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Asymptotically the List Colouring Constants Are 1

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In this paper we prove the following result about vertex list colourings, which shows that a conjecture of the first author (1999, *J. Graph Theory* **31**, 149–153) is asymptotically correct. Let G be a graph with the sets of lists $S(v)$, satisfying that for every vertex $|S(v)| = (1 + o(1))d$ and for each colour $c \in S(v)$, the number of neighbours of v that have c in their list is at most d. Then there exists a proper colouring of G from these lists.

1. INTRODUCTION

An instance of a *list colouring* problem consists of a graph G and a set of lists $S(v)$ of available colours, one for each vertex. We are asked if there is an *acceptable colouring* of G, that is, a choice for each vertex v of a colour $c(v)$ from $S(v)$ such that if u and v are adjacent then $c(u) \neq c(v)$.

List colourings arise naturally in a number of ways. If we want to complete a partial colouring of a graph using some set of colours to a complete colouring then we have a list colouring problem in the graph induced by the uncoloured vertices where $S(v)$ consists of those colours not appearing on any of the coloured neighbours of v. A famous conjecture due to Dinitz on completing Latin Squares can be phrased in this way. It was recently resolved by Galvin [4], as a consequence of a more general result on list colourings.

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List colourings can also arise from the consideration of more practical problems. For example, assigning frequencies to transmitters so as to avoid interference can be modeled as a graph colouring problem where vertices are transmitters, colours are frequencies, and two transmitters are adjacent if assigning them the same frequency could result in interference. If some transmitter is restricted to some subset of the frequencies, because of hardware capabilities or other practical difficulties, then we have a list colouring problem. For a survey on list colouring problems and results consult [6].

Now, if for every vertex v, we have that the size of $|S(v)|$ is at least $\Delta(G) + 1$ where Δ is the maximum vertex degree of G, then we can obtain an acceptable colouring greedily by simply assigning each vertex in turn a colour which does not yet appear on any of its neighbours. The same argument actually shows that a similar local condition ensures that we can find an acceptable colouring, we need only insist that $|S(v)|$ is at least one more than the degree of v.

We note that if colours occur infrequently, then we can improve on this bound. For example, if every colour appears in at most one $S(v)$, then, as long as each $S(v)$ is non-empty, any choice of a colour $c(v)$ for each v yields an acceptable colouring. Less trivially, if each colour appears at most d times then, provided each list has length at least d, we can find an acceptable colouring in which each colour is used on at most one vertex. That is, we can find a system of distinct representatives (SDR) for the sets $S(v)$. This follows from a classical result of Hall on SDRs.

We want to investigate the possibility that this global condition can also be weakened to obtain a local condition. In [10], Reed conjectured that if each colour appears at most d times in each neighbourhood and each list has at least $d + 1$ elements then an acceptable colouring exists.

Reed [10] proved the weakening of this conjecture where the bound on the list sizes is equal to $2ed$, here e is the base of the natural logarithm. In [5], Haxell, using completely different methods, proved that actually a bound of $2d$ on the lists size suffices. Reed's conjecture has recently been disproved by Bohman and Holzman [2]. They present an example in which each list have $d + 1$ elements and yet there is no acceptable colouring. On the other hand, in this paper we show that Reed's conjecture is true asymptotically, i.e., we obtain the following result.

THEOREM 1.1. *For every constant* $\varepsilon > 0$, *there exist a* d_0 *such that if* $d \geqslant d_0$ *the following is true. If* $G = (V, E)$ *is a graph with the sets of lists* $S(v)$, *one for each vertex* v *of* G, *satisfying that*

(1) *for every vertex* $|S(v)| = \lceil (1 + \varepsilon)d \rceil$ *and*

(2) *for each colour* $c \in S(v)$, *the number of neighbours of* v *that have* c *in their list is at most* d. *Then there exist a proper colouring of* G *from these lists.*

We prove this theorem using the semi-random method (which is also called the Rödl nibble). This method had a great influence on Combinatorics in the last decade (see, e.g., [7]). Many other graph coloring problems also been resolved using this technique. For an overview, including history, we refer the interested reader to the recent book [8].

