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Multicoloured extremal problems

Béla Bollobás,[a,b,1] Peter Keevash,[c] and Benny Sudakov[c,2]

[a] Department of Mathematical Sciences, University of Memphis, Memphis TN 38152, USA
[b] Trinity College, Cambridge CB2 1TQ, UK
[c] Department of Mathematics, Princeton University, Princeton, NJ 08544, USA

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Abstract

Many problems in extremal set theory can be formulated as finding the largest set system (or r-uniform set system) on a fixed ground set X that does not contain some forbidden configuration of sets. We shall consider multicoloured versions of such problems, defined as follows. Given a list of set systems, which we think of as colours, we call another set system multicoloured if for each of its sets we can choose one of the colours it belongs to in such a way that each set gets a different colour. Given an integer k and some forbidden configurations, the multicoloured extremal problem is to choose k colours with total size as large as possible subject to containing no multicoloured forbidden configuration.

Let f be the number of sets in the smallest forbidden configuration. For $k \leqslant f-1$ we can take all colours to consist of all subsets of X (or all r-subsets in the uniform case), and this is trivially the best possible construction. Even for $k \geqslant f-1$, one possible construction is to take $f-1$ colours to consist of all subsets, and the other colours empty. Another construction is to take all k colours to be equal to a fixed family that is as large as possible subject to not containing a forbidden configuration. We shall consider a variety of problems in extremal set theory, for which we show that one of these two constructions is always optimal. This was shown for the multicoloured version of Sperner's theorem by Daykin, Frankl, Greene and Hilton. We shall extend their result to some other Sperner problems, and also

E-mail addresses: bollobas@msci.memphis.edu b.bollobas@dpmms.cam.ac.uk (B. Bollobás), keevash@math.princeton.edu (P. Keevash), bsudakov@math.princeton.edu (B. Sudakov).

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prove multicoloured versions of the generalized Erdős–Ko–Rado theorem and the Sauer–Shelah theorem.

Keywords: Extremal set theory; Erdős–Ko–Rado; Sauer–Shelah

1. Introduction

Many problems in extremal set theory can be formulated as finding the largest set system (or r-uniform set system) on a fixed ground set X that does not contain some forbidden configuration of sets. For example Sperner's theorem determines the largest set system containing no pair of comparable sets, and the Erdős–Ko–Rado theorem determines the largest r-uniform set system containing no pair of disjoint r-tuples. For such an extremal problem we can formulate a multicoloured version as in [12]. Given a list of set systems, which we think of as *colours*, we call another set system *multicoloured* if for each of its sets we can choose one of the colours it belongs to in such a way that each set gets a different colour. Given an integer k and some forbidden configurations, the multicoloured extremal problem is to choose k colours with total size as large as possible subject to containing no multicoloured forbidden configuration.

Let f be the number of sets in the smallest forbidden configuration. For $k \leqslant f-1$ we can take all colours to consist of all subsets of X (or all r-subsets in the uniform case), and this is trivially the best possible construction. Even for $k \geqslant f-1$, one possible construction is to take $f-1$ colours to consist of all subsets, and the other colours empty. Another construction is to take all k colours to be equal to a fixed family that is as large as possible subject to not containing a forbidden configuration.

Multicoloured extremal graph theory problems (the case $r=2$) were studied in [12]. When the forbidden configuration is a complete graph, it was found that one of the two constructions described above is always optimal. There were indications that this phenomenon occurs quite generally, but an example was given when it does not occur (two triangles intersecting in a vertex). We shall consider three groups of problems in extremal set theory for which this phenomenon occurs.

1.1. Chains

We start by recalling Sperner's theorem [18] on the maximum size of an antichain. Let X be a set of size n. An *antichain* $\mathcal{A}$ is a set system on X for which there is no pair of sets $A, B \in \mathcal{A}$ with $A \subset B$. Any *level* $X^{(i)}$ (the collection of subsets of X of size i) is an antichain, and Sperner's theorem states that one of these is of maximum size. In fact, if n is even then $X^{(n/2)}$ is the unique maximum antichain, and if n is odd then $X^{((n-1)/2)}$ or $X^{((n+1)/2)}$ are the only maximum antichains.

More generally, Erdős [6] considered the problem of maximizing the size of a set system that contains no chain of length $t+1$ (the case $t=1$ is Sperner's theorem). He showed that the maximum is equal to the sum of the t largest binomial coefficients $\binom{n}{i}$, which we denote by $f(n,t)$. This is obtained by taking the t largest levels $X^{(i)}$. We prove the following multicoloured version of this result.

Theorem 1.1. *Let $\mathcal{G}_1, \cdots, \mathcal{G}_k$ be set systems on a set X of size n with no multicoloured chain of length $t+1$. Then $\sum_{i=1}^{k}|\mathcal{G}_i| \leqslant \max\{k \cdot f(n,t), t \cdot 2^n\}$ for $n > 4t^4$.*

Note that equality can be achieved. Depending on the value of k we may take all the colours consist of the t largest levels of X, or t colours equal to all subsets of X and the others empty. The case $t=1$ was solved by Daykin et al. [5] for any n. We also adapt their argument to prove this result for normal posets and give a generalization of the LYM inequality (see [2,14,15,20]), which may be of independent interest.

