SIAM J. DISCRETE MATH.
Vol. 35, No. 3, pp. 1557–1568

LOWER BOUNDS FOR MAX-CUT IN H-FREE GRAPHS VIA SEMIDEFINITE PROGRAMMING*

CHARLES CARLSON†, ALEXANDRA KOLLA†, RAY LI‡, NITYA MANI§, BENNY SUDAKOV¶, AND LUCA TREVISAN‖

Abstract. For a graph G, let $f(G)$ denote the size of the maximum cut in G. The problem of estimating $f(G)$ as a function of the number of vertices and edges of G has a long history and was extensively studied in the last fifty years. In this paper we propose an approach, based on semidefinite programming, to prove lower bounds on $f(G)$. We use this approach to find large cuts in graphs with few triangles and in K_r-free graphs.

Key words. semidefinite programming, Max Cut, extremal combinatorics, H free graphs

AMS subject classifications. 05C35, 90C22

DOI. 10.1137/20M1333985

1. Introduction.

The celebrated Max-Cut problem asks for the largest bipartite subgraph of a graph G, i.e., for a partition of the vertex set of G into disjoint sets V_1 and V_2 so that the number of edges of G crossing V_1 and V_2 is maximal. This problem has been the subject of extensive research, both from a largely algorithmic perspective in computer science and from an extremal perspective in combinatorics. Throughout, let G denote a graph with n vertices and m edges with maximal cut of size $f(G)$. The extremal version of Max-Cut problem asks to give bounds on $f(G)$ solely as a function of m and n. This question was first raised more than fifty years ago by Erdős [10] and has attracted a lot of attention since then (see, e.g., [8, 11, 12, 1, 19, 5, 3, 20, 7, 21, 16] and their references).

It is well known that every graph G with m edges has a cut of size at least $m/2$. To see this, consider a random partition of vertices of the vertices G into two parts V_1, V_2 and estimate the expected number of edges between V_1 and V_2. On the other hand, already in 1960's Erdős [10] observed that the constant $1/2$ cannot be improved even if we consider very restricted families of graphs, e.g., graphs that contain no short cycles. Therefore the main question, which has been studied by many researchers, is

*Received by the editors April 26, 2020; accepted for publication (in revised form) March 16, 2021; published electronically July 9, 2021. A conference version of this paper appeared at LATIN '20.

https://doi.org/10.1137/20M1333985

Funding: The second author's research supported by NSF CAREER grant 1452923, as well as NSF AF grant 1814385. The third author's research supported by an NSF GRF grant DGE-1656518 and by NSF grant CCF-1814629. The fourth author's research supported in part by a Stanford Undergraduate Advising and Research Major grant. The fifth author's research supported in part by SNSF grant 200021-196965. The sixth author's research supported by NSF grant CCF 1815434. Work on this project has also received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant 834861).

†Department of Computer Science, University of Colorado Boulder, Boulder, CO USA (charles.carlson@colorado.edu, https://cacarlson.github.io; alexandra.kolla@colorado.edu).

‡Department of Computer Science, Stanford University, Stanford, CA 94305 USA (rayyli@cs.stanford.edu).

§Department of Mathematics, MIT, Cambridge, MA 02138 USA (nmani@mit.edu).

¶Department of Mathematics, ETH, 8092 Zurich, Switzerland (benjamin.sudakov@math.ethz.ch).

‖Department of Decision Sciences, Bocconi University, Milano, 94720-1776, Italy (l.trevisan@unibocconi.it).

to estimate the error term $f(G) - m/2$, which we call *surplus*, for various families of graphs G.

The elementary bound $f(G) \geq m/2$ was improved by Edwards [8, 9] who showed that every graph with m edges has a cut of size at least $\frac{m}{2} + (\sqrt{8m+1} - 1)/8$. This result is easily seen to be tight in case G is a complete graph on an odd number of vertices, that is, whenever $m = \binom{k}{2}$ for some odd integer k. Estimates on the second error term for other values of m can be found in [4] and [5].

Although the $\sqrt{m}$ error term is tight in general, it was observed by Erdős and Lovász [11] that for triangle-free graph it can be improved to at least $m^{2/3+o(1)}$. This naturally yields a motivating question: what is the best surplus which can always be achieved if we assume that our family of graphs is H*-free*, i.e., no graph contains a fixed graph H as a subgraph. It is not difficult to show (see, e.g., [2]) that for every fixed graph H there is some $\epsilon = \epsilon(H) > 0$ such that $f(G) \geq \frac{m}{2} + \Omega(m^{1/2+\epsilon})$ for all H-free graphs with m edges. However, the problem of estimating the error term more precisely is not easy, even for relatively simple graphs H. It is plausible to conjecture (see [3]) that for every fixed graph H there is a constant c_H such that every H-free graph G with m edges has a cut with surplus at least $\Theta(m^{c_H})$; i.e., there is both a lower bound and an infinite sequence of example showing that exponent c_H cannot be improved. This conjecture is very difficult. Even in the case when H is a triangle, determining the correct error term took almost twenty years. Following the works of [11, 18, 19], Alon [1] proved that every m-edge triangle free graph has a cut with surplus of order $m^{4/5}$ and that this is tight up to constant factors. There are several other forbidden graphs H for which we know quite accurately the error term for the extremal Max-Cut problem in H-free graphs. For example, it was proved in [3] that if $H = C_r$ for $r = 4, 6, 10$, then $c_H = \frac{r+1}{r+2}$. The answer is also known in the case when H is a complete bipartite graph $K_{2,s}$ or $K_{3,s}$ (see [3] for details).

New approach to Max-Cut using semidefinite programming. Many extremal results for the Max-Cut problem rely on quite elaborate probabilistic arguments. A well known example of such an argument is a proof by Shearer [19] that if G is a triangle-free graph with n vertices and m edges, and if $d_1, d_2, \ldots, d_n$ are the degrees of its vertices, then $f(G) \geq \frac{m}{2} + O(\sum_{i=1}^{n} \sqrt{d_i})$. The proof is quite intricate and is based on first choosing a random cut and then randomly redistributing some of the vertices, depending on how many their neighbors are on the same side as the chosen vertex in the initial cut. Shearer's arguments were further extended, with more technically involved proofs, in [3] to show that the same lower bound remains valid for graphs G with relatively sparse neighborhoods (i.e., graphs which locally have few triangles).