The rest of this short paper is organized as follows. The next section contains a description of probabilistic tools which we will use to prove Theorem 1.1. The proof of our main result appears in Section 3. Finally, the last section contains some concluding remarks.

2. PROBABILISTIC TOOLS

In this section we describe few results which we will need later. We begin with two classical results in discrete probability. The first one is the symmetric version of the Lovász Local Lemma.

LOVÁSZ LOCAL LEMMA [1]. *Let $A_1, \ldots, A_n$ be events in a probability space. Suppose that each event A_i is mutually independent of a set of all the other events A_j but at most d, and that $Prob[A_i] \leqslant p$ for all i. If $ep(d+1) \leqslant 1$, $(e = 2.71\ldots)$, then $Prob[\wedge \bar{A}_i] > 0$.*

To state the next result we need the following definitions. Let $\Omega = \prod_{i=1}^{n} \Omega_i$ where each Ω_i is a probability space and Ω has the product measure and let $h : \Omega \to \mathbb{R}$. Let r be an integer then h is *r-certifiable* if whenever $h(x) \geqslant s$ there exist a subset $I \subset \{1, \ldots, n\}$ of size at most rs so that all $y \in \Omega$ that agree with x on the coordinates from I have $h(y) \geqslant s$. We also call h, *c-Lipschitz* if $|h(x) - h(y)| \leqslant c$ whenever x, y differ in at most one coordinate.

TALAGRAND'S INEQUALITY [8]. *Let X be a random variable determined by n independent trials $t_1, \ldots, t_n$, $t_i \in \Omega_i$ and let $E(X)$ be the expected value of X. If for some c and r, X is c-Lipschitz and r-certifiable then*

$$\Pr(|X - E(X)| > t + 30c\sqrt{rE(X)}) \leqslant 4e^{-\frac{t^2}{8c^2 rE(X)}}.$$

This inequality in slightly different, but essentially equivalent form appears also in [1]. Finally we present the following weaker version of Theorem 1.1 from [10] (see also Proposition 5.5.3 in [1]) whose easy prove we include for the sake of completeness.

PROPOSITION 2.1. *Let G be a graph with the sets of lists $S(v)$, satisfying that for every vertex $|S(v)| = \lceil 2ed \rceil$, $(e = 2.71\ldots)$ and for each colour $c \in S(v)$, the number of neighbours of v that have c in their list is at most d. Then there exist a proper colouring of G from these lists.*

Proof. For every vertex $v \in V(G)$ choose a colour $c \in S(v)$ uniformly at random, making independent choices for different vertices. For an edge $e = (u, v)$ and a colour c such that c appears in the colour lists of both u and v let $A_{e,c}$ be the event that "the vertices u and v are coloured c." As $|S(v)| = \lceil 2ed \rceil$ for every v, we get $Pr[A_{e,c}] = 1/\lceil 2ed \rceil^2$. The event $A_{e,c}$ is independent of all other events $A_{c',e'}$ but those for which $e \cap e' \neq \emptyset$ and both endpoints of e' contain c' in their lists. It is easy to see that the number of such events can be estimated from above by $2d\lceil 2ed \rceil - 1$. Hence by the symmetric version of the Local Lemma, with positive probability none of the events $A_{e,c}$ happens. Therefore there exists a proper colouring of G from the lists. ∎

3. PROOF OF THE MAIN THEOREM

In this section we present a proof of our main theorem. We may and will assume whenever this needed that the parameter d is sufficiently large and ε is less than 1.