Call a partially ordered set P *normal* if it has a rank function and a set of chains $\mathcal{C}$, so that each chain in $\mathcal{C}$ contains an element of every rank, and each element of rank i belongs to the same number of chains in $\mathcal{C}$. Kleitman [10] showed that the property of being normal is equivalent to satisfying the LYM inequality, i.e., if A is an antichain then $\sum_i |A \cap P_i|/|P_i| \leqslant 1$, where P_i denotes the set of elements of rank i. There are many interesting normal posets, of which we list a few below.

- The poset of all subsets of X (ordered by inclusion): if we take the set $\mathcal{C}$ to consist of all maximal chains then every set of size i belongs to $i!(n-i)!$ chains in $\mathcal{C}$.
- The set of all subspaces of some vector/affine/projective space.
- The *cubical poset*: all faces of the Boolean cube.
- The *function poset*: all partially defined functions between two fixed sets, where $f \leqslant g$ if g agrees with f on its domain of definition.

For $A \subset P$ we let $C_t(A)$ denote all elements x for which there is a chain that contains x and t elements of A.

Lemma 1.2. *Suppose P is a normal poset and $A \subset P$ contains no chain of length $t+1$. Then*

$$|C_t(A)| \geqslant \left(\sum_i \frac{|A \cap P_i|}{|P_i|} - (t-1) \right) |\, P \,|.$$

Note that this is a generalization of the LYM inequality, as with $t=1$ we have $\sum_i |A \cap P_i|/|P_i| \leqslant |C_t(A)|/|P| \leqslant 1$. We use this lemma to generalize the result of [5] to normal posets.

Theorem 1.3. *Let $G_1, \ldots, G_k$ be subsets of a normal poset P for which there are no two distinct comparable elements x, y such that there is $i \neq j$ with $x \in G_i$ and $y \in G_j$. Choose m so that $|P_m|$ is as large as possible. Then $\sum_{i=1}^{k}|G_i| \leqslant \max\{k|P_m|, |P|\}$.*

Depending on k, equality can either be achieved by taking all G_i equal to P_m, or by taking one G_i equal to P and the others empty.

1.2. Matchings

The Erdős–Ko–Rado theorem [8] states that if $n \geqslant 2r$ and $\mathcal{A} \subset [n]^{(r)}$ is *intersecting* (i.e. $A, B \in \mathcal{A} \Rightarrow A \cap B \neq \emptyset$) then $|\mathcal{A}| \leqslant \binom{n-1}{r-1}$, with equality only when $\mathcal{A}$ consists of all r-sets containing some fixed element. We prove the following multicoloured version of this question.

Theorem 1.4. *Let $\mathcal{G}_1, \ldots, \mathcal{G}_k$ be r-uniform set systems on a set X of size $n \geqslant 2r$ with no multicoloured pair of disjoint sets. Then* $\sum_{i=1}^{k} |\mathcal{G}_i| \leqslant \max\left\{k\binom{n-1}{r-1}, \binom{n}{r}\right\}$.

Depending on k, equality can either be achieved by taking all $\mathcal{G}_i$ equal to all r-sets containing some fixed element, or by taking one $\mathcal{G}_i$ equal to $X^{(r)}$ and the others empty.

Next we consider the more general problem in which the forbidden configuration is a matching of size $t+1$, for some $t \geqslant 1$. This question was considered by Erdős [7], who showed that for n sufficiently large, a family $\mathcal{A} \subset [n]^{(r)}$ with no matching of size $t+1$ satisfies $|\mathcal{A}| \leqslant \binom{n}{r} - \binom{n-t}{r}$, with equality when $\mathcal{A}$ consists of all r-sets that hit some particular set of size t. Bollobás et al. [4] showed that this is true for $n > 2r^3(t+1)$. We prove the following multicoloured version of this result.

Theorem 1.5. *Let $\mathcal{G}_1, \ldots, \mathcal{G}_k$* be *r-uniform set systems on a set X of size n with no multicoloured matching of size $t+1$. Suppose $n > 4r^3 t$. Then* $\sum_{i=1}^{k} |\mathcal{G}_i| \leqslant \max\left\{k\left(\binom{n}{r} - \binom{n-t}{r}\right), t\binom{n}{r}\right\}$.

Depending on k, equality can either be achieved by taking all $\mathcal{G}_i$ equal to all r-sets hitting some particular set of size t, or by taking t of the $\mathcal{G}_i$ equal to $X^{(r)}$ and the others empty.

1.3. Shattered sets

First we recall the result of Sauer [16], Perles, Shelah [17], Vapnik and Chervonenkis [19], that is frequently referred to as the Sauer–Shelah theorem. Let $\mathcal{A}$ be a family on a set X with n elements. A set $Y \subset X$ is said to be *shattered* by $\mathcal{A}$ if for every $Z \subset Y$ there is a set A in $\mathcal{A}$ such that $A \cap Y = Z$. The theorem states that if $\mathcal{A}$ is a family that does not shatter any set of size r then $|\mathcal{A}| \leqslant \sum_{i=0}^{r-1} \binom{n}{i}$. Note that equality can occur when $\mathcal{A}$ consists of all subsets of X with size at most $r-1$. We prove the following multicoloured version of this theorem.

