower-regularity and pigeonhole principle to find some color c_e with $G'_{c_e}[X_u,X_v]$ having density at least $p/2q$. Then for each edge e we assign $C(e):=c_e$ and apply Theorem 4 (with $\epsilon=\eta$) to get sets X_v^* (for vertices v that are not covered by the matching we can take $X_v^*\subset X_v$ arbitrarily). □

Sketch of Proposition 4.1. Set $\epsilon_0:=\eta$ and $\lambda_0:=(p/2q)^{13/\epsilon_0}\leq \frac{1}{2}(p/2q)^{12/\epsilon_0}$, and for $i=1,\dots,\Delta$ set $\epsilon_i:=\epsilon_{i-1}\lambda_{i-1}$ and $\lambda_i:=(p/2q)^{13/\epsilon_i}$. For $t=0,\dots,\Delta$, write $\lambda^{\leq t}:=\prod_{i=0}^{t}\lambda_i$; inductively we get that $\epsilon_t=\eta\lambda^{\leq t-1}$. We shall show that one can take $\lambda(p,\Delta,q,\eta)\geq\lambda^{\leq\Delta}$.

Indeed, to do this, we first apply Vizing's theorem to partition the edges of G into $\Delta+1$ matchings $M_\Delta, M_{\Delta-1},\dots,M_0$ (since G has maximum degree Δ). Now fix some q-coloring of the edges of G'. We shall find our sets X_v^* in $\Delta+1$ stages.

We initialize with $X_v^{(\Delta+1)}:=X_v$ for each $v\in V(G)$. For $t=\Delta,\dots,0$, write $s_t:=(\prod_{i=t}^{\Delta}\lambda_t)s$. The idea is that at stage $t=\Delta,\dots,0$, we can pass to a subset $X_v^{(t)}\subset X_v^{(t+1)}$ of size $\lambda_t|X_v^{(t+1)}|=s_t$, so that for each $e=uv\in M_t$ and some choice of $C(e)$, we have $G'_{C(e)}[X_u^{(t)},X_v^{(t)}]$ is $(\epsilon_t s_t, p/4q)$-lower-regular. This is done by applying Lemma 4.2 (with $\eta=\epsilon_t$).

To finish, since $\epsilon_t=\eta\lambda^{\leq t-1}$, we get that $\epsilon_t s_t=\eta\prod_{i=0}^{\Delta}\lambda_i s=\eta s_0$. Whence, at the end of the process, we will have a collection of subsets $X_v^{(0)}\subset X_v^{(1)}\subset\cdots\subset X_v^{(\Delta+1)}$, so that for any edge $e\in M_t$, $G'_{C(e)}[X_u^{(t)},X_v^{(t)}]$ (and thus $G'_{C(e)}[X_u^{(0)},X_v^{(0)}]$) is $(\eta s_0,p/4q)$-lower-regular and $X_v^{(0)}$ has size s_0. Taking $X_v^*:=X_v^{(0)}$ for $v\in V(G)$ and the coloring C will then complete the proof (with $s^*=s_0$).

It is not hard to see that $\lambda^{\leq\Delta}=\lambda^{\leq\Delta-1}\lambda_\Delta\geq\epsilon_\Delta\lambda_\Delta$. Using that $\epsilon_t\geq(1/2)^{1/\epsilon_t}>(p/2q)^{1/\epsilon_t}$, one then gets the tower-type bound by recursively noting that $\epsilon_t\lambda_t\geq(p/2q)^{1/\epsilon_t}\cdot(p/2q)^{13/\epsilon_t}=(p/2q)^{14/\epsilon_t}=(p/2q)^{14/(\epsilon_{t-1}\lambda_{t-1})}$. □

We now move on to our more efficient cleaning process, which we prove for bipartite graphs.