In this article we propose a different approach to give lower bounds on the Max-Cut of sparse H-free graphs using approximation by semidefinite programming (SDP). This approach is intuitive and computationally simple. The main idea was inspired by the celebrated approximation algorithm of Goemans and Williamson [15] of the Max-Cut: given a graph G with m edges, we first construct an explicit solution for the standard Max-Cut SDP relaxation of G which has value at least $(\frac{1}{2} + W)m$ for some positive surplus W. We then apply a Goemans–Williamson randomized rounding, based on the sign of the scalar product with random unit vector, to extract a cut in G whose surplus is within constant factor of W. Using this approach we prove the following result.

THEOREM 1.1. *Let $G = (V, E)$ be a graph with n vertices and m edges. For every $i \in [n]$, let V_i be any subset of neighbors of vertex i and $\varepsilon_i \leq 1/\sqrt{|V_i|}$. Then,*

$$f(G) \geq \frac{m}{2} + \sum_{i=1}^{n} \frac{\varepsilon_i |V_i|}{2\pi} - \sum_{(i,j)\in E} \frac{\varepsilon_i \varepsilon_j |V_i \cap V_j|}{2}. \tag{1.1}$$

This results implies Shearer's bound [19]. To see this, set V_i to the neighbors of i and $\varepsilon_i = \frac{1}{\sqrt{d_i}}$ for all i. Then, if G is triangle-free graph, then $|V_i \cap V_j| = 0$ for every pair of adjacent vertices i, j.

The fact that we apply Goemans–Williamson SDP rounding in this setting is perhaps surprising for a few reasons. In general, our result obtains a surplus of $\Omega(W)$ from an SDP solution with surplus W, which is not possible in general. The best cut that can be guaranteed from any kind of rounding of a Max-Cut SDP solution with value $(\frac{1}{2} + W)m$ is $(\frac{1}{2} + \Omega(\frac{W}{\log W}))m$ (see [17]). Furthermore, this is achieved using the RPR^2 rounding algorithm, not the Goemans–Williamson rounding algorithm. Nevertheless, we show that our explicit Max-Cut solution has additional properties that circumvents these issues and permits a better analysis.

New lower bound for Max-Cut of triangle sparse graphs. Using Theorem 1.1, we give a new result on the Max-Cut of triangle sparse graphs that is more convenient to use than previous similar results. A graph G is *d-degenerate* if there exists an ordering of the vertices $1, \ldots, n$ such that vertex i has at most d neighbors $j < i$. Equivalently, a graph is d-degenerate if every induced subgraph has a vertex of degree at most d. Degeneracy is a broader notion of graph sparseness than maximum degree: all maximum degree d graphs are d-degenerate, but the star graph is 1-degenerate while having maximum degree $n-1$. Theorem 1.1 gives the following useful corollary on the Max-Cut of d-degenerate graphs.

COROLLARY 1.2. *Let $\varepsilon \leq \frac{1}{\sqrt{d}}$. Let G be a d-degenerate graph with m edges and t triangles. Then*

$$f(G) \geq \frac{m}{2} + \frac{\varepsilon m}{2\pi} - \frac{\varepsilon^2 t}{2}. \tag{1.2}$$

As all max-degree-d graphs are d-degenerate, (1.2) holds in particular for max-degree-d graphs. To see the corollary, let $1, \ldots, n$ be an ordering of the vertices such that any i has at most d neighbors $j < i$, and let V_i be this set of neighbors. Let $\varepsilon_i = \varepsilon$ for all i. In this way, $\sum_i |V_i|$ counts every edge exactly once and $\sum_{(i,j)\in E} |V_i \cap V_j|$ counts every triangle exactly once, and the result follows. This shows that graphs with few triangles have cuts with surplus similar to triangle-free graphs.

This result is new and more convenient to use than existing results in this vein, because it relies only on the global count of the number of triangles, rather than a local triangle sparseness property assumed by prior results. For example, it was shown that (using Lemma 3.3 of [3]) a d-degenerate graph with a local triangle-sparseness property, namely, that every large induced subgraph with a common neighbor is sparse, has Max-Cut at least $\frac{m}{2} + \Omega(\frac{m}{\sqrt{d}})$. However, we can achieve the same result with only the guarantee that the global number of triangles is small. In particular, when there are at most $O(m\sqrt{d})$ triangles, which is always the case with the local triangle-sparseness assumption above, setting $\varepsilon = \Theta(\frac{1}{\sqrt{d}})$ in Corollary 1.2 gives that the Max-Cut is again at least $\frac{m}{2} + \Omega(\frac{m}{\sqrt{d}})$.

Corollary: Lower bounds for Max-Cut of H-free degenerate graphs. We illustrate usefulness of the above results by giving the following lower bound on the Max-Cut of K_r-free graphs.

THEOREM 1.3. *Let $r \geq 3$. There exists a constant $c = c(r) > 0$ such that, for all K_r-free d-degenerate graphs G with m edges,*

$$f(G) \geq \left(\frac{1}{2} + \frac{c}{d^{1-1/(2r-4)}}\right) m. \tag{1.3}$$

Lower bounds such as Theorem 1.3 giving a surplus of the form $c \cdot \frac{m}{d^\alpha}$ are more fine-grained than those that depend only on the number of edges. Accordingly, they are useful for obtaining lower bounds the Max-Cut independent of the degeneracy: many tight Max-Cut lower bounds in H-free graphs of the form $\frac{m}{2} + cm^\alpha$ first establish that $f(G) \geq \frac{m}{2} + c \cdot \frac{m}{\sqrt{d}}$ for all H-free graphs, and their results follow by case-working on the degeneracy [3].