Theorem 1.6. *Let $\mathcal{G}_1, \ldots, \mathcal{G}_k$ be r-uniform set systems on a set X of size n with no multicoloured collection of sets that shatter a set of size r, and suppose $n > 10^{4r}$. Then*

$$\sum_{i=1}^{k} |\mathcal{G}_i| \leqslant \max\left\{k \sum_{i=0}^{r-1} \binom{n}{i}, (2^r - 1)2^n\right\}.$$

Depending on k, equality can either be achieved by taking all $\mathcal{G}_i$ equal to the subsets of X of size at most $r-1$, or by taking 2^r-1 of the $\mathcal{G}_i$ equal to all subsets of X and the others empty.

The rest of this paper is organised as follows. In the next section we make some preliminary observations that give some useful properties that we can use when proving our results. The proofs of our results on multicoloured chains are in Section 3. Multicoloured matchings are in Section 4, and the multicoloured Sauer-Shelah theorem is in Section 5. The last section contains some concluding remarks.

2. Preliminaries

The purpose of this section is to describe some useful properties that we can assume, without loss of generality, when proving our results. The proofs are similar to those in [12], but they are short, so we include them for the convenience of the reader. The following key lemma is used throughout the paper.

Lemma 2.1. *Suppose $\mathcal{G}_1, \ldots, \mathcal{G}_k$ are set systems that do not contain a multicoloured copy of some set system $\mathcal{F}$. Then there exist set systems $\mathcal{H}_1, \ldots, \mathcal{H}_k$ satisfying*

(1) *For any set A we have $|\{i : A \in \mathcal{H}_i\}| = |\{i : A \in \mathcal{G}_i\}|$,*
(2) $\mathcal{H}_1 \subset \cdots \subset \mathcal{H}_k$,
(3) *$\mathcal{H}_1, \ldots, \mathcal{H}_k$ do not contain a multicoloured copy of $\mathcal{F}$.*

Proof. For any $1 \leqslant i < j \leqslant k$ we consider the operation of replacing $\mathcal{G}_i$ by $\mathcal{G}_i \cap \mathcal{G}_j$ and $\mathcal{G}_j$ by $\mathcal{G}_i \cup \mathcal{G}_j$. This does not change the number of times any set appears, so property (1) holds. Suppose, for a contradiction, that this operation creates a multicoloured copy of $\mathcal{F}$. This copy of $\mathcal{F}$ was not originally multicoloured, so must contain a set $A \in \mathcal{G}_i \cup \mathcal{G}_j$ and a set $B \in \mathcal{G}_i \cap \mathcal{G}_j$. We may assume $A \in \mathcal{G}_i$. Then in the original sequence we can colour A with colour i and B with colour j, so this $\mathcal{F}$ is in fact multicoloured originally, a contradiction. This proves condition (3). Repeatedly applying the above transformation of colours, after a finite number of steps, we obtain a sequence of set systems in which (2) is satisfied. This completes the proof. □

This lemma shows that in any multicoloured extremal problem we can assume that the colours are nested. It is convenient to reformulate our problem as follows. We say that $\mathcal{G}$ is a *k-family* on X if it is a multiset whose elements are subsets of X, each appearing with multiplicity at most k. If we have k set systems $\mathcal{G}_1, \ldots, \mathcal{G}_k$ then the multiset sum $\mathcal{G}_1 + \cdots + \mathcal{G}_k$ is a k-family. Conversely any k-family $\mathcal{G}$ has a unique partition into k nested colours: if the colours are $\mathcal{G}_1 \subset \cdots \subset \mathcal{G}_k$ then $\mathcal{G}_i$ consists of all sets of multiplicity at least $k+1-i$. We say that $\mathcal{G}$ contains a multicoloured copy of $\mathcal{F}$ if its nested k-colouring does. Then we can reformulate our extremal problem as finding the largest k-family with no multicoloured forbidden configuration.

The following Hall-type condition characterises this property by reference only to multiplicities. (The proof is immediate from Hall's theorem.)

Proposition 2.2. *Let $\mathcal{G}$ be a k-family. Then $\mathcal{F}$ is not multicoloured in $\mathcal{G}$ if and only if there is some integer w, for which at least $w+1$ sets in $\mathcal{F}$ have multiplicity at most w in $\mathcal{G}$.*

The following proposition gives some further properties of an optimal k-family.