Proposition 4.3. *Let $G=(A,B,E)$ be a bipartite graph with maximum degree Δ. Let G' be a graph with a homomorphism $\phi:V(G')\to V(G)$, and define $X_a:=\phi^{-1}(a)$ for $a\in A$ and $Y_b:=\phi^{-1}(b)$ for $b\in B$. For integers $r,q,L\geq 2$, $p\in(0,1)$ and all $a\in A, b\in B$, suppose that $|Y_b|=s_0\geq L(2q/p)^{5\Delta^2 r}$ and $|X_a|=s\geq 4\,(2q/p)^{5\Delta}\,\Delta s_0$. Also suppose that $G'[X_a,Y_b]$ is (L,p)-regular for $ab\in E(G)$. Then given any q-edge-coloring χ' of G', we can produce an q-edge-coloring χ of G and subsets $Y_b^*\subset Y_b$ for $b\in B$ with $|Y_b^*|\geq\left(\frac{p}{2q}\right)^{5\Delta^2 r}|Y_b|$, so that for all choices of $a\in A$, $b_1,\dots,b_r\in N_G(a)$, and $y_i\in Y_{b_i}^*$ for $i\in[r]$, we have that*

$$|\{x\in X_a:\chi'(xy_i)=\chi(ab_i) \text{ for each } i=1,\dots,r\}|\geq\sqrt{|X_a|/2q^\Delta}.$$

For our applications of finding w-blowups of H, it is important that in this statement $b_1, \dots, b_r$ might not all be distinct.

While Proposition 4.1 worked by iterating upon matchings (cf. Lemma 4.2), we now shall instead iterate upon stars.

Lemma 4.4. *Consider sets X and $Y'_1, \dots, Y'_\ell$ in a graph G' so that all $|Y'_i| \geq L$, $|X| \geq 2\ell L$ and $|X| \geq (2q^\ell)(2(\frac{2q}{p})^{4\ell}) \sum_i |Y'_i|$. Also suppose that all $G'[X, Y'_i]$ are (L,p)-regular. Then given any q-edge-coloring χ' of G', we can find for each $i \in [\ell]$, a choice of $c_i \in [q]$ and $Y''_i \subset Y'_i$ with $|Y''_i| \geq \frac{1}{2}(\frac{p}{2q})^{4r\ell}|Y'_i|$ so that for all choices of $i_1, \dots, i_r \in [\ell]$, and $y_j \in Y''_{i_j}$ for $j \in [r]$, we have that*

$$|\{x \in X : \chi'(xy_j) = c_{i_j} \text{ for each } j = 1, \dots, r\}| \geq \sqrt{|X|/2q^\ell}.$$

Proof of Proposition 4.3 assuming Lemma 4.4. Enumerate the vertices of A as $a_1, \dots, a_{|A|}$ in some arbitrary order. For $t = 0, \dots, |A|$, let $G_t := G[\{a_1, \dots, a_t\}, B]$ denote the induced subgraph between the first t vertices in our ordering of A and B. Set $\delta := \left(\frac{p}{2q}\right)^{5\Delta r} < \frac{1}{2}\left(\frac{p}{2q}\right)^{4\Delta r}$.

For $t = 0, \dots, |A|$, we will inductively construct a q-coloring χ_t of $E(G_t)$ and subsets $Y'_{b,t} \subset Y_b$ for $b \in B$ satisfying the following properties. All $|Y'_{b,t}| \geq \delta^{d_{G_t}(b)} \cdot s_0$ and for any $a \in \{a_1, \dots, a_t\}$, $b_1, \dots, b_r \in N(a)$, and choices $y_i \in Y'_{b_i,t}$ for $1 \leq i \leq r$, we have that

$$|\{x \in X_a : \chi'(xy_i) = \chi_t(ab_i) \text{ for each } i = 1, \dots, r\}| \geq \sqrt{|X_a|/2q^\Delta}. \tag{4.1}$$

Clearly, taking $t = |A|$ will give our result, since $d_{G_{|A|}}(b) = d_G(b) \leq \Delta$ for all $b \in B$.

At time $t = 0$, take $Y'_{b,0} := Y_b$ for all $b \in B$. Then at time $t > 0$, assuming the conditions are satisfied, do the following. We shall find subsets $Y''_b \subset Y'_{b,t-1}$ with $|Y''_b| \geq \frac{1}{2}(p/2q)^{4\Delta r}|Y'_{b,t-1}| \geq \delta|Y'_{b,t-1}|$ and colors $c_b \in [q]$ for $b \in N(a_t)$, so that for any choice of $b_1, \dots, b_r \in N(a_t)$, and $y_i \in Y''_{b_i}$ we have that

$$|\{x \in X_{a_t} : \chi'(xy_i) = c_{b_i} \text{ for each } i = 1, \dots, r\}| \geq \sqrt{|X_a|/2q^\Delta}.$$

Assigning $Y'_{b,t} := Y''_b$ for $b \in N(a_t)$ and $Y'_{b,t} = Y'_{b,t-1}$ otherwise, and assigning $\chi_t(a_t b) = c_b$ for $b \in N(a_t)$ (and $\chi_t(e) = \chi_{t-1}(e)$ otherwise) shall allow us to continue the induction.