We note that Theorem 1.3 is stronger than the result in [21], which says that K_r-free graphs have surplus at least $\tilde{\Omega}(m^{(r-1)/(2r-3)})$. Theorem 1.3 is stronger than [21, Theorem 1.2] both in that it is more fine grained, depending on the degeneracy d, and that when one plugs in $d \leq 2\sqrt{m}$, we get a stronger bound of $\Omega(m^{(2r-3)/(4r-8)})$.

In the case of $r = 4$ one can use our arguments together with Alon's result on Max-Cut in triangle-free graphs to improve Theorem 1.3 further to $m/2 + cm/d^{2/3}$. While Theorem 1.3 gives nontrivial bounds for K_r-free graphs, we believe that a stronger statement is true and propose the following conjecture.

CONJECTURE 1.4. *For any graph H, there exists a constant $c = c(H) > 0$ such that, for all H-free d-degenerate graphs with $m \geq 1$ edges,*

$$f(G) \geq \left(\frac{1}{2} + \frac{c}{\sqrt{d}}\right) m. \tag{1.4}$$

Our Theorem 1.1 implies this conjecture for various graphs H, e.g., $K_{2,s}, K_{3,s}, C_r$ and for any graph H which contains a vertex whose deletion makes it acyclic. This was already observed in [3] using the weaker, locally triangle-sparse form of Corollary 1.2 described earlier.

Conjecture 1.4 provides a natural route to proving a closely related conjecture proposed by Alon et al. [2].

CONJECTURE 1.5 (see [2]). *For any graph H, there exists constants $\varepsilon = \varepsilon(H) > 0$ and $c = c(H) > 0$ such that, for all H-free graphs with $m \geq 1$ edges,*

$$f(G) \geq \frac{m}{2} + cm^{3/4+\varepsilon}. \tag{1.5}$$

Since every graph with m edges is obviously $\sqrt{2m}$-degenerate, the Conjecture 1.4 implies immediately a weaker form of Conjecture 1.5 with surplus of order $m^{3/4}$. With some extra technical work we can show that it actually implies the full conjecture, achieving a surplus of $m^{3/4+\varepsilon}$ for any graph H (for brevity, we omit the proof, which can be found in [6]). For many graphs H for which Conjecture 1.5 is known, (1.4) was implicitly established for H-free graphs [3], making Conjecture 1.4 a plausible stepping stone to Conjecture 1.5. As further evidence of the plausibility of Conjecture 1.4, we show that Conjecture 1.5 implies a weaker form of Conjecture 1.4, namely, that any H-free graph has Max-Cut $\frac{m}{2} + cm \cdot d^{-5/7}$. Using similar techniques, we can obtain nontrivial, unconditional results on the Max-Cut of d-degenerate H-free graphs for particular graphs H.

Conjecture 1.4, if true, gives a surplus of $\Omega(\frac{m}{\sqrt{d}})$ that is optimal up to a multiplicative constant factor for every fixed graph H which contains a cycle. To see this, consider an Erdős–Rényi random graph $G(n, p)$ with $p = n^{-1+\delta}$. Using standard Chernoff-type estimates, one can easily show that with high probability that this graph is $O(np)$-degenerate and its Max-Cut has size at most $\frac{1}{4}\binom{n}{2}p + O(n\sqrt{np})$. Moreover, if $\delta = \delta(H) > 0$ is small enough, then with high probability $G(n, p)$ contains only very few copies of H which can be destroyed by deleting few vertices, without changing the degeneracy and surplus of the Max-Cut.

2. Lower bounds for Max-Cut using SDP.

In this section we give a lower bound for $f(G)$ in graphs with few triangles, showing Theorem 1.1. To prove this result, we make heavy use of the SDP relaxation of the Max-Cut problem, formulated below for a graph $G = (V, E)$:

$$\begin{aligned} &\text{maximize} \quad && \sum_{(i,j)\in E} \frac{1}{2}(1 - \langle v^{(i)}, v^{(j)} \rangle) \\ &\text{subject to} \quad && \|v^{(i)}\|^2 = 1 \,\forall i \in V. \end{aligned} \tag{2.1}$$

We leverage the classical Goemans–Williamson [15] rounding algorithm which that gives an integral solution from a vector solution to the Max-Cut SDP.

Proof of Theorem 1.1. For $i \in [n]$, define $\tilde{v}^{(i)} \in \mathbb{R}^n$ by

$$\tilde{v}^{(i)}_j = \begin{cases} 1, & i = j, \\ -\varepsilon_i, & j \in V_i, \\ 0, & \text{otherwise}. \end{cases} \tag{2.2}$$

Then $1 \le \|\tilde{v}^{(i)}\|^2 \le 1 + \varepsilon_i^2 |V_i| \le 2$ for all i. For $i \in [n]$, let $v^{(i)} \overset{\text{def}}{=} \frac{\tilde{v}^{(i)}}{\|\tilde{v}^{(i)}\|} \in \mathbb{R}^n$. For each edge (i, j) with $j \in V_i$, we have

$$v^{(i)}_i v^{(j)}_i = \frac{1}{\|\tilde{v}^{(i)}\|} \cdot \frac{-\varepsilon_j}{\|\tilde{v}^{(j)}\|} \le \frac{-\varepsilon_j}{2}. \tag{2.3}$$

For $k \in V_i \cap V_j$, we have $v^{(i)}_k v^{(j)}_k \le \varepsilon_i \varepsilon_j$. For $k \notin \{i, j\} \cup (V_i \cap V_j)$, we have $v^{(i)}_k v^{(j)}_k = 0$ as $v^{(i)}_k = 0$ or $v^{(j)}_k = 0$. Thus, for all edges (i, j),

$$\langle v^{(i)}, v^{(j)} \rangle \le -\frac{\varepsilon_i}{2} \mathbb{1}_{V_i}(j) - \frac{\varepsilon_j}{2} \mathbb{1}_{V_j}(i) + |V_i \cap V_j| \varepsilon_i \varepsilon_j. \tag{2.4}$$