3.1. *Random Colouring*

Here we describe the colouring procedure that we use to obtain a proper list colouring of a graph G, which satisfies the properties (1) and (2) from Theorem 1.1. Our algorithm will consist of several iterations of the wasteful colouring procedure described below. During the iterations, for each vertex v, we maintain a list L_v of available colours. Initially, $L_v = S(v)$ is the original list of permitted colours for vertex v and has size $\lceil (1+\varepsilon)d \rceil$.

Wasteful Colouring Procedure (ith iteration).

(1) For each uncoloured vertex v, activate v with probability $\frac{1}{\ln d}$.

(2) For each activated vertex v, assign to v a colour chosen uniformly at random from L_v.

(3) For each activated vertex v, remove the colour assigned to v from the list of every neighbour of v.

(4) Uncolour every vertex which receives the same colour as a neighbour.

During the execution of the above procedure we keep track of two parameters: the size of the lists and the number of neighbours of a vertex which have a certain colour in their list. Let $l_i(v)$ be the size of L_v at the beginning of iteration i. Also, let $N_i(v, c)$ denote the set of uncoloured neighbours u of v with $c \in L_u$ at the beginning of iteration i, and let $t_i(v, c)$ be the size of $N_i(v, c)$. Also, let $z_i(v, c)$ be the number of uncoloured neighbours u of v such that at the beginning of iteration i, $c \in L_i(u)$ and u was successfully coloured during the iteration i. Clearly $t_{i+1}(v, c) \leqslant t_i(v, c)$

$-z_i(v,c)$. This fact will be extremely useful since $t_{i+1}(v,c)$ is not concentrated, but $z_i(v,c)$ is.

Rather than keeping track of the parameters $l_i(v)$ and $t_i(v,c)$ for every v and c, we focus instead on their extreme values. In particular we estimate $l_i(v)$ and $t_i(v,c)$ by L_i and T_i respectively. The values of L_i and T_i are defined by the following recurrences.

- $L_1 = (1+\varepsilon)d, \quad T_1 = d$
- $L_{i+1} = (1 - \frac{1}{(1+3\varepsilon/4)\ln d})L_i$
- $T_{i+1} = (1 - \frac{1}{(1+\varepsilon/4)\ln d})T_i.$

We will show by induction that with positive probability the following property holds for every i in some specific range.

Property P(i). At the start of iteration i for every uncoloured vertex v of the graph G and each colour $c \in L_v$ the following is true,

$$l_i(v) \geqslant L_i, \qquad t_i(v,c) \leqslant T_i.$$

More precisely in the next section we prove the following statement.

PROPOSITION 3.1. *With positive probability*, *P*(i) *holds for every* $i \leqslant \frac{10}{\varepsilon}\ln d + 2$.

We finish this section by showing how this proposition implies the main result of the paper.

Proof of Theorem 1.1. We carry out the wasteful colouring procedure for up to $i' = \lceil \frac{10}{\varepsilon} \ln d \rceil$ iterations. By Proposition 3.1, with positive probability, at the end of iteration i' the size of the each list is at least $L_{i'+1}$, each colour $c \in L_v$ appears in the list of at most $T_{i'+1}$ neighbours of v, and

$$\frac{T_{i'+1}}{L_{i'+1}} = \frac{\left(1 - \frac{1}{(1+\varepsilon/4)\ln d}\right)^{i'} T_1}{\left(1 - \frac{1}{(1+3\varepsilon/4)\ln d}\right)^{i'} L_1} \leqslant \left(\frac{1 - \frac{1}{(1+\varepsilon/4)\ln d}}{1 - \frac{1}{(1+3\varepsilon/4)\ln d}}\right)^{i'}$$

$$\leqslant \left(1 - \frac{\varepsilon}{5 \ln d}\right)^{\frac{10 \ln d}{\varepsilon}} \leqslant e^{-2}.$$

Now we can complete the colouring of the graph G using Proposition 2.1. ∎

3.2. Concentration Lemma

In this section we prove Proposition 3.1. The crucial part of the proof is the following statement.