Indeed, one easily checks that the bound on the size of the sets $Y'_{b,t}$ stay satisfied by induction. Meanwhile, by definition of the Y''_b and c_b, the inequality Eq. 4.1 shall be satisfied when $a = a_t$. Lastly, for $a = a_{t'}$ for $t' < t$, the inequality Eq. 4.1 follows from the inductive hypothesis, since shrinking the sets $Y'_{b,t-1}$ cannot invalidate this condition.

It remains to find the Y''_b and c_b. We will enumerate $N(a_t)$ as $b_1, \dots, b_\ell$ for some $\ell \leq \Delta$. We then simply apply Lemma 4.4 (with $X = X_{a_t}, Y'_i := Y'_{b_i,t-1}$). This can be done since $|Y'_{b,t-1}| \geq \delta^\Delta s_0 = \left(\frac{p}{2q}\right)^{5\Delta^2 r} s_0 \geq L$ for each b, and additionally we have

$$\frac{|X_{a_t}|}{\sum_i |Y_{b_i,t-1}|} \geq \frac{s}{\Delta s_0} \geq 4\,(2q/p)^{5\Delta} \geq (2q^\Delta)(2(2q/p)^{4\Delta}).$$

□

It remains to prove the star cleaning statement. We start with a simple lemma, which follows directly from the definition of regularity.

Lemma 4.5. *Let X and $Y_1, \dots, Y_\ell$ be subsets of G' such that all $|Y_i| \geq L$ for $i \in [\ell]$, $|X| \geq 2\ell L$ and all $G'[X, Y_i]$ are (L,p)-regular. Then for any q-edge-coloring of G', we can find $X^* \subset X$ of size $|X^*| \geq \frac{|X|}{2q^\ell}$ and choices of $c_1, \dots, c_\ell \in [q]$ so that for $x \in X^*, i \in [\ell]$, $|N_{c_i}(x) \cap Y_i| \geq \frac{p}{2q}|Y_i|$.*

Proof. For $i = 1, \ldots, \ell$, let $\tilde{X}_i$ be the set of $x \in X$ where $|N_{G'}(x) \cap Y_i| < \frac{p}{2}|Y_i|$. Using that $G'[X, Y_i]$ is (L, p)-regular together with $|Y_i| \geq L$ we have that $\tilde{X}_i \leq L$ for each $i \in [\ell]$. Setting $X' := X \setminus \bigcup_{i=1}^{t} \tilde{X}_i$, implies $|X'| \geq |X| - \ell L \geq \frac{1}{2}|X|$.

Next, for $x \in X'$ and $i \in [\ell]$, let $c_{i,x}$ be the color class that maximizes $|N_{c_{i,x}}(x) \cap Y_i|$. This implies that $|N_{c_i^*}(x) \cap Y_i| \geq (p/2q)|Y_i|$ for each $x \in X'', i \in [\ell]$. By pigeonhole, there must be some tuple of colors $\vec{c}^* \in [q]^\ell$ so that the set

$$X'' := \{x \in X' : c_{i,x} = c_i^* \text{ for all } i \in [\ell]\},$$

satisfies $|X''| \geq |X'|/q^\ell$. Thus, taking $X^* = X''$ and $c_i = c_i^*$ for $i \in [\ell]$ completes the proof. □

Proof of Lemma 4.4. Fix any q-coloring χ' of $E(G')$. Using Lemma 4.5, we can pass to some $X^* \subset X$ of size $|X^*| \geq |X|/(2q^\ell) \geq 2(2q/p)^{4\ell} \sum_i |Y_i'|$, along with colors $c_1, \ldots, c_\ell$ so that $|N_{c_i}(x) \cap Y_i'| \geq \frac{p}{2q}|Y_i'|$ for each $i \in [\ell]$ and $x \in X^*$.