Here, $\mathbb{1}_S(i)$ is 1 if $i \in S$ and 0 otherwise. Vectors $v^{(1)}, \dots, v^{(n)}$ form a vector solution to the SDP (2.1). We now round this solution using the Goemans–Williamson [15] rounding algorithm. Let w denote a uniformly random unit vector, $A = \{i \in [n] : \langle v^{(i)}, w \rangle \ge 0\}$, and $B = [n] \setminus A$. Note that the angle between vectors $v^{(i)}, v^{(j)}$ is equal to $\cos^{-1}(\langle v^{(i)}, v^{(j)} \rangle)$, so the probability an edge (i, j) is cut is

$$\begin{aligned} \Pr[(i,j) \text{ cut}] &= \frac{\cos^{-1}(\langle v^{(i)}, v^{(j)} \rangle)}{\pi} \\ &= \frac{1}{2} - \frac{\sin^{-1}(\langle v^{(i)}, v^{(j)} \rangle)}{\pi} \\ &\ge \frac{1}{2} - \frac{1}{\pi} \sin^{-1}\left(|V_i \cap V_j| \varepsilon_i \varepsilon_j - \frac{\varepsilon_i}{2} \mathbb{1}_{V_i}(j) - \frac{\varepsilon_j}{2} \mathbb{1}_{V_j}(i) \right) \\ &\ge \frac{1}{2} - \frac{1}{\pi} \cdot \left(\frac{\pi}{2} \cdot |V_i \cap V_j| \varepsilon_i \varepsilon_j - \frac{\varepsilon_i}{2} \mathbb{1}_{V_i}(j) - \frac{\varepsilon_j}{2} \mathbb{1}_{V_j}(i) \right) \\ &= \frac{1}{2} + \frac{\varepsilon_i}{2\pi} \mathbb{1}_{V_i}(j) + \frac{\varepsilon_j}{2\pi} \mathbb{1}_{V_j}(i) - \frac{|V_i \cap V_j| \varepsilon_i \varepsilon_j}{2}. \end{aligned}$$

In the last inequality, we used that, for $a, b \in [0, 1]$, we have $\sin^{-1}(a - b) \le \frac{\pi}{2} a - b$. This is true as $\sin^{-1}(x) \le \frac{\pi}{2} x$ when x is positive and $\sin^{-1}(x) \le x$ when x is negative. Thus, the expected size of the cut given by $A \sqcup B$ is, by linearity of expectation,

$$\sum_{(i,j)\in E} \Pr[(i,j) \text{ cut}] \;\ge\; \sum_{\substack{(i,j)\in E\\ i<j}} \left(\frac{1}{2} + \frac{\varepsilon_i}{2\pi}\mathbb{1}_{V_i}(j) + \frac{\varepsilon_j}{2\pi}\mathbb{1}_{V_j}(i) - \frac{|V_i\cap V_j|\varepsilon_i\varepsilon_j}{2}\right)$$

$$(2.5) \qquad = \frac{m}{2} + \sum_{i=1}^{n} \frac{|V_i|\varepsilon_i}{2\pi} - \sum_{(i,j)\in E} \frac{|V_i\cap V_j|\varepsilon_i\varepsilon_j}{2}. \qquad \square$$

In the proof of Theorem 1.3 we use the following consequence of Corollary 1.2.

COROLLARY 2.1. *There exists an absolute constant $c > 0$ such that the following holds. For all $d \ge 1$ and $\varepsilon \le \frac{1}{\sqrt{d}}$, if a d-degenerate graph $G = (V, E)$ has m edges and at most $\frac{m}{8\varepsilon}$ triangles, then*

$$(2.6) \qquad f(G) \ge \left(\frac{1}{2} + c\varepsilon\right) \cdot m.$$

3. Decomposition of degenerate graphs. In a graph $G = (V, E)$, let $n(G)$ and $m(G)$ denote the number of vertices and edges, respectively. For a vertex subset $V' \subset V$, let $G[V']$ denote the subgraph induced by V'. We show that d-degenerate graphs with few triangles have small subsets of neighborhoods with many edges.

LEMMA 3.1. *Let $d \ge 1$ and $\varepsilon > 0$, and let $G = (V, E)$ be a d-degenerate graph with at least $m(G)/\varepsilon$ triangles. Then there exists a subset V' of at most d vertices with a common neighbor in G such that the induced subgraph $G[V']$ has at least $|V'|/\varepsilon$ edges.*

Proof. Since G is d-degenerate, we fix an ordering $1, \ldots, n$ of the vertices such that $d_<(i) \le d$ for all $i \in [n]$, where $d_<(i)$ denotes the number of neighbors $j < i$ of i. Then, if $t_<(i)$ denotes the number of triangles $\{i, j, k\}$ of G where $j, k < i$, we have

$$(3.1) \qquad \sum_i t_<(i) \;=\; t(G) \;\ge\; \frac{m(G)}{\varepsilon} \;=\; \sum_{i=1}^{n} \frac{d_<(i)}{\varepsilon}.$$

Hence, there must exist some i such that $t_<(i) \ge d_<(i)/\varepsilon$. Let V' denote the neighbors of i with index less than i. By definition, the vertices of V' have common neighbor i. Additionally, $G[V']$ has at least $d_<(i)/\varepsilon$ edges and $d_<(i) \le d$ vertices, proving the lemma. $\square$

We use this lemma to partition the vertices of any d-degenerate graph in a useful way.