LEMMA 3.2. *If property $P(i)$ holds and $i \leqslant \frac{10}{\varepsilon} \ln d + 1$, then for any uncoloured vertex v and colour $c \in L_v$,*

(a) $\Pr(l_{i+1}(v) < L_{i+1}) < d^{-\ln d}$
(b) $\Pr(t_{i+1}(v, c) > T_{i+1}) < d^{-\ln d}$.

Proof. If $P(i)$ holds and $i \leqslant \frac{10}{\varepsilon} \ln d + 1$, then by definition we have the following inequalities

$$l_i(v) \geqslant L_i = \left(1 - \frac{1}{(1 + 3\varepsilon/4) \ln d}\right)^{i-1} \cdot L_1$$
$$\geqslant \left(1 - \frac{1}{(1 + 3\varepsilon/4) \ln d}\right)^{\frac{10}{\varepsilon} \ln d} \cdot d = \Theta(d),$$

$$T_i = \left(1 - \frac{1}{(1 + \varepsilon/4) \ln d}\right)^{i-1} \cdot T_1$$
$$\geqslant \left(1 - \frac{1}{(1 + \varepsilon/4) \ln d}\right)^{\frac{10}{\varepsilon} \ln d} \cdot d = \Theta(d).$$

Also note that $T_i / L_i < T_1 / L_1 = 1/(1 + \varepsilon)$ for all i. We will use these facts later in the proof. Now consider the values $l_{i+1}(v)$ and $t_{i+1}(v, c)$ after the ith step of the colouring procedure.

(a) It is easy to see that the expected value of $l_{i+1}(v)$ is equal to

$$E(l_{i+1}(v)) = \sum_{c \in L_v} \prod_{u \in N(v,c)} \left(1 - \frac{1}{\ln d} \frac{1}{l_i(u)}\right) \geqslant \left(1 - \frac{1}{\ln d} \frac{1}{L_i}\right)^{T_i} l_i(v)$$
$$\geqslant \left(1 - \frac{T_i}{L_i \ln d} - \Theta\left(\frac{T_i^2}{L_i^2 \ln^2 d}\right)\right) l_i(v)$$
$$\geqslant \left(1 - \frac{1}{(1 + \varepsilon) \ln d} - \Theta\left(\frac{1}{\ln^2 d}\right)\right) l_i(v).$$

Next we show that this variable is concentrated. More precisely, instead of proving that $l_{i+1}(v)$ i.e., the number of colours which remain in L_v after the ith iteration is concentrated, we prove that $l_i(v) - l_{i+1}(v)$, i.e., the number of colours which are removed from L_v in the ith iteration is concentrated. Denote this random variable by R. Clearly changing the assignment of any vertex adjacent to v can change R by at most one and changing the assignment of any other vertex cannot affect R at all. Also if $R \geqslant s$ then there are s neighbours of v who were assigned different colours form L_v, so the outcomes of this trials certify that $R \geqslant s$. Therefore Talagrand's concentration inequality (with $r = c = 1$) yields

$$\Pr\left(|R - E(R)| > \frac{l_i(v)}{\ln^2 d}\right) < e^{-\Theta\left(\frac{l_i(v)}{\ln^4 d}\right)} < d^{-\ln d}.$$

By linearity of expectation $E(l_{i+1}(v)) = l_i(v) - E(R)$, and thus

$$\Pr\left(|l_{i+1} - E(l_{i+1}(v))| > \frac{l_i(v)}{\ln^2 d}\right) = \Pr\left(|R - E(R)| > \frac{l_i(v)}{\ln^2 d}\right) < d^{-\ln d}.$$

Note that since d is large enough

$$\begin{aligned} L_{i+1} &\leqslant \left(1 - \frac{1}{(1 + 3\varepsilon/4)\ln d}\right) l_i(v) \leqslant \left(1 - \frac{1}{(1+\varepsilon)\ln d} - \Theta\left(\frac{1}{\ln^2 d}\right)\right) l_i(v) \\ &\leqslant E(l_{i+1}(v)) - \frac{l_i(v)}{\ln^2 d}. \end{aligned}$$

