Induced Ramsey-type theorems

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Definition:

A subset of vertices of a graph G is *homogeneous* if it is either a clique or an independent set.

$hom(G)$ is the size of the largest homogeneous set in G.

Theorem: (*Ramsey-Erdős-Szekeres, Erdős*)

- For every graph G on n vertices, $hom(G) \geq \frac{1}{2} \log n$.
- There is an n-vertex graph G with $hom(G) \leq 2 \log n$.

Definition:

A *Ramsey graph* is a graph G on n vertices with $hom(G) \leq C \log n$.

Theorem: (*Erdős-Szemerédi*)

If an n-vertex graph G has *edge density* $\epsilon < \frac{1}{2}$ (i.e., $\epsilon\binom{n}{2}$ edges), then

$$\hom(G) \geq \frac{c \log n}{\epsilon \log 1/\epsilon}.$$

Definition:

A graph is *k-universal* if it contains every graph on k vertices as induced subgraph.

Theorem: (*Prömel-Rödl*)

If G is an n-vertex graph with $\hom(G) \leq C \log n$ then it is $c \log n$-universal, where c depends on C.

Forbidden induced subgraphs

Definition:

A graph is *H-free* if it does not contain H as an induced subgraph.

Theorem: (*Erdős-Hajnal*)

For each H there is $c(H) > 0$ such that every H-free graph G on n vertices has

$$\hom(G) \geq 2^{c(H)\sqrt{\log n}}.$$

Conjecture: (*Erdős-Hajnal*)

Every H-free graph G on n vertices has

$$\hom(G) \geq n^{c(H)}.$$

Forbidden induced subgraphs

Theorem: (*Rödl*)

For each $\epsilon > 0$ and H there is $\delta = \delta(\epsilon, H) > 0$ such that every H-free graph on n vertices contains an induced subgraph on at least δn vertices with edge density at most ϵ or at least $1 - \epsilon$.

Remarks:

- Demonstrates that H-free graphs are far from having uniform edge distribution.
- Rödl's proof uses Szemerëdi's regularity lemma and therefore gives a very weak bound on $\delta(\epsilon, H)$.

New results

Theorem:

For each $\epsilon > 0$ and k-vertex graph H, every H-free graph on n vertices contains an induced subgraph on at least

$$2^{-ck \log^2 1/\epsilon} n$$

vertices with edge density at most ϵ or at least $1 - \epsilon$.

Corollary:

Every n-vertex graph G which is not k-universal has

$$\hom(G) \geq 2^{c\sqrt{(\log n)/k}} \log n.$$

Remarks:

- Implies results of Erdős-Hajnal and Prömel-Rödl.
- Simple proofs.

Edge distribution in H-free graphs

Theorem: (*Chung-Graham-Wilson*)

For a graph G on n vertices the following properties are equivalent:

- For every subset S of G, $e(S) = \frac{1}{4}|S|^2 + o(n^2)$.
- For every fixed k-vertex graph H, the number of labeled copies of H in G is $(1 + o(1))2^{-\binom{k}{2}}n^k$.

Question: (*Chung-Graham*)

If a graph G on n vertices has much fewer than $2^{-\binom{k}{2}}n^k$ induced copies of some k-vertex graph H, how far is the edge distribution of G from being uniform with density $1/2$?

Theorem: (*Chung-Graham*)

If a graph H on n vertices is not k-universal, then it has a subset S of $n/2$ vertices with $|e(S) - \frac{1}{16}n^2| > 2^{-2k^2+54}n^2$.

Quasirandomness and induced subgraphs

Theorem:

Let $G = (V, E)$ be a graph on n vertices with $(1-\epsilon)2^{-\binom{k}{2}}n^k$ labeled induced copies of a k-vertex graph H. Then there is a subset $S \subset V$ with $|S| = n/2$ and

$$\left|e(S) - \frac{n^2}{16}\right| \geq \epsilon c^{-k} n^2.$$

Remarks:

- It is tight, since for all $n \geq 2^{k/2}$, there is a K_k-free graph on n vertices such that for every subset S of size $n/2$,
$$\left|e(S) - \frac{n^2}{16}\right| < c2^{-k/4} n^2.$$
- Same is true if we replace the $(1-\epsilon)$ factor by $(1+\epsilon)$.
- This answers the original question of Chung and Graham in a very strong sense.

Induced Ramsey numbers

Definition:

The *induced Ramsey number* $r_{\mathrm{ind}}(H)$ of a graph H is the minimum n for which there is a graph G on n vertices such that for every 2-edge-coloring of G, one can find an induced copy of H in G whose edges are monochromatic.

Theorem: (*Deuber; Erdős-Hajnal-Posa; Rödl*)

The induced Ramsey number $r_{\mathrm{ind}}(H)$ exists for each graph H.

Remark:

Early proofs of this theorem gave huge upper bounds on $r_{\mathrm{ind}}(H)$.

Bounds on induced Ramsey numbers

Theorem: (*Kohayakawa-Prömel-Rödl*)

Every graph H on k vertices and chromatic number q has

$$r_{\mathrm{ind}}(H) \le k^{ck \log q}.$$

Theorem: (*Łuczak-Rödl*)

For each Δ there is $c(\Delta)$ such that every k-vertex graph H with maximum degree Δ has

$$r_{\mathrm{ind}}(H) \le k^{c(\Delta)}.$$

Remark:

- The theorems of Łuczak-Rödl and Kohayakawa-Prömel-Rödl are based on complicated random constructions.
- Łuczak and Rödl gave an upper bound on $c(\Delta)$ that grows as a tower of 2's with height proportional to Δ^2.

New result

Definition:

H is d-degenerate if every subgraph of H has minimum degree $\leq d$.

Theorem:

For each d-degenerate graph H on k vertices and chromatic number q,

$$r_{\mathrm{ind}}(H) \leq k^{cd \log q}.$$

Remarks:

- First polynomial upper bound on induced Ramsey numbers for degenerate graphs. Implies earlier results of Łuczak-Rödl and Kohayakawa-Prömel-Rödl.
- Proof shows that pseudo-random graphs (i.e., graphs with random-like edge distribution) have strong induced Ramsey properties. This leads to explicit constructions.