Proposition 2.3. *Suppose $\mathcal{G}$ is a k-family with no multicoloured copy of $\mathcal{F}$, where $k \geqslant |\mathcal{F}|$. Then there is a k-family $\mathcal{H}$ such that*

(1) $\mathcal{H} \supset \mathcal{G}$,
(2) *$\mathcal{H}$ contains no multicoloured copy of $\mathcal{F}$,*
(3) *Every set in $\mathcal{H}$ either has multiplicity at most $|\mathcal{F}|-1$ or exactly equal to k,*
(4) *The sets in $\mathcal{H}$ of multiplicity k form a family with no subfamily isomorphic to $\mathcal{F}$.*

Proof. Form $\mathcal{H}$ from $\mathcal{G}$ by the following rule: increase any set of multiplicity at least $|\mathcal{F}|$ to multiplicity k. Then (1) and (3) hold by construction. Consider a copy of $\mathcal{F}$ in $\mathcal{G}$. It is not multicoloured, so by Proposition 2.2, there is an integer w and a set of $w+1$ sets $\mathcal{W} \subset \mathcal{F}$ so that each set of $\mathcal{W}$ has multiplicity at most w in $\mathcal{G}$. Since $\mathcal{W} \subset \mathcal{F}$ we have $w \leqslant |\mathcal{F}|-1$, so the above rule has no effect on sets of $\mathcal{W}$, i.e. they have the same multiplicities in $\mathcal{H}$. It follows that $\mathcal{H}$ contains no multicoloured copy of $\mathcal{F}$, proving (2). As $k \geqslant |\mathcal{F}|$, (4) is immediate. □

In the problems that we consider, we shall show that there is some critical multiplicity k_c that divides two regimes of behaviour for the size of the largest k-family $\mathcal{G}$ not containing a multicoloured copy of $\mathcal{F}$. For $|\mathcal{F}| \leqslant k \leqslant k_c$, the size of $\mathcal{G}$ is at most that of $|\mathcal{F}|-1$ copies of the system consisting of all subsets of X (or all r-subsets in the uniform case.) For $k > k_c$, the size of $\mathcal{G}$ is at most that of k copies of a fixed set system of maximum size not containing $\mathcal{F}$. Note that if we can prove this statement for $k = k_c$ and $k = k_c + 1$ then it is true for all k. This is clear for $|\mathcal{F}| \leqslant k \leqslant k_c$. For $k > k_c$ we have the following easy induction argument. Let m be the size of the largest set system not containing $\mathcal{F}$. The $k-1$ largest colours of $\mathcal{G}$ form a $(k-1)$-family with no multicoloured $\mathcal{F}$, so have size at most $(k-1)m$ by induction hypothesis. Therefore $\mathcal{G}$ has size at most $\frac{k}{k-1} \cdot (k-1)m = km$.

Finally we remark that if we can show that the only k-family with nested colours achieving maximum size among k-families not containing a multicoloured copy of $\mathcal{F}$ is one of the two examples mentioned above, then in fact this is the only k-family achieving maximum size, even without the assumption that the colours are nested. This follows in many cases from the proof of Lemma 2.1. Suppose first that the only k-family with nested colours achieving maximum size has all colours equal. Then starting with any k-family achieving maximum size, we can apply some sequence of intersection/union transformations until the colours are nested, and then all colours are equal. It is clear that a k-family in which all colours are equal cannot be obtained by these transformations from any different k-family, so in fact any k-family achieving maximum size must have all colours equal. Now suppose that the only k-family with nested colours achieving maximum size has $|\mathcal{F}|-1$ colours equal to the system consisting of all subsets of X (or all r-subsets in the uniform case), and all other colours empty. If there is a different non-nested k-family achieving maximum size, we can apply some sequence of intersection/union transformations to end up with this nested configuration. One step before the nested configuration we have $|\mathcal{F}|-2$ colours consisting of all subsets, 2 colours which partition all subsets, and the remaining colours empty. It will be clear in all of our examples that we can choose a copy of $\mathcal{F}$ that uses both of the 2 colours that partition all subsets. The other sets of this copy of $\mathcal{F}$ can be arbitrarily assigned different colours from the $|\mathcal{F}|-2$ that are complete, so it is multicoloured. This contradiction shows that there is no non-nested k-family achieving maximum size, as required.

3. Multicoloured chains

In this section we shall find the size of the largest k-family $\mathcal{G}$ not containing a multicoloured chain of length $t+1$. The case $t=1$ was solved by Daykin et al. [5] for any n. We also adapt their argument to prove this result for normal posets and give a generalization of the LYM inequality of [2,15,20] and [14].

Recall that a partially ordered set P is *normal* if it has a rank function and a set of chains $\mathcal{C}$, so that each chain in $\mathcal{C}$ contains an element of every rank, and each element of rank i belongs to same number of chains in $\mathcal{C}$. Kleitman [10] showed that the property of being normal is equivalent to satisfying the LYM inequality, i.e. if A is an antichain then $\sum_i |A\cap P_i|/|P_i|\leqslant 1$, where P_i is the set of elements with rank i. It follows easily that a normal poset has the *Sperner property*, i.e. the size of the largest antichain is achieved by P_i for some i. For $A\subset P$ we let $C_t(A)$ denote the set of all elements x for which there is a chain in $\mathcal{C}$ that contains x and t elements of A.

We shall first prove Lemma 1.2, which states that if A is a subset of a normal poset P that contains no chain of length $t+1$ then $|C_t(A)|\geqslant \left(\sum_i \frac{|A\cap P_i|}{|P_i|}-(t-1)\right)|P|$.