This finally implies that

$$\Pr(l_{i+1}(v) < L_{i+1}) \leqslant \Pr\left(l_{i+1}(v) < E(l_{i+1}(v)) - \frac{l_i(v)}{\ln^2 d}\right) < d^{-\ln d}.$$

(b) If initially $t_i(v,c) \leqslant T_{i+1}$ then there is nothing to prove, so suppose that $t_i(v,c) > T_{i+1} = \Theta(d)$. Note that since d is large enough, we have that if $z_i(v,c)$ is at least $t_i(v,c)/\ln d - \Theta(t_i(v,c)/\ln^2 d)$ then

$$\begin{aligned} t_{i+1}(v,c) &\leqslant t_i(v,c) - z_i(v,c) \\ &\leqslant \left(1 - \frac{1}{\ln d} - \Theta\left(\frac{1}{\ln^2 d}\right)\right) t_i(v,c) \leqslant \left(1 - \frac{1}{\ln d} - \Theta\left(\frac{1}{\ln^2 d}\right)\right) T_i \\ &\leqslant \left(1 - \frac{1}{(1+\varepsilon/4)\ln d}\right) T_i = T_{i+1}. \end{aligned}$$

This yields that

$$\Pr(t_{i+1}(v,c) > T_{i+1}) \leqslant \Pr\left(z_i(v,c) < \frac{t_i(v,c)}{\ln d} - \Theta\left(\frac{t_i(v,c)}{\ln^2 d}\right)\right)$$

and therefore it is enough to bound the probability on the right.

To analyze the behavior of $z_i(v,c)$ we divide the ith step of the colouring procedure into three independent phases. First we activate all vertices in $N_i(v,c)$ uniformly at random with probability $1/\ln d$. Let X_1 be the set of activated vertices. Then the size of X_1 is binomially distributed random variable with parameters $t_i(v,c)$ and $1/\ln d$. Thus by the standard estimates for Binomial distributions (see, e.g., [1, Appendix A]) and since $t_i(v,c) = \Theta(d)$ it easily follows that

$$\Pr\left(\left||X_1| - \frac{t_i(v,c)}{\ln d}\right| > \frac{t_i(v,c)}{\ln^2 d}\right) < \frac{1}{3} d^{-\ln d}.$$

Next, conditioned on the event that $||X_1| - t_i(v,c)/\ln d| < t_i(v,c)/\ln^2 d$, we arbitrarily order vertices from X_1 and in this particular order we pick for each vertex a random colour from its lists. Let $X_2 \subset X_1$ be the set of vertices whose colour is unique in this colouring and hence this vertices will retain their colour at this step. For each of the vertices in X_1 the probability that it gets repeated colour is equal to the number of colours already assign to other vertices in X_1 (which is at most $|X_1| \leqslant t_i(v,c)(1/\ln d + 1/\ln^2 d)$) divided by the size of its list (which is at least $L_i \geqslant (1+\varepsilon)t_i(v,c)$). Therefore this probability is at most $1/\ln d$. Thus the probability that there are more than $4t_i(v,c)/\ln^2 d$ vertices which get repeated colours is at most

$$\binom{\frac{t_i(v,c)}{\ln d}}{4\frac{t_i(v,c)}{\ln^2 d}}\left(\frac{1}{\ln d}\right)^{4\frac{t_i(v,c)}{\ln^2 d}} \leqslant \left(\frac{et_i(v,c)/\ln d}{4t_i(v,c)/\ln^2 d}\frac{1}{\ln d}\right)^{4\frac{t_i(v,c)}{\ln^2 d}}$$

$$= \left(\frac{e}{4}\right)^{4\frac{t_i(v,c)}{\ln^2 d}} < \frac{1}{3} d^{-\ln d}.$$