LEMMA 3.2. *Let $\varepsilon > 0$. Let $G = (V, E)$ be a d-degenerate graph on n vertices with m edges. Then there exists a partition $V_1, \ldots, V_{k+1}$ of the vertex set V with the following properties.*

1. *For $i = 1, \ldots, k$, the vertex subset V_i has at most d vertices and has a common neighbor, and the induced subgraph $G[V_i]$ has at least $|V_i|/\varepsilon$ edges.*
2. *The induced subgraph $G[V_{k+1}]$ has at most $m(G[V_{k+1}])/\varepsilon$ triangles.*

Proof. We construct the partition iteratively. Let $V_0^* = V$. For $i \ge 1$, we partition the vertex subset V_{i-1}^* into $V_i \sqcup V_i^*$ as follows. If $G[V_{i-1}^*]$ has at least $m(G[V_{i-1}^*])/\varepsilon$ triangles, then by applying Lemma 3.1 to the induced subgraph $G[V_{i-1}^*]$, there exists a vertex subset V_i with a common neighbor in V_{i-1}^* such that $|V_i| \le d$ and the induced subgraph $G[V_i]$ has at most $|V_i|/\varepsilon$ edges. In this case, let $V_i^* \stackrel{\text{def}}{=} V_{i-1}^* \setminus V_i$. Let k denote the maximum index such that V_k^* is defined, and let $V_{k+1} \stackrel{\text{def}}{=} V_k^*$. By construction,

$V_1, \ldots, V_k$ satisfy the desired conditions. By definition of k, the induced subgraph $G[V_k^*]$ has at most $m(G[V_k^*])/\varepsilon$ triangles, so for $V_{k+1} = V_k^*$, we obtain the desired result. ∎

3.1. Large Max-Cut from decompositions. For a d-degenerate graph $G = (V, E)$, in a partition $V_1, \ldots, V_{k+1}$ of V given by Lemma 3.2, the induced subgraph $G[V_{k+1}]$ has few triangles, and thus, by Corollary 1.2, has a cut with good surplus. This allows us to obtain the following technical result regarding the Max-Cut of H-free d-degenerate graphs.

LEMMA 3.3. *There exists an absolute constant $c > 0$ such that the following holds. Let $0 < \varepsilon < \frac{1}{\sqrt{d}}$. For any H-free d-degenerate graph $G = (V, E)$, one of the following holds:*

- *We have*

$$f(G) \geq \left(\frac{1}{2} + c\varepsilon\right) m. \tag{3.2}$$

- *There exist graphs $G_1, \ldots, G_k$ such that five conditions hold:* (i) *graphs G_i are H'-free for all i and all graphs H' obtained by deleting one vertex from H,* (ii) *$n(G_i) \leq d$ for all i,* (iii) *$m(G_i) \geq \frac{n(G_i)}{8\varepsilon}$ for all i,* (iv) *$n(G_1) + \cdots + n(G_k) \geq \frac{m}{6d}$, and* (v)

$$f(G) \geq \frac{m(G)}{2} + \sum_{i=1}^{k} \left(f(G_i) - \frac{m(G_i)}{2}\right). \tag{3.3}$$

Proof. Let $c_1 < 1$ be the parameter given by Corollary 2.1. Let $c = c_1/6$. Let $G = (V, E)$ be a d-degenerate H-free graph. Applying Lemma 3.2 with parameter 8ε, we can find a partition $V_1, \ldots, V_{k+1}$ of the vertex set V with the following properties.

1. For $i = 1, \ldots, k$, the vertex subset V_i has at most d vertices and has a common neighbor, and the induced subgraph $G[V_i]$ at least $|V_i|/8\varepsilon$ edges.
2. The subgraph $G[V_{k+1}]$ has at most $m(G[V_{k+1}])/8\varepsilon$ triangles.

For $i = 1, \ldots, k+1$, let $G_i \stackrel{\text{def}}{=} G[V_i]$, and let $m_i \stackrel{\text{def}}{=} m(G_i)$. For $i = 1, \ldots, k$, since G is H-free and each V_i is a subset of some vertex neighborhood in G, the graphs G_i are H'-free for all H' obtained by deleting one vertex from H. For $i = 1, \ldots, k$, fix a maximal cut of G_i with associated vertex partition $V_i = A_i \sqcup B_i$. By the second property above, the graph G_{k+1} has at most $m_{k+1}/8\varepsilon$ triangles. Applying Corollary 2.1 with parameter ε, we can find a cut of G_{k+1} of size at least $(\frac{1}{2} + c_1\varepsilon)m_{k+1}$ with associated vertex partition $V_{k+1} = A_{k+1} \sqcup B_{k+1}$.

We now construct a cut of G by randomly combining the cuts obtained above for each G_i. Independently, for each $i = 1, \ldots, k+1$, we add either A_i or B_i to vertex set A, each with probability $\frac{1}{2}$. Setting $B = V \setminus A$, gives a cut of G. As $V_1, \ldots, V_{k+1}$ partition V, each of the $m - (m_1 + \cdots + m_{k+1})$ edges that is not in one of the induced graphs $G_1, \ldots, G_{k+1}$ has exactly one endpoint in each of A, B with probability $1/2$. This allows us to compute the expected size of the cut (a lower bound on $f(G)$ as there is some instantiation of this random process that achieves this expected size):

$$\begin{aligned} f(G) &\geq \frac{1}{2}(m - (m_1 + \cdots + m_{k+1})) + \left(\frac{1}{2} + c_1\varepsilon\right) \cdot m_{k+1} + \sum_{i=1}^{k} f(G_i) \\ &= \frac{m}{2} + c_1\varepsilon m_{k+1} + \sum_{i=1}^{k} \left(f(G_i) - \frac{m_i}{2}\right). \end{aligned} \tag{3.4}$$

We bound (3.4) based on the distribution of edges in G in 3 cases:

- $m_{k+1} \geq \frac{m}{6}$. Since $f(G_i) \geq \frac{m_i}{2}$ for all $i = 1, \ldots, k$, (3.2) holds:

$$f(G) \;\geq\; \frac{m}{2} + c_1 \varepsilon m_{k+1} \;\geq\; \left(\frac{1}{2} + c\varepsilon\right) \cdot m.$$

- The number of edges between $V_1 \cup \cdots \cup V_k$ and V_{k+1} is at least $\frac{2m}{3}$. Then, the cut given by vertex partition $V = A' \sqcup B'$ with $A' = V_1 \cup \cdots \cup V_k$ and $B' = V_{k+1}$ has at least $\frac{2m}{3}$ edges, in which case $f(G) \geq \frac{2m}{3} > (\frac{1}{2} + \frac{c_1\varepsilon}{6}) \cdot m$, so (3.2) holds.
- $G' = G[V_1 \cup \cdots \cup V_k]$ has at least $\frac{m}{6}$ edges. We show (3.3) holds. By construction, for $i = 1, \ldots, k$, the graph G_i is H'-free for all graphs H' obtained by deleting one vertex from H, has at most d vertices, and has at least $\frac{m_i}{8\varepsilon}$ edges. Since G is d-degenerate, G' is as well, so

$$\frac{m}{6} \leq m(G') \leq d \cdot n(G') = d \cdot \sum_{i=1}^{k} n(G_i). \tag{3.5}$$