Setting $h := 4r$, we have that

$$(\sum_i |Y_i'|)^r |X^*|^{-h/2} \leq |X^*|^{-r} < \frac{1}{2}(p/2q)^{4r\ell} = \frac{1}{2}(p/2q)^{h\ell},$$

where we used that $|X^*| \geq 2(2q/p)^{4\ell}$ for the last step.

Let Γ be the bipartite graph with vertex sets X^* and $\overline{Y} := \bigsqcup_i Y_i'$, where (for $x \in X^*, y_i \in Y_i'$) we have $x \sim_\Gamma y_i$ if $\chi'(xy_i) = c_i$. Applying Lemma 2.5 to Γ with $h = 4r$, gives subsets $Y_i'' \subset Y_i'$ of size $|Y_i''| \geq \frac{1}{2}(p/2q)^{h\ell}|Y_i'|$ which, by recalling the definition of Γ and that $|X^*| \geq |X|/(2q^\ell)$, have the desired common neighborhood property. □

5 Proof of main theorems

5.1 The two reductions

Proof of Theorem 3. Suppose we have been given some host graph G so that $G \to (H)_q$. Recall $\Delta(H) \leq k$ and $\Delta(G) \leq \Delta$. Set $p := \frac{1}{100}, \rho := \frac{p}{4q}$ and let $\eta := \frac{(\rho/2)^k (1-2p)^\Delta}{\Delta + k}$. Take $\lambda := \lambda(q, \Delta, p, \eta)$ which is defined in Lemma 4.1.

By Proposition 2.1, we can find some bipartite graph Γ with parts of size $s = C(\lambda\eta)^{-2}$ for some absolute constant C which is $\left(\frac{48}{p}\ln(2s), p\right)$-regular. By choosing C large enough we can make $\frac{48}{p}\ln(2s) \leq s^{1/2} \leq \lambda\eta s$. Therefore Γ is $(\lambda\eta s, p)$-regular and thus obviously $(\lambda s, p)$-lower-regular. Take $s^* := \lambda s$, and note Γ is $(\eta s^*, p)$-regular.

Define G' to have (disjoint) vertex sets X_v of size s for $v \in V(G)$. For every $uv \in E(G)$, we add a copy of Γ between the sets X_u and X_v (i.e., $G'[X_u, X_v]$ is isomorphic to Γ). We do not add any other edges to G', thus G' is an s-blowup of G. Now consider any q-coloring χ' of $E(G')$. Since $G'[X_u, X_v] \cong \Gamma$ is $(\lambda s, p)$-lower-regular for $uv \in E(G)$, we may apply Lemma 4.1 to pass to subsets $X_v^* \subset X_v$ each of size $s^* = \lambda s$ and get an auxiliary q-coloring χ of $E(G)$ so that $G'_{\chi(uv)}[X_u^*, X_v^*]$ is $(\eta s^*, \rho)$-lower-regular for all $uv \in E(G)$ (recall that given a color $c \in [q]$, we write G_c' to denote the c-monochromatic subgraph of G' in the coloring χ').

Since $G \to (H)_q$, we can find some monochromatic copy of H inside χ, say in color c. Let $f : V(H) \to V(G)$ denote a c-monochromatic copy of H. We now intend to apply our embedding procedure (Proposition 3.1). We define H^* to be the graph on vertex set $\bigcup_{v \in V(H)} X_{f(v)}^*$, with the edges $\bigcup_{uv \in E(H)} G_c'[X_{f(u)}^*, X_{f(v)}^*]$.

Let $G^* = G'[\bigcup_v X_v^*]$ and without loss of generality assume that $f(v) = v$ for $v \in V(H)$. Then, we have $|X_v^*| = s^*$ for $v \in V(H)$, and $G^*[X_u^*, X_v^*]$ is $(\eta s^*, p)$-regular for all $uv \in E(G)$. We also have $H^*[X_u^*, X_v^*]$ is $(\eta s^*, \rho)$-lower-regular for $uv \in E(H)$. Taking $L = L' = \eta s^*$ and using the definition of η, we can verify that $s^* \frac{(\rho/2)^k(1-2p)^\Delta}{\Delta+k} = \eta s^* = L$, i.e., Eq. 3.1 holds. Thus we may apply Proposition 3.1 to find an induced copy of H in H^*. This copy corresponds to a monochromatic induced copy of H within G', completing the proof. □

We now establish our second reduction, whose proof is rather similar to the above arguments.