Therefore with probability at least $1 - \frac{2}{3}d^{-\ln d}$ we get that $|X_2| \geqslant |X_1| - 8t_i(v,c)/\ln^2 d = t_i(v,c)/\ln d - \Theta(t_i(v,c)/\ln^2 d)$. Finally, conditioned on this event, we complete our colouring procedure for the rest of the vertices and uncolour vertices from X_2 if their neighbours outside $N_i(v,c)$ got the same colour. Denote by $X_3 \subset X_2$ the set of vertices which retain their colour after this phase. Clearly the size of X_3 is a lower bound on $z_i(v,c)$. It is easy to see that the expected value of $|X_3|$, conditioned on the event that $|X_2| > t_i(v,c)/\ln d - \Theta(t_i(v,c)/\ln^2 d)$, is at least

$$E(|X_3|) \geqslant \left(1 - \frac{1}{\ln d}\frac{1}{L_i}\right)^{T_i} |X_2|$$

$$\geqslant \left(1 - \frac{T_i}{L_i \ln d} - \Theta\left(\frac{T_i^2}{L_i^2 \ln^2 d}\right)\right)\left(\frac{t_i(v,c)}{\ln d} - \Theta\left(\frac{t_i(v,c)}{\ln^2 d}\right)\right)$$

$$\geqslant \left(1 - \frac{1}{(1+\varepsilon)\ln d} - \Theta\left(\frac{1}{\ln^2 d}\right)\right)\left(\frac{t_i(v,c)}{\ln d} - \Theta\left(\frac{t_i(v,c)}{\ln^2 d}\right)\right)$$

$$\geqslant \frac{t_i(v,c)}{\ln d} - \Theta\left(\frac{t_i(v,c)}{\ln^2 d}\right).$$

Note also that again by Talagrand's inequality $|X_2| - |X_3|$, i.e., the number of vertices in X_2 which lost their colour, is highly concentrated. Indeed, all vertices in X_2 have different colours and thus changing an assignment to any vertex from outside $N_i(v,c)$ can change $|X_2| - |X_3|$ by at most one. Also it is easy to see that if $|X_2| - |X_3| > s$, then there is a set of neighbours of vertices from X_2 of size s, each assigned a different colour, which force this change. Also by linearity of expectation $|X_3| - E(|X_3|) = -(|X_2| - |X_3|) - E(|X_2| - |X_3|)$. Therefore

$$\Pr\left(||X_3| - E(|X_3|)| > \frac{t_i(v,c)}{\ln^2 d}\right) < \frac{1}{3} d^{-\ln d}.$$

So finally we obtain that

$$z_i(v,c) \geqslant |X_3| \geqslant E(|X_3|) - \frac{t_i(v,c)}{\ln^2 d} = \frac{t_i(v,c)}{\ln d} - \Theta\left(\frac{t_i(v,c)}{\ln^2 d}\right),$$

happens with probability at least $1 - 3(d^{-\ln d}/3) = 1 - d^{-\ln d}$. This completes the proof. ∎

Proposition 3.1 now follows easily.

Proof of Proposition 3.1. We use induction on i. The base of induction is trivially true. Suppose now that $P(i)$ holds. Let A_v be the event that $l_{i+1}(v) < L_{i+1}$ and $B_{v,c}$ be the event that $t_{i+1}(v,c) > T_{i+1}$. Then by Lemma 3.2 the probability of each of these events is less than $d^{-\ln d}$. On the other hand it is easy to see that both A_v and $B_{v,c}$ are independent from all but at most $d^{O(1)}$ other such events. Indeed, they are independent from all A_u and $B_{u,c'}$ corresponding to vertices u whose distance form v is bigger than 4 and it is easy to see that the maximum degree of the graph G is at most $O(d^2)$. Therefore by the Local Lemma, with positive probability, no event A_v or $B_{v,c}$ happens. This implies that the property $P(i+1)$ holds and completes the induction step. ∎