Hence $n(G_1) + \cdots + n(G_k) \geq \frac{m}{6d}$. Lastly, by (3.4), we have

$$f(G) \geq \frac{m}{2} + \sum_{i=1}^{k} \left(f(G_i) - \frac{m_i}{2}\right).$$

This covers all possible cases, and in each case we showed either (3.2) or (3.3) holds. ∎

Remark 3.4. In Corollary 2.1 we can take $c = \frac{1}{11}$, and in Lemma 3.3 we can take $c = \frac{1}{66}$.

Lemma 3.3 allows us to convert Max-Cut lower bounds on H'-free graphs to Max-Cut lower bounds on H-free d-degenerate graphs.

LEMMA 3.5. *Let H be a graph. Suppose that there exist constants $a = a(H) \in [\frac{1}{2}, 1]$ and $c' = c'(H) > 0$ such that for all graphs G with $m' \geq 1$ edges that are H'-free for all graphs H' obtained by removing one vertex of H, we have $f(G) \geq \frac{m'}{2} + c' \cdot (m')^a$. Then there exists a constant $c = c(H) > 0$ such that for all H-free d-degenerate graphs G with $m \geq 1$ edges,*

$$f(G) \geq \left(\frac{1}{2} + c d^{-\frac{2-a}{1+a}}\right) \cdot m.$$

Proof. Let c_2 be the parameter in Lemma 3.3. We may assume without loss of generality that $c' \leq 1$. Let G be a d-degenerate H-free graph. Let $\varepsilon \stackrel{\text{def}}{=} c' d^{-\frac{2-a}{1+a}} < d^{-1/2}$ and $c \stackrel{\text{def}}{=} \min(c' c_2, \frac{c'}{48})$.

Applying Lemma 3.3 with parameter ε, either (3.2) or (3.3) holds. If (3.2) holds, then, as desired,

$$f(G) \geq \left(\frac{1}{2} + c_2 \varepsilon\right) m \geq \left(\frac{1}{2} + c d^{-\frac{2-a}{1+a}}\right) m.$$

Else (3.3) holds. Let $G_1, \ldots, G_k$ be the induced subgraphs satisfying the properties in Lemma 3.3, so that $G_1, \ldots, G_k$ are H'-free for all graphs H' obtained by removing a vertex from H, and

$$f(G) \geq \frac{m}{2} + \sum_{i=1}^{k}\left(f(G_i) - \frac{m(G_i)}{2}\right) \geq \frac{m}{2} + \sum_{i=1}^{k} c' \cdot m(G_i)^a.$$

For all i, we have

$$c' \cdot m(G_i)^a \overset{(*)}{\geq} \frac{c'\varepsilon}{8\varepsilon^{1+a}} \cdot n(G_i)^a \overset{(**)}{\geq} \frac{\varepsilon d}{8(c')^a} \cdot n(G_i) \overset{(+)}{\geq} \frac{\varepsilon d}{8} \cdot n(G_i),$$

where $(*)$ follows since $m(G_i) \geq n(G_i)/8\varepsilon$, $(**)$ follows since $n(G_i)^{a-1} \geq d^{a-1}$ and $\varepsilon^{1+a} = (c')^{1+a}d^{a-2}$, and $(+)$ follows since $c' \leq 1$. Hence, as $n(G_1)+\cdots+n(G_k) \geq \frac{m}{6d}$, we have

$$f(G) \geq \frac{m}{2} + \varepsilon d \sum_{i=1}^{k} \frac{n(G_i)}{8} \geq \frac{m}{2} + \frac{\varepsilon m}{48} \geq \left(\frac{1}{2} + cd^{-\frac{2-a}{1+a}}\right) \cdot m,$$

as desired. ∎

Remark 3.6. If Conjecture 1.5 is true, then applying Lemma 3.5 with an arbitrary H and $a = 3/4$ yields that $f(G) \geq \frac{m}{2} + cm \cdot d^{-5/7}$ for all d-degenerate H-free graphs.

Remark 3.7. By repeatedly applying Lemma 3.5 with results from [3], we obtain nontrivial surplus lower bounds for d-degenerate H-free graphs, given in the following table. Here, forest+1 means that H is some forbidden subgraph such that one vertex can be removed from H to give a forest, and forest+2 means that two vertices can be removed to give a forest. As an example, for all $s > 0$ there exists $c = c(s)$ such that any d-degenerate $K_{4,s}$-free graph G always satisfies $f(G) \geq \frac{m}{2} + cd^{-2/3}m$.

H	H'	H'-free surplus [3]	a	$\frac{2-a}{1+a}$	d-deg. H-free surplus
forest+1	forest	$c'm$	1	$\frac{1}{2}$	$cd^{-1/2}m$
forest+2	forest+1	$c'm^{4/5}$	$\frac{4}{5}$	$\frac{2}{3}$	$cd^{-2/3}m$
W_r (r odd)	C_{r-1}	$c'm^{r/(r+1)}$	$\frac{r}{r+1}$	$\frac{r+2}{2r+1}$	$cd^{-(r+2)/(2r+1)}m$
$K_{4,s}$	$K_{3,s}$	$c'm^{4/5}$	$\frac{4}{5}$	$\frac{2}{3}$	$cd^{-2/3}m.$

4. Max-Cut in K_r-free graphs. In this section we specialize Lemmas 3.3 and 3.5 to the case $H = K_r$ to prove Theorem 1.3. Let $\chi(G)$ denote the chromatic number of a graph G, the minimum number of colors needed to properly color the vertices of the graph so that no two adjacent vertices receive the same color. We first obtain a nontrivial upper bound on the chromatic number of a K_r-free graph G, giving an lower bound (Lemma 4.4) on the Max-Cut of K_r-free graphs. This lower bound was implicit in [2], but we provide a proof for completeness. The lower bound on the Max-Cut of general K_r-free graphs enables us to apply Lemma 3.3 to give a lower bound on the Max-Cut of d-degenerate K_r-free graphs per Theorem 1.3. The following well-known lemma gives a lower bound on the Max-Cut using the chromatic number.