Theorem 5. *There exists an absolute constant C so that the following holds. Let H, G be bipartite graphs with $G \to (H)_q$. Suppose $\Delta(H) \le k, \Delta(G) \le \Delta$ and let H' be some w-blowup of H. Then, for $s := (\Delta q)^{Ck\Delta^2 w}$, there is an s-blowup G' of G, so that $G' \to_{ind} (H')_q$. Consequently,*

$$r_{ind}(H'; q) \le s|V(G)|,$$

and

$$\hat{r}_{ind}(H'; q) \le s^2 e(G) \le s^2 \Delta |V(G)|.$$

Proof. Let $G = (A, B, E)$ be a bipartite graph so that $G \to (H)_q$ and $\Delta(H) \le k, \Delta(G) \le \Delta$. Note that we must have $k \le \Delta$, since H is a subgraph of G. Set $p := \frac{1}{4\Delta}$, and let $s := (\frac{2q}{p})^{C\Delta^2 kw}$. In the rest of the proof we assume that the constant C is sufficiently large so that all the inequalities which we will use are satisfied. Fix $s_0 := s^{1/3}$ and use Corollary 2.1, to find some (L, p)-regular graph Γ with parts of size s, s_0 and $L = \frac{48}{p}\ln(2s) \le 200\Delta \ln(2s) \le s^{1/9} = s_0^{1/3}$. Form a graph G' by fixing a vertex set X_a of size s for all $a \in A$, and Y_b of size s_0 for all $b \in B$. For $e = (a, b) \in E(G)$, add a copy of Γ between X_a, Y_b.

We claim that the graph G' satisfies the assertion of the theorem. Suppose we are given a q-edge-coloring χ' of G'. Use Proposition 4.3 with $r := kw$ and s, s_0 defined above. Note that since the constant C is large the conditions of this proposition are satisfied. Indeed $L(2q/p)^{5\Delta^2 r} \le s_0^{1/3}(2q/p)^{5\Delta^2 kw} \le s_0 = |Y_b|$ and $4\,(2q/p)^{5\Delta}\,\Delta s_0 \le (2q/p)^{6\Delta^2 kw} s^{1/3} \le s = |X_a|$ for all $b \in B$ and $a \in A$. Therefore, by Proposition 4.3, there is some q-edge-coloring χ of G and subsets $Y_b^* \subset Y_b$ of size $\left(\frac{p}{2q}\right)^{5\Delta^2 kw} s_0 \ge w s_0^{1/2}$, so that for any $a \in A$, $b_1, \dots, b_{kw} \in N(a)$ and $y_i \in Y_{b_i}$ for $i \in [kw]$:

$$|\{x \in X_a : \chi(xy_i) = \chi'(ab_i) \text{ for all } i \in [kw]\}| \ge \sqrt{|X_a|/2q^\Delta} \ge s^{1/2}/q^\Delta \ge s^{1/4}.$$

Since $G \to (H)_q$ we can find a monochromatic copy of H inside the q-edge-coloring χ of G, say in color c. Consider the induced c-monochromatic subgraph H^* of G'_c whose vertex set is the union of sets X_a, Y_b^* for a, b that correspond to the vertices of the monochromatic copy of H which we found in χ. Let $s^* := s_0^{1/2}$. We have that $|Y_b^*| \ge ws^*$, $G'[X_a, Y_b^*]$ is (L, p)-regular for any edge $ab \in E(G)$ and for any $a \in A$, $b_1, \dots, b_{kw} \in N_H(a)$ and $y_i \in Y_{b_i}^*$ we have $\left|X_a \cap \bigcap_{i=1}^{wk} N_{H^*}(y_i)\right| \ge s^{1/4}$. Using that $s = (\frac{2q}{p})^{C\Delta^2 kw}$, $s_0 = s^{1/3}$ and $L \le s^{1/9} = s_0^{1/3}$, it is easy to see that $(1-2p)^{\Delta w} s^{1/4} \ge 2^{-\Delta w} s^{1/4} \ge s^{1/5} \ge wL$ and that $w\Delta \frac{L}{s^*} \le w\Delta s_0^{-1/6} < s_0^{-1/7} < \frac{1}{e(w\Delta^2+1)}$. Therefore we can apply Lemma 3.2 to find an induced monochromatic copy of H' inside G'. □