Proof of Lemma 1.2. Choose a chain C in $\mathcal{C}$ uniformly at random. As x ranges over elements of P_i, the events 'C contains x' are mutually disjoint and equiprobable, so each has probability $1/|P_i|$. If $F\subset P_i$ then C hits F with probability $|F|/|P_i|$. Let D be the complement of $C_t(A)$ in P. Define a random variable $Z=|A\cap C|+\chi(C \text{ hits } D)$, where χ denotes the characteristic function of an event. Then $Z\leqslant t$ by definition of $C_t(A)$. Taking expectations we get

$$t\geqslant \sum_{x\in A}\mathbb{P}(x\in C)+\mathbb{P}(C \text{ hits } D)=\sum_i \frac{|A\cap P_i|}{|P_i|}+\mathbb{P}(C \text{ hits } D).$$

By averaging there is some i for which $|D\cap P_i|/|P_i|\geqslant |D|/|P|$. The probability that C hits D is at least the probability it hits $D\cap P_i$, which is $|D\cap P_i|/|P_i|\geqslant |D|/|P|$. Therefore $\sum_i \frac{|A\cap P_i|}{|P_i|}+|D|/|P|\leqslant t$, which gives the result. □

This implies Theorem 1.3, which is the following generalization of the result of [5] to normal posets. Let $G_1, \dots, G_k$ be subsets of a normal poset P for which there are no two comparable elements x, y such that there is $i \neq j$ with $x \in G_i$ and $y \in G_j$. Choose m so that $|P_m|$ is as large as possible. Then $\sum_{i=1}^{k} |G_i| \leqslant \max\{k|P_m|, |P|\}$.

Proof of Theorem 1.3. Let A be the set of elements of multiplicity $\geqslant 2$ and B those with multiplicity 1. Note that A contains no chain of length 2, as this would certainly be multicoloured. Similarly B is disjoint from $C_1(A)$, so by Lemma 1.2 $|B| \leqslant |P| \cdot (1 - |A|/|P_m|)$. Therefore $\sum_{i=1}^{k} |G_i| \leqslant k|A| + |B| \leqslant (k - |P|/|P_m|)|A| + |P|$. Now we see that the critical multiplicity is $k_c = \lfloor |P|/|P_m| \rfloor$. For $k \leqslant k_c$ we should take $|A| = 0$, which gives a maximum of $|P|$. For $k > k_c$ should take A to be as large as possible, i.e. $|A| = |P_m|$ by the Sperner property. Then B is empty, and the maximum is $k|P_m|$. This proves the result. $\square$

It is clear from the proof that equality can only be achieved when either all colours are equal to some antichain of maximum size, or one colour is equal to P and the others are empty.

Our next result is Theorem 1.1, in which we consider the case of general t, and return to the case when the poset P is that of all subsets of a set X. Let $f(n,t)$ denote the sum of the t largest binomial coefficients $\binom{n}{i}$. We show that if $\mathcal{G}$ is a k-family on a set X of size n with no multicoloured chain of length $t+1$, then $|\mathcal{G}| \leqslant \max\{k \cdot f(n,t), t \cdot 2^n\}$ for $n > 4t^4$. Before giving the proof, we recall the defect form of Hall's theorem (see, e.g., [3, p. 7]).

Proposition 3.1. *Let H be a bipartite graph with bipartition (X, Y). If H has no matching of size $m+1$ then there is $U \subset X$ for which $|N(U)| \leqslant |U| + m - |X|$.*

Proof of Theorem 1.1. As noted in the previous section (see Proposition 2.3), we can assume that all multiplicities are equal to k or at most t. Also, it suffices to prove the theorem in the case when k is either k_c or $k_c + 1$, where $k_c = \lfloor 2^n t / f(n,t) \rfloor$. We can crudely bound this as

$$k_c < \frac{2^n t}{t\binom{n}{\lfloor (n-t)/2 \rfloor}} < 2\sqrt{n}.$$

Let $m(A)$ denote the multiplicity of A in $\mathcal{G}$ and define weights $w(A) = |A|!^{-1} (n - |A|!)^{-1}$. Then

$$|\mathcal{G}| = \sum_{A} m(A) = \sum_{A} m(A) \sum_{\mathcal{C}\,:\, A \in \mathcal{C}} w(A) = \sum_{\mathcal{C}} \sum_{A \in \mathcal{C}} m(A) w(A),$$

where the sum is taken over all maximal chains $\mathcal{C}$. Put the set of possible weights in the order $w_1 \geqslant \cdots \geqslant w_{n+1}$ and let $W_i = w_1 + \cdots + w_i$.