4. CONCLUDING REMARKS

- A more careful analysis of our colouring procedure yields that the value of ε in the assertion of Theorem 1.1 can be made as small as $1/\ln^{1-\delta} d$ for any positive constant δ.
- The results of Molloy and Reed [9], about the algorithmic version of the local lemma, can be combined with our method here to design an efficient algorithm for finding the desired colourings of graphs, which satisfy conditions of Theorem 1.1.
- Consider a graph $G = (V, E)$ and a set of lists $S(v)$, $v \in V$ satisfying that for each colour $c \in S(v)$, the number of neighbours of v that have c in

their list is at most d. We define an auxiliary graph H by taking vertex set $V(H)$ to be $\{(v,c) \mid v \in V,\ c \in S(v)\}$ and joining two vertices (v,c) and (u,c') by an edge if and only if $c = c'$ and $(v,u) \in E(G)$. By definition, the maximum degree of H is at most d and $V(H)$ is the disjoint union of independent sets $V_v = \{(v,c) \mid c \in S(v)\}$, each of size $|S(v)|$. Clearly in order to list-colour G it is enough to find an independent set in H which intersects each independent set V_v in one vertex.In [5] Haxell proved that if graph H has maximal degree d and it's vertex set has a partition into disjoint independent sets $V_1, \ldots, V_n$, each of size $2d$, then H contains an independent set which intersects each V_i in one vertex. By the above discussion this immediately implies that the lists of size $2d$ are enough to colour the vertices of graph G. Therefore it is natural to ask if the $2d$ bound on the size of the independent sets V_i can be improved, thus proving a slight generalization of Reed's conjecture? Unfortunately, there are examples (see, e.g., [3, 11]) which show that the above bound is tight.

- Consider again a graph $G = (V, E)$ and a set of lists $S(v)$, $v \in V$ satisfying that for each colour $c \in S(v)$, the number of neighbours of v that have c in their list is at most d. Let $f(d)$ be the smallest integer such that if every list $S(v)$ has a size at least $f(d)$, then G has an acceptable coloring from these lists. It remains an interesting problem to give a better estimate on the function $f(d)$. So far we know that $d + 2 \leqslant f(d) \leqslant (1 + o(1))d$. Hence an intriguing open question is to determine whether $f(d) - d$ is bounded by a constant which does not depends on d.

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REFERENCES

1. N. Alon and J. Spencer, "The Probabilistic Method," 2nd ed., Wiley, New York, 2000.
2. T. Bohman and R. Holzman, On a list coloring conjecture of Reed, preprint.
3. B. Bollobás, P. Erdös, and E. Szemerédi, On complete subgraphs of r-chromatic graphs, *Discrete Math.* **13** (1975), 97–107.
4. F. Galvin, The list chromatic index of a bipartite multigraph, *J. Combin. Theory Ser. B* **63** (1995), 153–158.
5. P. Haxell, A note on vertex list colouring, preprint.
6. T. Jensen and B. Toft, "Graph Colouring Problems," Wiley, New York, 1995.
7. J. Kahn, On some hypergraph problems of Paul Erdős and the asymptotics of matchings, covers and colorings, *in* "The mathematics of Paul Erdős I," pp. 345–371, Springer-Verlag, Berlin, 1997.

8. M. Molloy and B. Reed, "Graph Colouring and the Probabilistic Method," Springer-Verlag, Berlin, 2001.
9. M. Molloy and B. Reed, Further algorithmic aspects of the local lemma, *in* "Proc. of the 30th Annual ACM STOC," pp. 524–529, Assoc. Comput. Mach., New York, 1998.
10. B. Reed, The list colouring constants, *J. Graph Theory* **31** (1999), 149–153.
11. R. Yuster, Independent transversals in r-partite graphs, *Discrete Math.* **176** (1997), 255–261.