LEMMA 4.1 (see, e.g., Lemma 2.1 of [2]). *Given a graph $G = (V, E)$ with m edges and chromatic number $\chi(G) \leq t$, we have $f(G) \geq (\frac{1}{2} + \frac{1}{2t})m$.*

Proof. Since $\chi(G) \leq t$, we can decompose V into independent subsets $V = V_1, \ldots, V_t$. Partition the subsets randomly into two parts containing $\lfloor \frac{t}{2} \rfloor$ and $\lceil \frac{t}{2} \rceil$ subsets V_i, respectively, to obtain a cut. The probability any edge is cut is $\lfloor t/2 \rfloor \cdot \lceil t/2 \rceil / \binom{t}{2} \geq (t+1)/2t$, so the result follows from linearity of expectation. ∎

LEMMA 4.2. *Let $r \geq 3$ and $G = (V, E)$ be a K_r-free graph on n vertices. Then,*

$$\chi(G) \leq 4n^{(r-2)/(r-1)}.$$

Proof. We proceed by induction on n. For $n \leq 4^{r-1}$, the statement is trivial as the chromatic number is always at most the number of vertices. Now assume $G = (V, E)$ has $n > 4^{r-1}$ vertices and that $\chi(G) \leq 4n_0^{(r-2)/(r-1)}$ for all K_r-free graphs on $n_0 \leq n-1$ vertices. The off-diagonal Ramsey number $R(r, s)$ satisfies $R(r, s) \leq \binom{r+s-2}{s-1} \leq s^{r-1}$ [14]. Hence, G has an independent set I of size $s = \lfloor n^{1/(r-1)} \rfloor$. The induced subgraph $G[V \setminus I]$ is K_r-free and has fewer than n vertices, so its chromatic number is at most $4(n-s)^{(r-2)/(r-1)}$. Hence, G has chromatic number at most

$$\begin{aligned} 1 + 4(n-s)^{(r-2)/(r-1)} &= 1 + 4n^{(r-2)/(r-1)} \left(1 - \frac{s}{n}\right)^{(r-2)/(r-1)} \\ &\overset{(*)}{\leq} 1 + 4n^{(r-2)/(r-1)} - 4n^{(r-2)/(r-1)} \cdot \frac{s}{3n} \overset{(**)}{<} 4n^{(r-2)/(r-1)}. \end{aligned} \tag{4.1}$$

In $(*)$, we used that $\frac{r-2}{r-1} \geq \frac{1}{2}$, that $\frac{s}{n} \leq \frac{1}{4}$, and that $(1-x)^a \leq 1 - \frac{x}{3}$ for $a \geq \frac{1}{2}$ and $x \leq \frac{1}{4}$. In $(**)$, we used that $s \geq 4$ and hence $\frac{3s}{4} < n^{1/(r-1)}$. This completes the induction, completing the proof. ∎

Remark 4.3. The upper bound on the off-diagonal Ramsey number $R(r, k^{1/(r-1)})$ has an extra logarithmic factor which suggests that the upper bound on $\chi(G)$ of Lemma 4.2 can be improved by a logarithmic factor with a more careful analysis.

LEMMA 4.4. *If G is a K_r-free graph with at most n vertices and m edges, then*

$$f(G) \geq \left(\frac{1}{2} + \frac{1}{8n^{(r-2)/(r-1)}}\right) m.$$

Proof. This follows immediately via Lemma 4.1 and Lemma 4.2. ∎

The above bounds allow us to prove Theorem 1.3.

Proof of Theorem 1.3. Let G be a d-degenerate K_r-free graph and $\varepsilon = d^{-1+\frac{1}{2r-4}}$. Let c_2 be the parameter given by Lemma 3.3. Let $c = \min(c_2, \frac{1}{388})$.

Applying Lemma 3.3 with parameter ε, one of two properties hold. If (3.2) holds, then

$$f(G) \geq \left(\frac{1}{2} + c_2\varepsilon\right) m \geq \left(\frac{1}{2} + cd^{-1+\frac{1}{2r-4}}\right) m \tag{4.2}$$

as desired. If (3.3) holds, there exist graphs $G_1, \ldots, G_k$ that are K_{r-1}-free with at most d vertices such that G_i has at least $\frac{n(G_i)}{8\varepsilon}$ edges, $n(G_1) + \cdots + n(G_k) \geq \frac{m}{6d}$, and

$$f(G) \geq \frac{m}{2} + \sum_{i=1}^{k} \left(f(G_i) - \frac{m(G_i)}{2}\right).$$

For all i, we have

$$\begin{aligned} f(G_i) - \frac{m(G_i)}{2} &\geq \frac{m(G_i)}{8n(G_i)^{(r-3)/(r-2)}} \\ &\geq \frac{n(G_i)}{64\varepsilon n(G_i)^{(r-3)/(r-2)}} \geq \frac{n(G_i)}{64\varepsilon d^{(r-3)/(r-2)}} = \frac{\varepsilon d n(G_i)}{64}. \end{aligned}$$

In the first inequality, we used Lemma 4.4. In the second inequality, we used that $m(G_i) \geq \frac{n(G_i)}{8\varepsilon}$. In the third inequality, we used that $n(G_i) \leq d$. Hence, as $d(n(G_1) + \cdots + n(G_k)) \geq \frac{m}{6}$, we have as desired that

$$f(G) \geq \frac{m}{2} + \sum_{i=1}^{k} \frac{\varepsilon d n(G_i)}{64} \geq \frac{m}{2} + \frac{\varepsilon m}{388} \geq \left(\frac{1}{2} + cd^{-1+\frac{1}{2r-4}}\right) \cdot m. \tag{4.3}$$ □

Remark 4.5. As we already mentioned in the introduction, we can improve the result of Theorem 1.3 in the case that $r = 4$ using Lemma 3.5. By Remark 3.7, as $H = K_4$ falls under the case forest+2, for an absolute $c > 0$, we have $f(G) \geq cmd^{-2/3}$ for d-degenerate K_4-free graphs G.