We claim that for each maximal chain $\mathcal{C}$, $\sum_{A\in\mathcal{C}} m(A)w(A) \leqslant \max\{kW_t, tW_{n+1}\}$. This suffices to prove the theorem, as then

$$|\mathcal{G}| = \sum_{\mathcal{C}} \sum_{A\in\mathcal{C}} m(A)w(A) \leqslant n! \max\{kW_t, tW_{n+1}\} = \max\{k \cdot f(n,t), t \cdot 2^n\}.$$

Consider a maximal chain $\mathcal{C}$. Let H be the bipartite graph with bipartition $(\mathcal{C}, [k])$, where (C, i) is an edge iff the set C is in colour i. Since $\mathcal{C}$ contains no multicoloured chain of length $t+1$, there is no matching of size $t+1$ in H. Then Proposition 3.1 tells us that there is some i for which $\mathcal{C}$ has at least $n+1-t+i$ sets with multiplicity $\leqslant i$. This leaves $t-i$ multiplicities which could be as large as k. To maximize the weighted sum the largest multiplicities should have highest weight, so $\sum_{A\in\mathcal{C}} m(A)w(A) \leqslant k(w_1 + \cdots + w_{t-i}) + i(w_{t-i+1} + \cdots + w_{n+1}) = (k-i)W_{t-i} + iW_{n+1}$.

To prove the claim it suffices to show that $(k-i)W_{t-i} + iW_{n+1} \leqslant \frac{t-i}{t} \cdot kW_t + \frac{i}{t} \cdot tW_{n+1}$ (which is clearly at most $\max\{kW_t, tW_{n+1}\}$). Rearranging, we need to show that $(k-i)tW_{t-i} \leqslant (t-i)kW_t$. For $i=0$ or t we have equality. Otherwise, using the estimates $W_{t-i} \leqslant (t-i)w_1$ and $W_t \geqslant tw_t$ it suffices to show that $(k-i)w_1 \leqslant kw_t$. To write these expressions more explicitly we need to divide into cases depending on the parities of n and t. For brevity we shall just deal with the case when n and t are even; the other cases are similar. Suppose then that $n = 2m$ and $t = 2s$. Then $w_1 = (1/m!)^2$ and $w_t = 1/(m+s)!(m-s)!$. Now, recalling that $n > 4t^4$, $i \geqslant 1$ and $k < 2\sqrt{n}$, we get

$$w_1/w_t = \prod_{j=1}^{s} \frac{m+j}{m-s+j} < \left(\frac{m}{m-s}\right)^s = \left(1 + \frac{t}{n-t}\right)^{t/2}$$
$$< 1 + \frac{t^2}{2(n-t)} \sum_{j=0}^{\infty} \left(\frac{t^2}{n}\right)^j < 1 + t^2/n < \frac{k}{k-1} \leqslant \frac{k}{k-i},$$

as required. □

We have shown that $\sum_{A\in\mathcal{C}} m(A)w(A) \leqslant \max\{kW_t, tW_{n+1}\}$ for any maximal chain $\mathcal{C}$. It is clear from the proof that equality can only occur when either the sets in the chain with weights $w_1, \cdots, w_t$ have multiplicity k and the others have multiplicity 0, or all sets in the chain have multiplicity t. We need equality to occur for every chain in order to achieve equality in the theorem. Note that even if $kW_t = tW_{n+1}$ all of the chains must have the same type of configuration. For if one chain has all sets of multiplicity t then in particular the empty set has multiplicity t. Since all chains contain the empty set, now there can be no chain with t sets of multiplicity k and the others of multiplicity 0. Therefore there are only two configurations in which equality can occur. One is that in which all sets have multiplicity t. The other is that in which any set with positive multiplicity has multiplicity k, so the sets of multiplicity k form a maximum size set system with no chain of length $t+1$.

4. Multicoloured matchings

We first consider the problem of finding the maximum size of an r-uniform k-family with no multicoloured pair of disjoint sets. This is answered by Theorem 1.4, which states that for $n \geqslant 2r$ the maximum size is $\max\{k\binom{n-1}{r-1}, \binom{n}{r}\}$. Our solution will be a simple adaptation of Katona's proof of the Erdős–Ko–Rado theorem by the permutation method.

Proof of Theorem 1.4. Fix a cyclic ordering of $[n]$, i.e. a labelling of n points on a circle by $[n]$. We shall count (with multiplicity) the number of sets of $\mathcal{G}$ that appear as consecutive elements in the ordering. Of these sets, let $\mathcal{A}$ be the set of those of multiplicity $\geqslant 2$ and let $\mathcal{B}$ be those of multiplicity 1. Write $|\mathcal{A}| = a$. If $a = 0$ then this ordering contributes $|\mathcal{B}| \leqslant n$. Note that each pair of sets in $\mathcal{A}$ intersect, so the sets in $\mathcal{A}$ have a common intersection I, where $1 \leqslant |I| \leqslant r - a + 1$. Each set in $\mathcal{B}$ must contain a point of I, to avoid a multicoloured pair of disjoint sets. There are $r - 1 + |I|$ sets that contain a point of I, of which a belong to $\mathcal{A}$, so $|\mathcal{B}| \leqslant r - 1 + |I| - a \leqslant 2(r - a)$. Therefore, the ordering contributes at most $k|\mathcal{A}| + |\mathcal{B}| \leqslant ka + 2(r - a) \leqslant kr$. There are $(n - 1)!$ distinguishable cyclic orderings of $[n]$, and each set appears consecutively in $r!(n - r)!$ of them, so summing over all cyclic orderings gives

$$|\mathcal{G}| \leqslant \max\{kr, n\} \cdot \frac{(n-1)!}{r!(n-r)!} = \max\left\{k\binom{n-1}{r-1}, \binom{n}{r}\right\}. \qquad \square$$