5.2 Obtaining Theorems 1 and 2.

Definition. Given graph G, H and $\gamma > 0$, we say $G \to_\gamma H$ if for every $\tilde{G} \subset G$ with $e(\tilde{G}) \ge \gamma e(G)$, we have that $\tilde{G}$ contains a copy of H.

By using Friedman-Pippenger embedding techniques, Haxell and Kohayakawa proved (cf. [23, Lemma 6 and the proof of Theorem 9]):

Theorem 6. *There exists an absolute constant C so that the following holds. Let T be an n-vertex tree with maximum degree k and let $\gamma \in (0,1)$. If $N := C\gamma^{-2}n$ and $p := \frac{C\gamma^{-2}k}{N}$, then for a random bipartite graph $G \sim G(N, N, p)$, we have that $\mathbb{P}(G \to_\gamma H) \geq 1/2$.*

Note that the maximum degree of such random graph is not bounded. But the results of [23] are more general and say that any G with appropriate expansion properties will have $G \to_\gamma T$. In particular, we can take G to be a random bipartite $O(\gamma^{-2}k)$-regular graph on $O(\gamma^{-2}n)$ vertices. By taking $\gamma = 1/q$ we have that $G \to (T)_q$ and has maximum degree $O(q^2k)$. Whence Theorem 2 follows from Theorem 5. Alternatively, one can use explicit bounded degree expanders, as described in [23, Section 5], to make G (and the proof of Theorem 2) constructive.

Next, to prove Theorem 2, we recall a very recent result [4, Corollary 2] for size-Ramsey of graphs with bounded degree and bounded tree-width. We actually use a slightly more precise statement, [12, Theorem 3.1], which observes the construction from [4] has bounded maximum degree, rather than just linearly many edges.

Theorem 7. *For every k, w, q there exists a constant $D = D_{k,w,q}$ so that the following holds. Let H be an n-vertex graph with maximum degree $\leq k$ and tree-width $\leq w$. Then there is some G with at most Dn vertices and $\Delta(G) \leq D$, so that $G \to (H)_q$.*

Theorem 2 now can be obtained using the above statement followed by Theorem 3.

Remark 5.1. The statements of Theorems 3 and 5 require the host graph G to have linearly many vertices and bounded maximum degree. A priori, this may sound stronger than having a host graph G with linearly many edges, but these properties are almost equivalent due to a simple general reduction. Indeed, suppose H is an n-vertex graph with maximum degree k and no isolated vertices, and G_0 is a graph with Dn edges so that $G_0 \to (H)_{q+1}$. Then letting $G \subset G_0$ be the graph induced by the non-isolated vertices in G_0 with degree at most $4kD$, one can check that $G \to (H)_q$, $|V(G)| \leq 2Dn$, and obviously $\Delta(G) \leq 4kD$.

6 Conclusion

As noted in the introduction, there exist n-vertex trees T with maximum degree k so that $\hat{r}_{\text{ind}}(T) \geq \hat{r}(T) = \Omega(nk)$. But for any tree T, we know that $r(T) = O(n)$, so one might wonder if we could have the linear bound $r_{\text{ind}}(T) = O(n)$ (where the constant does not depend on the maximum degree k). This is refuted by a result of Fox and Sudakov [17, Theorem 1.7], which says that for every constant C, there exist n-vertex trees T with $r_{\text{ind}}(T) \geq Cn$ for all sufficiently large values of n. However, the argument of [17] uses trees with maximum degree growing with n (they have $k = \Omega(n)$). We are not aware of a counterexample to the claim that $r_{\text{ind}}(T) \leq O(n) + O_k(1)$; which in words would mean there is some absolute constant C, so that for fixed k there are only finitely many n-vertex trees T with maximum degree k violating $r_{\text{ind}}(T) \leq Cn$.