5. Concluding remarks. In this paper we presented an approach, based on semidefinite programming, to prove lower bounds on Max-Cut and used it to find large cuts in graphs with few triangles and in K_r-free graphs. A closely related problem of interest is bounding the Max-t-Cut of a graph, i.e., the largest t-colorable (t-partite) subgraph of a given graph. Our results imply good lower bounds for this problem as well. Indeed, by taking a cut for a graph G with m edges and surplus W, one can produce a t-cut for G of size $\frac{t-1}{t}m + \Omega(W)$ as follows. Let A, B be the two parts of the original cut. If $t = 2s$ is even, simply split randomly both A, B into s parts. If $t = 2s + 1$ is odd, then put every vertex of A randomly in the parts $1, \ldots, s$ with probability $2/(2s+1)$ and in the part $2s+1$ with probability $1/(2s+1)$. Similarly, put every vertex of B randomly in the parts $s + 1, \ldots, 2s$ with probability $2/(2s + 1)$ and in the part $2s + 1$ with probability $1/(2s + 1)$. An easy computation (which we omit here) shows that the expected size of the resulting t-cut is $\frac{t-1}{t}m + \Omega(W)$.

The main open question left by our work is Conjecture 1.4. Proving this conjecture will require some major new ideas. Even showing that any d-degenerate H-free graph with m edges has a cut with surplus at least $m/d^{1-\delta}$ for some fixed δ (independent of H) is out of reach of current techniques.

Acknowledgments. The authors thank anonymous reviewers for helpful feedback. The authors thank Jacob Fox and Matthew Kwan for helpful discussions and feedback. The authors thank Joshua Brakensiek for pointing out an error in an earlier draft of this paper. The authors thank Joshua Brakensiek and Yuval Wigderson for helpful comments on an earlier draft of the paper.

REFERENCES

[1] N. Alon, *Bipartite subgraphs*, Combinatorica, 16 (1996), pp. 301–311.
[2] N. Alon, B. Bollobás, M. Krivelevich, and B. Sudakov, *Maximum cuts and judicious partitions in graphs without short cycles*, J. Combin. Theory, Ser. B, 88 (2003), pp. 329–346.
[3] N. Alon, M. Krivelevich, and B. Sudakov, *Max Cut in H-Free graphs*, Combin. Prob. Comput., 14 (2005), pp. 629–647.
[4] N. Alon and E. Halperin, *Bipartite subgraphs of integer weighted graphs*, Discrete Math., 181 (1998), pp. 19–29.
[5] B. Bollobás and A. D. Scott, *Better bounds for max cut*, in Contemporary Combinatorics, Bolyai Society Mathematical Studies, B. Bollobás, ed., Springer, New York, 2002, pp. 185–246.
[6] C. Carlson, A. Kolla, R. Li, N. Mani, B. Sudakov, and L. Trevisan, *Lower bounds for max-cut in H-free graphs via semidefinite programming*, preprint, arXiv:1810.10044.
[7] D. Conlon, J. Fox, M. Kwan, and B. Sudakov, *Hypergraph cuts above the average*, Israel J. Math., 233 (2019), pp. 67–111.

[8] C. S. Edwards, *Some extremal properties of bipartite subgraphs*, Canad. J. Math., 3 (1973), pp. 475–485.

[9] C. S. Edwards, *An improved lower bound for the number of edges in a largest bipartite subgraph*, in Proceedings of Second Czechoslovak Symposium on Graph Theory, Prague, 1975, pp. 167–181.

[10] P. Erdős, *On even subgraphs of graphs*, Mat. Lapok, 18 (1967), pp. 283–288.

[11] P. Erdős, *Problems and results in graph theory and combinatorial analysis*, Proc. Fifth British Comb. Conf., 15 (1975), pp. 169–192.

[12] P. Erdős, R. Faudree, J. Pach, and J. Spencer, *How to make a graph bipartite*, J. Combin. Theory Ser. B, 45 (1988), pp. 86–98.

[13] P. Erdős, E. Győri, and M. Simonovits, *How many edges should be deleted to make a triangle-free graph bipartite?*, in Sets, Graphs and Numbers (Budapest, 1991), Colloq. Math. Soc. János Bolyai 60, North-Holland, Amsterdam, 1992, pp. 239–263.

[14] P. Erdős and G. Szekeres, *A combinatorial problem in geometry*, Compos. Math., 2 (1935), pp. 463–470.

[15] M. X. Goemans and D. P. Williamson, *Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming*, J. ACM, 42 (1995), pp. 1115–1145.

[16] J. Lin, Q. Zeng, and F. Chen, *Maximum cuts in graphs without wheels*, B. Aust. Math. Soc., 100 (2019), pp. 13–26.

[17] R. O'Donnell and Y. Wu, *An optimal SDP algorithm for Max-Cut, and equally optimal long code tests*, in Proceedings of the 40th Symposium on Theory Computing, 2008, pp. 335–344.

[18] S. Poljak and Zs. Tuza, *Bipartite subgraphs of triangle-free graphs*, SIAM J. Discrete Math., 7 (1994), pp. 307–313.

[19] J. Shearer, *A note on bipartite subgraphs of triangle-free graphs*, Rand. Struct. Alg., 3 (1992), pp. 223–226.

[20] B. Sudakov, *Making a K_4-free graph bipartite*, Combinatorica, 27 (2007), pp. 509–518.

[21] Q. Zeng and J. Hou, *Bipartite subgraphs of H-free graphs*, B. Aust. Math. Soc., 96 (2017), pp. 1–13.