To achieve equality, every cyclic ordering must either have all consecutive sets appearing with multiplicity 1, or all consecutive sets containing some particular point appearing with multiplicity k and the others with multiplicity 0. It follows that there is only one of these possibilities that applies to every cyclic ordering. For if not we can find two cyclic orderings that differ by a transposition in which one has all sets appearing with multiplicity 1 and the other has some sets with multiplicity k and others with multiplicity 0. Since they differ by a transposition they share at least one consecutive set, so this is impossible. It follows that there are only two constructions that can achieve equality. One construction is to take all sets with multiplicity one. The other construction has all multiplicites equal to 0 or k. Then the sets with multiplicity k form a maximum size family with no pair of disjoint r-tuples, i.e. they all contain some fixed point.

Now we consider the more general problem in which the forbidden configuration is a matching of size $t + 1$, for some $t \geqslant 1$. This question was considered by Erdős [7], who showed that for n sufficiently large, a family $\mathcal{A} \subset [n]^{(r)}$ with no matching of size $t + 1$ satisfies $|\mathcal{A}| \leqslant \binom{n}{r} - \binom{n-t}{r}$, with equality when $\mathcal{A}$ consists of all r-sets that hit some particular set of size t. Bollobás et al. [4] showed that this is true for $n > 2r^3(t + 1)$.

We prove Theorem 1.5, which is the following multicoloured version of this result. Let $\mathcal{G}$ be an r-uniform k-family on a set X of size n with no multicoloured matching of size $t + 1$. If $n > 4r^3 t$ then $|\mathcal{G}| \leqslant \max\{k(\binom{n}{r} - \binom{n-t}{r}),\ t\binom{n}{r}\}$.

Proof of Theorem 1.5. We shall argue by induction on t, starting from the case $t = 1$, which we have already proved as Theorem 1.4. Suppose $\mathcal{G}$ is an r-uniform k-family with $|\mathcal{G}| > \max\{k(\binom{n}{r} - \binom{n-t}{r}), t\binom{n}{r}\}$. By Section 2 we can suppose that the colours of $\mathcal{G}$ are nested, and all sets have multiplicity at most t or equal to k. We also recall from that section that it is sufficient to prove the result when k is equal to k_c or $k_c + 1$, where

$$k_c = \left\lfloor \frac{t\binom{n}{r}}{\binom{n}{r} - \binom{n-t}{r}} \right\rfloor. \tag{1}$$

Define the *degree* $d(x)$ of an element $x \in X$ to be the number of sets of $\mathcal{G}$ containing x, counted with multiplicity. Choose $x \in X$ of maximum degree and let $\mathcal{G}'$ be the simply k-coloured r-uniform multifamily on $X \backslash x$ obtained by taking those sets of $\mathcal{G}$ that do not contain x. We claim that $\mathcal{G}'$ contains a multicoloured matching of size t. This will follow from our induction hypothesis if we can prove the following slightly technical claim.

Claim 4.1. $|\mathcal{G}'| > \max\left\{k\left(\binom{n-1}{r} - \binom{n-t}{r}\right),\ (t-1)\binom{n-1}{r}\right\}$.

Proof of claim. Since $d(x) \leqslant k\binom{n-1}{r-1}$ we have $|\mathcal{G}'| \geqslant |\mathcal{G}| - k\binom{n-1}{r-1}$. Since $|\mathcal{G}| > k(\binom{n}{r} - \binom{n-t}{r})$ this immediately gives

$$|\mathcal{G}'| > k\left(\binom{n-1}{r} - \binom{n-t}{r}\right). \tag{2}$$

To show that $|\mathcal{G}'| > (t-1)\binom{n-1}{r}$ we divide into cases depending on whether $k = k_c$ or $k = k_c + 1$.

If $k = k_c + 1$ then Eq. (1) gives $k > \frac{t\binom{n}{r}}{\binom{n}{r} - \binom{n-t}{r}}$. We claim that

$$\frac{t\binom{n}{r}}{\binom{n}{r} - \binom{n-t}{r}} > \frac{(t-1)\binom{n-1}{r}}{\binom{n-1}{r} - \binom{n-t}{r}}. \tag{3}$$

Cross-multiplying this inequality and rearranging would give $\binom{n}{r}\binom{n-1}{r} > \binom{n-t}{r}\left(t\binom{n}{r} - (t-1)\binom{n-1}{r}\right)$, so it suffices to show that

$$\binom{n}{r} > \left(1 + \frac{tr}{n-r}\right)\binom{n-t}{r}.$$

This in turn would follow from $\left(1 + \frac{t}{n-t}\right)^r > 1 + \frac{tr}{n-r}$. Now

$$\begin{aligned}\left(1 + \frac{t}{n-t}\right)^r - \left(1 + \frac{tr}{n-r}\right) &> \frac{rt}{n-t} + \binom{r}{2}\left(\frac{t}{n-t}\right)^2 - \frac{tr}{n-r} \\ &= \frac{rt}{2(n-t)^2(n-r)}((r-1)t(n-r) - 2(r-t)(n-t))\end{aligned}$$

and $(r-1)t(n-r)-2(r-t)(n-t)=((t-2)r+t)n-(r^2t+2t^2-3rt)>0$ as $t\geqslant 2$ and $n>4r^3t$. This proves Eq. (3). Combining this with Eq. (2) we have

$$|\mathcal{G}'|>k\left(\binom{n-1}{r}-\binom{n-t}{r}\right)>(t-1)\binom{n-1}{r},$$

as required.