When H is bipartite, one could naturally ask for a density version of our results. Recall that $G \to_\epsilon H$ if for every $\tilde{G} \subset G$ with $e(\tilde{G}) \geq \epsilon e(G)$, we have that H is a subgraph of $\tilde{G}$. The result of Beck [2] proved that for each $\epsilon > 0$, there is a G with $O_\epsilon(n)$ vertices so that $G \to_\epsilon P_n$; likewise the bounds of [19] and [23] for arbitrary bounded degree trees also were density results. We now write $G \to_{\text{ind},\epsilon} H$ if for every $\tilde{G} \subset G$ with $e(\tilde{G}) \geq \epsilon e(G)$, there is some set of vertices S so that $\tilde{G}[S] \cong H \cong G[S]$.

A minor generalization of our proof of Theorem 3 gives the following.

Theorem 8. *Given $\Delta, k \geq 1$ and $\epsilon' > \epsilon > 0$, there exists some s so that the following holds. Let G, H be graphs so that $\Delta(G) \leq \Delta, \Delta(H) \leq k$ and $G \to_\epsilon H$. Then there exists an s-blowup G' of G so that $G' \to_{ind,\epsilon'} H$.*

To construct G', we again replace each $v \in V(G)$ by a set X_v of s vertices and put a pseudorandom graph Γ inside $G'[X_u, X_v]$ for each $uv \in E(G)$. And like before, we shall ultimately use Proposition 3.1 to embed H. But for the cleaning, we must use the full strength of graph regularity, rather than the weaker form stated in Theorem 4. Here is the necessary cleaning result.

Proposition 6.1. *Fix $\Delta \geq 1$, and some $\eta, c > 0$. There exists a $\lambda = \lambda(\Delta, \eta, c)$ so that the following holds. Let G be a graph with maximum degree Δ and let $\tilde{G}$ be an s-blowup of G with $(X_v)_{v \in V(G)}$ being sets of size s that were mapped to the vertices of G. Then there is randomized procedure to choose subsets $X_v^* \subset X_v$ of size $s^* := \lambda s$, so that, for each $e = uv \in E(G)$, we have*

$$\mathbb{P}(\tilde{G}[X_u^*, X_v^*] \text{ is } (\eta s^*, p_{uv})\text{-regular for some } p_{uv} \geq c) \geq \frac{1}{s^2} e(\tilde{G}[X_u, X_v]) - c - \eta.$$

In particular, if $e(\tilde{G}) = \epsilon' s^2 e(G)$, then there is some outcome of $(X_u^*)_{v \in V(G)}$ where there are $\geq (\epsilon' - c - \eta)e(G)$ edges $e = uv \in E(G)$ where $\tilde{G}[X_u^*, X_v^*]$ is $(\eta s^*, c)$-lower-regular.

This implies that, if G' is a pseudorandom s-blowup with $ps^2 e(G)$ edges, and $\tilde{G} \subset G'$ with $e(\tilde{G}) \geq \epsilon' e(G')$, then this proposition yields a "regular" s^*-blowup $G^* \subset \tilde{G}$, where $G^*[X_u^*, X_v^*]$ is lower-regular for at least an $(p\epsilon' - c - \eta)$-fraction of edges $e = uv \in E(G)$. Assuming we took $\epsilon' > \epsilon/p$ and η, c sufficiently small, then by definition of G there should be a copy of H inside G using only these regular edges. Letting $H^* \subset G^*$ be the appropriate subgraph associated with H, we can apply Proposition 3.1 to find an induced copy of H.

Here we sketch how to find the necessary X_v^*. We let $\delta_1, \dots, \delta_{\Delta+1}, \ell_1, \dots, \ell_{\Delta+1}$ be appropriately chosen constants. Firstly, $\delta_{\Delta+1}$ should equal η. Next, ℓ_i should be large with respect to δ_i, c so that we can apply the Regularity Lemma in order to get a "δ_i-regular partition" into ℓ_i equal-sized parts. Lastly we require $\delta_{i-1} < \delta_i/\ell_i$ for $i > 1$. Take $s := s^* \prod_{i=1}^{\Delta+1} \ell_i$ for some s^* sufficiently large. Note that we have $\delta_i \frac{s}{\prod_{j \leq i} \ell_j} \leq \delta_{\Delta+1} s^*$. We run the following process.

