

# Complete minors and average degree – a short proof

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## Abstract

We provide a short and self-contained proof of the classical result of Kostochka and of Thomason, ensuring that every graph of average degree  $d$  has a complete minor of order  $\Omega(d/\sqrt{\log d})$ .

Let  $G = (V, E)$  be a graph with  $|E|/|V| \geq d$ . How large a complete minor are we guaranteed to find in  $G$ ? This classical question, closely related to the famed Hadwiger’s conjecture, has been thoroughly studied over the last half a century. It is quite easy to see the answer is at least logarithmic in  $d$ . Mader [3] proved it is of order at least  $d/\log d$ . The right order of magnitude was established independently by Kostochka [1, 2] and by Thomason [4] to be  $d/\sqrt{\log d}$ , its tightness follows by considering random graphs. Finally, Thomason found in [5] the asymptotic value of this extremal function.

Here we provide a short and self-contained proof of the celebrated Kostochka–Thomason bound.

**Theorem 1.** *Let  $G = (V, E)$  be a graph with  $|E|/|V| \geq d$ , where  $d$  is a sufficiently large integer. Then  $G$  contains a minor of the complete graph on at least  $\frac{d}{10\sqrt{\ln d}}$  vertices.*

The constant  $1/10$  in the above statement is inferior to the best constant  $3.13\dots$  found by Thomason [5] (yet is better than the constants in [1, 2]); we did not make any serious attempt to optimize it in our arguments. The main point here is to give a short proof of the tight  $\Omega(d/\sqrt{\log d})$  bound for this classical extremal problem.

Throughout the proof we assume, whenever this is needed, that the parameters  $n$  and  $d$  are sufficiently large. To simplify the presentation we omit all floor and ceiling signs in several places. For a graph  $G = (V, E)$ , its minimum degree is denoted by  $\delta(G)$ , and for  $v \in V$  we use  $N_G(v)$  for the external neighborhood of  $v$  in  $G$ .

We need the following lemma proven by simple probabilistic arguments.

**Lemma 2.** *Let  $H = (V, E)$  be a graph on at most  $n$  vertices with  $\delta(H) \geq n/6$ . Let  $t \leq n/\sqrt{\ln n}$ , and let  $A_1, \dots, A_t \subset V$  with  $|A_j| \leq ne^{-\sqrt{\ln n}/3}$  for all  $1 \leq j \leq t$ . Then there is  $B \subset V$  of size  $|B| \leq 3.1\sqrt{\ln n}$*

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such that  $B$  dominates all but at most  $ne^{-\sqrt{\ln n}/3}$  vertices of  $V$ ,  $B \setminus A_j \neq \emptyset$  for all  $j = 1, \dots, t$ , and the induced subgraph  $G[B]$  has at most six connected components.

*Proof.* Set  $s = 3.1\sqrt{\ln n}$  and choose  $s$  vertices of  $V$  independently at random with repetitions. Let  $B$  be the set of chosen vertices. Observe that for every vertex  $v \in V$ ,

$$\Pr[N(v) \cap B = \emptyset] \leq \left(1 - \frac{d(v)}{n}\right)^s \leq e^{-\frac{sd(v)}{n}} \leq e^{-s/6}.$$

Hence the expected number of vertices not dominated by  $B$  is at most  $ne^{-s/6} < ne^{-3.1\sqrt{\ln n}/6} < ne^{-\sqrt{\ln n}/2}$ , and by Markov's inequality, it is at most  $ne^{-\sqrt{\ln n}/3}$  with probability exceeding  $1/2$  (with room to spare). Also, since  $|V| > \delta(H) \geq n/6$ , for every subset  $A_j$ ,

$$\Pr[B \subseteq A_j] = \left(\frac{|A_j|}{|V|}\right)^s < \left(\frac{6|A_j|}{n}\right)^s \leq 6^s e^{-s\sqrt{\ln n}/3} = 6^{\Theta(\sqrt{\log n})} e^{-3.1\ln n/3} < \frac{1}{n}.$$

Therefore the probability that  $B \setminus A_j \neq \emptyset$  for all  $j$  is at least  $1 - t/n \geq 1 - 1/\sqrt{\ln n}$ .

We now argue about the number of connected components in  $G[B]$ . Writing  $B = (v_1, \dots, v_s)$ , for  $1 \leq i \leq s$  let  $x_i$  be the random variable counting the number of indices  $1 \leq j \neq i \leq s$  for which  $v_j$  is a neighbor of  $v_i$ . Conditioning on  $v_i$ , we see that  $x_i$  is distributed as a binomial random variable with parameters  $s - 1$  and  $d(v_i)/|V| > 1/6$ . Hence invoking the standard Chernoff-type bound on the lower tail of the binomial distribution, we derive that  $\Pr[x_i < s/7] \leq e^{-\Theta(s)}$ . Applying the union bound over all  $1 \leq i \leq s$ , we conclude that with probability  $1 - o(1)$ , we have  $x_i \geq s/7$  for all  $i$ . Finally, observe that since  $s \ll \sqrt{|V|}$ , with probability  $1 - o(1)$  there are no repetitions in  $B$ , and hence  $d(v_i, B) = x_i \geq s/7$  for all  $1 \leq i \leq s$ . But then all connected components of  $G[B]$  are of size exceeding  $s/7$ , and therefore  $G[B]$  has at most six connected components.

