

Complete minors and average degree – a short proof

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Abstract

We provide a short and self-contained proof of the classical result of Kostochka and of Thomason, ensuring that every graph of average degree d has a complete minor of order $\Omega(d/\sqrt{\log d})$.

Let $G = (V, E)$ be a graph with $|E|/|V| \geq d$. How large a complete minor are we guaranteed to find in G ? This classical question, closely related to the famed Hadwiger's conjecture, has been thoroughly studied over the last half a century. It is quite easy to see the answer is at least logarithmic in d . Mader [3] proved it is of order at least $d/\log d$. The right order of magnitude has been established independently by Kostochka [1, 2] and by Thomason [4] to be $d/\sqrt{\log d}$, its tightness follows by looking into a random graph. Finally, Thomason found in [5] the asymptotic value of this extremal function.

Here we provide a short and self-contained proof of the celebrated Kostochka-Thomason bound.

Theorem 1. *Let $G = (V, E)$ be a graph with $|E|/|V| \geq d$, where d is a sufficiently large integer. Then G contains a minor of the complete graph on at least $\frac{d}{10\sqrt{\ln d}}$ vertices.*

Throughout the proof we assume, whenever this is needed, that the parameters n and d are sufficiently large. To simplify the presentation we omit all floor and ceiling signs.

We need the following lemma proven by simple probabilistic arguments.

Lemma 2. *Let $H = (V, E)$ be a graph on at most n vertices with $\delta(H) \geq n/6$. Let $t \leq \frac{n}{\sqrt{\ln n}}$, and let $A_1, \dots, A_t \subset V$ with $|A_j| \leq \frac{n}{e^{\sqrt{\ln n}/3}}$. Then there is $B \subset V$ of size at most $3.1\sqrt{\ln n}$ such that B dominates all but at most $\frac{n}{e^{\sqrt{\ln n}/3}}$ vertices of V , and $B \setminus A_j \neq \emptyset$ for all $j = 1, \dots, t$.*

Proof. Set $s = 3.1\sqrt{\ln n}$ and choose s vertices of V independently at random with repetitions. Let B be the set of chosen vertices. Observe that for every vertex $v \in V$,

$$\Pr[N(v) \cap B = \emptyset] \leq \left(1 - \frac{d(v)}{n}\right)^s \leq e^{-\frac{sd(v)}{n}} \leq e^{-s/6}.$$

Hence the expected number of vertices not dominated by B is at most $ne^{-\sqrt{\ln n}/2}$ and by Markov's inequality, it is $\leq ne^{-\sqrt{\ln n}/3}$ with probability exceeding $1/2$ (with room to spare). Also, since $|V| > \delta(H) \geq n/6$, for every subset A_j

$$\Pr[B \subseteq A_j] = \left(\frac{|A_j|}{|V|}\right)^s \leq \left(\frac{6|A_j|}{n}\right)^s \leq 6^s e^{-s\sqrt{\ln n}/3} < \frac{1}{n}.$$

Therefore the probability that $B \setminus A_j \neq \emptyset$ for all j is at least $1 - 1/\sqrt{\ln n}$, and the desired result follows by the union bound. \square

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Proof of Theorem 1. Let $G = (V, E)$ be a graph with $|E|/|V| \geq d$. Let G' be a minor of G such that $|E(G')| \geq d|V(G')|$ and $|V(G')| + |E(G')|$ is minimal. If an edge e of G' is contained in t triangles then contracting e gives a minor of G with one vertex and $t + 1$ edges less. By the minimality of G' we have $t + 1 > d$, implying $t \geq d$. The minimality of G' also implies $|E(G')| = d|V(G')|$, hence G' has a vertex v of degree at most $2d$. Let H be the subgraph of G' induced by the neighborhood of v . Then H has at most $2d$ vertices and minimum degree at least d . Obviously a minor of H is a minor of G as well.

We now argue that H contains a $d/3$ -connected subgraph H_0 with $\delta(H_0) \geq 2d/3$. If H itself is $d/3$ -connected this holds for $H_0 = H$. Otherwise there is a partition $V(H) = A \cup B \cup S$, where $A, B \neq \emptyset$, $|S| < d/3$, and H has no edges between A and B . Assume without loss of generality $|A| \leq |B|$. Then $|A| \leq d$, and since $\delta(H) \geq d$, every vertex $v \in A$ has at least $2d/3$ neighbors in A , implying that every pair of vertices of A has at least $d/3$ common neighbors in A . Hence the induced subgraph $H_0 := H[A]$ is $d/3$ -connected, has at most $2d$ vertices and satisfies $\delta(H_0) \geq 2d/3$.

Set $i = 0$ and repeat the following iteration $d/10\sqrt{\ln d}$ times. Let $H_i = (V_i, E_i) \subseteq H_0$ be the current graph, and suppose A_1, \dots, A_{i-1} are subsets of V_i of cardinalities $|A_j| \leq \frac{2d}{e^{\sqrt{\ln(2d)}/3}}$ (representing the non-neighbors of the previously found branch sets B_j in V_i). We assume (and justify it later) that H_i is connected and has $\delta(H_i) > d/3$. Then the diameter of H_i is at most 14, since on any shortest path $P = (v_0, v_1, \dots)$ in H_i the vertices at positions divisible by three have pairwise disjoint neighborhoods and $|V(H_i)|/\delta(H_i) < 6$. Applying Lemma 2 with $H := H_i$, $n := 2d$, $t := i - 1$ and A_1, \dots, A_{i-1} we get a subset B_i of cardinality $|B_i| \leq 3.1\sqrt{\ln(2d)}$ as promised by the lemma. We now turn B_i into a connected set by adding some more vertices of H_i . If $H_i[B_i]$ has at least $r \geq 15$ connected components, then there is a vertex v in H_i connected to at least $\lfloor \frac{r\delta(H_i)}{|V(H_i)|} \rfloor \geq \lfloor r/6 \rfloor \geq r/10$ components of B_i . Adding v to B_i decreases the number of connected components by a factor of 0.9. Therefore we can reduce the number of connected components to 15 by adding at most $O(\ln |B_i|) = O(\ln \ln d)$ vertices. The last 15 components can be connected using shortest paths of length at most 14, adding at most $14 \cdot 13 < 200$ additional vertices. Altogether we obtain a connected subset B_i of cardinality $|B_i| \leq (3.1 + o(1))\sqrt{\ln(2d)}$, dominating all but at most $\frac{2d}{e^{\sqrt{\ln(2d)}/3}}$ vertices of V_i and having a neighbor outside every A_j — meaning connected to every previous B_j . We now update $V_{i+1} := V_i - B_i$, $A_i := V_{i+1} - N_{H_i}(B_i)$, and $A_j := A_j \cap V_{i+1}$, $j = 1, \dots, i - 1$, and finally increment $i := i + 1$ and proceed to the next iteration. The total number of vertices deleted in all iterations satisfies:

$$|\cup_i B_i| \leq \frac{d}{10\sqrt{\ln d}} \cdot (3.1 + o(1))\sqrt{\ln(2d)} < \frac{d}{3},$$

and since we started with the $d/3$ -connected graph H_0 with $\delta(H_0) \geq 2d/3$, we indeed have that at each iteration the graph H_i is connected and has $\delta(H_i) > d/3$.

After having completed all $d/10\sqrt{\ln d}$ iterations, we get a family of $d/10\sqrt{\ln d}$ branch sets B_i , all connected, and with an edge of H_0 between every pair of branch sets. Hence they form a complete minor of order $d/10\sqrt{\ln d}$ as promised. \square

References

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