Given some $\tilde{G} \subset G'$, split $E(G)$ into matchings $M_1, \dots, M_{\Delta+1}$. For $v \in V(G)$, we write $X_{v,0} := X_v$, and then begin phase $t = 1$. In phase t, we have sets $(X_{v,t-1})_{v \in V(G)}$ of size $s_{t-1} := \frac{s}{\prod_{i<t} \ell_i}$. For every edge $e = uv \in E(M_t)$, by applying Regularity Lemma to the graph $\tilde{G}[X_{u,t-1}, X_{v,t-1}]$ we can get for every $v \in G$ an equipartition $X_{v,t-1} = X_v^{(1)} \cup \dots \cup X_v^{(\ell_t)}$ into ℓ_t parts of size s_t, so that for each $e = uv \in E(M_t)$, there are at most $\delta_t \ell_t^2$ "bad choices" of $i, j \in [\ell_t]^2$. Here we say that the choice is bad if writing $p_e^{(i,j,t)} := \frac{1}{s_t^2} \tilde{G}[X_u^{(i)}, X_v^{(j)}]$, we have $p_e^{(i,j,t)} \geq c$ but $\tilde{G}[X_u^{(i)}, X_v^{(j)}]$ is not $(\delta_t s_t, p_e^{(i,j,t)})$-regular. Then for each $v \in V(G)$, we randomly pick some $i_{v,t} \in [\ell_t]$ and set $X_{v,t} := X_{v,t-1}^{i_{v,t}}$. Repeat this process until phase $\Delta + 1$ completes. Afterwards, we return the vertex sets $(X_{v,\Delta+1})_{v \in V(G)}$ and also write $X_v^* := X_{v,\Delta+1}$ for each $v \in V(G)$.

Given an edge $e \in E(M_t)$, we say that $e = uv$ is *useful*, if in phase t, we picked indices $i := i_{u,t}$ and $j := i_{v,t}$ from $[\ell_t]$, so that $p_e^{(i,j,t)} \geq c$ and $\tilde{G}[X_{u,t}, X_{v,t}]$ is $(\delta_t s_t, p_e^{(i,j,t)})$-regular (meaning the choice i, j was not "bad", in the sense described above). Note that if e is useful, then, since $\delta_t s_t \leq \delta_{\Delta+1} s^*$, we will have that $\tilde{G}[X_u^*, X_v^*]$ is $(\eta s^*, p_e^{(i,j,t)})$-regular. So, it will suffice to show that for each $e = uv \in E(G)$, that $\mathbb{P}(e \text{ is useful}) \geq \frac{1}{s^2} e(\tilde{G}[X_u, X_v]) - c - \eta$. Note that

$$\mathbb{P}(e \text{ is useful}) = \mathbb{P}(e(\tilde{G}[X_{u,t}, X_{v,t}]) \geq cs_t^2) - \mathbb{P}(e \text{ is bad}).$$

By construction of the δ_t-regular partitions of $X_{u,t-1}$ and $X_{v,t-1}$, we have that $\mathbb{P}(e \text{ is bad}) \leq \delta_t \leq \eta$. Hence we just need to prove that $\mathbb{P}(e(\tilde{G}[X_{u,t}, X_{v,t}]) \geq cs_t^2) \geq \frac{1}{s^2} e(\tilde{G}[X_u, X_v]) - c$.

To prove this last bound, consider the sequence of random variables $Z_i := \frac{1}{s_i^2} e(\tilde{G}[X_{u,i}, X_{v,i}])$. From definitions, it is easy to see that $\mathbb{E}[Z_i | Z_{i-1}] = Z_{i-1}$ and $\mathbb{E}[Z_0] = p$. Therefore $\mathbb{E}[Z_i] = p$

for all $i \geq 0$. Since Z_i is $[0,1]$-valued, $p = \mathbb{E}[Z_t] \leq c + \mathbb{P}(Z_t \geq c)$. Rearranging then gives the result.

Acknowledgement. We would like to thank Rajko Nenadov for asking us about density version of our results, and suggesting that some density variant of Proposition 4.1 should be possible. This led us to the proof of Theorem 8. We also thank Nemanja Draganić and Rajko Nenadov for offering some helpful comments on a preliminary version of this manuscript.

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