Combining the above three estimates, the desired result follows.  $\square$

**Proof of Theorem 1.** Let  $G' = (V', E')$  be a minor of  $G$  such that  $|E'| \geq d|V'|$  and  $|V'| + |E'|$  is minimal. If an edge  $e$  of  $G'$  is contained in  $t$  triangles then contracting  $e$  gives a minor of  $G$  with one vertex and  $t + 1$  edges less. By the minimality of  $G'$  we have  $t + 1 > d$ , implying  $t \geq d$ . Hence for every edge  $e = (u, v) \in E(G')$ , the vertex  $u$  is connected by an edge of  $G'$  to at least  $d$  neighbors of  $v$ . The minimality of  $G'$  also implies  $|E'| = d|V'|$ , hence  $G'$  has a vertex  $v$  of degree at most  $2d$ . Let  $H$  be the subgraph of  $G'$  induced by  $N_{G'}(v)$ . Then  $H$  has at most  $2d$  vertices and minimum degree at least  $d$ . Obviously a minor of  $H$  is a minor of  $G$  as well.

We now argue that  $H$  contains a  $d/3$ -connected subgraph  $H_1$  with  $\delta(H_1) \geq 2d/3$ . If  $H$  itself is  $d/3$ -connected this holds for  $H_1 = H$ . Otherwise there is a partition  $V(H) = A \cup B \cup S$ , where  $A, B \neq \emptyset$ ,  $|S| < d/3$ , and  $H$  has no edges between  $A$  and  $B$ . Assume without loss of generality  $|A| \leq |B|$ . Then  $|A| \leq d$ , and since  $\delta(H) \geq d$ , every vertex  $v \in A$  has at least  $2d/3$  neighbors in  $A$ , implying that every pair of vertices of  $A$  has at least  $d/3$  common neighbors in  $A$ . Hence the induced subgraph  $H_1 := H[A]$  is  $d/3$ -connected, has at most  $2d$  vertices and satisfies  $\delta(H_1) \geq 2d/3$ .

Set  $i = 1$  and repeat the following iteration  $d/10\sqrt{\ln d}$  times. Let  $H_i = (V_i, E_i) \subseteq H_1$  be the current graph, and suppose  $A_1, \dots, A_{i-1}$  are subsets of  $V_i$  of cardinalities  $|A_j| \leq 2de^{-\sqrt{\ln(2d)}/3}$  (representing

the non-neighbors of the previously found branch sets  $B_j$  in  $V_i$ ). We assume (and justify it later) that  $H_i$  is connected and has  $\delta(H_i) > d/3$ . Then the diameter of  $H_i$  is at most 14, as on any shortest path  $P = (v_0, v_1, \dots)$  in  $H_i$  the vertices at positions divisible by three have pairwise disjoint neighborhoods. Since  $|V(H_i)|/\delta(H_i) < 6$ , the number of such neighborhoods is at most 5, and therefore any shortest path has at most 15 vertices. Applying Lemma 2 with  $H := H_i$ ,  $n := 2d$ ,  $t := i - 1$ , and  $A_1, \dots, A_{i-1}$  (for the initial step  $i = 1$  there are no  $A_j$ 's to plug into Lemma 2 — which of course does not hinder its application) we get a subset  $B_i$  of cardinality  $|B_i| \leq 3.1\sqrt{\ln(2d)}$  as promised by the lemma. We now turn  $B_i$  into a connected set by adding few vertices of  $H_i$  if necessary. Recall that  $H_i[B_i]$  has at most six connected components. Connecting one of them by shortest paths in  $H_i$  to all others and recalling that  $H_i$  has diameter at most 14, we conclude that by appending to  $B_i$  all the vertices of these paths we make it connected by adding to it at most  $13 \cdot 5 = 65$  vertices. Altogether we obtain a connected subset  $B_i$  of cardinality  $|B_i| \leq (3.1 + o(1))\sqrt{\ln(2d)}$ , dominating all but at most  $2de^{-\sqrt{\ln(2d)}/3}$  vertices of  $V_i$  and having a vertex outside every  $A_j$  (these properties are preserved under vertex addition when making  $B_i$  into a connected subset) — meaning connected to every previous  $B_j$ . We now update  $V_{i+1} := V_i - B_i$ ,  $A_i := V_{i+1} - N_{H_i}(B_i)$ , and  $A_j := A_j \cap V_{i+1}$ ,  $j = 1, \dots, i - 1$ , and finally increment  $i := i + 1$ , set  $H_i := H[V_i]$ , and proceed to the next iteration. The total number of vertices deleted in all iterations satisfies:

$$|\cup_i B_i| \leq \frac{d}{10\sqrt{\ln d}} \cdot (3.1 + o(1))\sqrt{\ln(2d)} < \frac{d}{3},$$

and since we started with the  $d/3$ -connected graph  $H_1$  with  $\delta(H_1) \geq 2d/3$ , we indeed have that at each iteration the graph  $H_i$  is connected and has  $\delta(H_i) > d/3$ .

After having completed all  $d/10\sqrt{\ln d}$  iterations, we get a family of  $d/10\sqrt{\ln d}$  branch sets  $B_i$ , all connected, and with an edge of  $H_1$  between every pair of branch sets. Hence they form a complete minor of order  $d/10\sqrt{\ln d}$  as promised.  $\square$

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