Now we deal with the case $k=k_c$, where by Eq. (1) we have $k\leqslant\frac{t\binom{n}{r}}{\binom{n}{r}-\binom{n-t}{r}}$. We claim that

$$k+1-t<n/r. \tag{4}$$

Using the bound on k, and the estimate

$$\frac{\binom{n}{r}-\binom{n-t}{r}}{\binom{n}{r}}>1-\left(1-\frac{t}{n}\right)^r>\frac{rt}{n}-\binom{r}{2}\left(\frac{t}{n}\right)^2 \tag{5}$$

(where in the last inequality we note that the terms of the binomial expansion are decreasing in magnitude) we see that it suffices to show that

$$t<(n/r+t-1)\left(\frac{rt}{n}-\binom{r}{2}\left(\frac{t}{n}\right)^2\right).$$

Now

$$\begin{aligned}\frac{2n^2}{t}\left[(n/r+t-1)\left(\frac{rt}{n}-\binom{r}{2}\left(\frac{t}{n}\right)^2\right)-t\right]&=((t-2)r+t)n-r(r-1)t(t-1)\\&>0,\end{aligned}$$

as $t\geqslant 2$ and $n>4r^3t$. This proves equation (4). Now we have

$$\begin{aligned}|\mathcal{G}'|&\geqslant|\mathcal{G}|-k\binom{n-1}{r-1}>t\binom{n}{r}-k\binom{n-1}{r-1}\\&=(t-1)\binom{n-1}{r}+(n/r-k-1+t)\binom{n-1}{r-1}\\&>(t-1)\binom{n-1}{r},\end{aligned}$$

as required. This completes the proof of the claim. $\square$

Returning to the proof of the theorem, we see from the claim and the induction hypothesis that $\mathcal{G}'$ contains a multicoloured matching $\mathcal{M}$ of size t. Since $\mathcal{G}$ does not contain a multicoloured matching of size $t+1$ we can bound the number of sets containing x, by noting that they must either have multiplicity at most t, or contain a point from $\mathcal{M}$. Since $\mathcal{M}$ contains rt points there are at most $rt\binom{n-2}{r-2}$ sets containing x and a point from $\mathcal{M}$. Therefore we have maximum degree

$$d(x)\leqslant t\binom{n-1}{r-1}+(k-t)rt\binom{n-2}{r-2}.$$

Let A_1 be a set of multiplicity k. (There must be such a set, or we would have $|\mathcal{G}| \leqslant t\binom{n}{r}$, which is contrary to assumption.) Now pick s as large as possible so that there is a sequence $A_1, \cdots, A_s$ of pairwise disjoint sets such that A_i has multiplicity at least $t+2-i$. Note that this sequence forms a multicoloured matching, so $1 \leqslant s \leqslant t$. By construction, any set disjoint from $\bigcup_1^s A_i$ has multiplicity at most $t-s$. Since $|\bigcup_1^s A_i| = rs$, at most $rs\left(t\binom{n-1}{r-1} + (k-t)rt\binom{n-2}{r-2}\right)$ sets are incident to $\bigcup_1^s A_i$, by our bound on the maximum degree. Therefore

$$|\mathcal{G}| \leqslant (t-s)\binom{n}{r} + rs\left(t\binom{n-1}{r-1} + (k-t)rt\binom{n-2}{r-2}\right).$$

Now, since $k - t < k_c$, from Eqs. (1) and (5) we have

$$\frac{(k-t)\binom{n-2}{r-2}}{\binom{n}{r}} < \frac{t\binom{n-2}{r-2}}{\binom{n}{r} - \binom{n-t}{r}} < \frac{tr(r-1)/n(n-1)}{\frac{rt}{n} - \binom{r}{2}(t/n)^2} < 2\,\frac{r-1}{n-1}.$$

Since $n > 4r^3 t$, we have

$$\begin{aligned} |\mathcal{G}| &< \left(t - s + \frac{r^2 st}{n} + \frac{2r^2(r-1)st}{n-1}\right)\binom{n}{r} \\ &< \left(t - s + \frac{s}{4r} + \frac{s}{2}\right)\binom{n}{r} \\ &< (t - s/4)\binom{n}{r} < t\binom{n}{r}. \end{aligned}$$

This contradiction completes the proof. □

Examining the proof, one can see that there are only two circumstances in which equality can hold. One possibility is that there is no set of multiplicity k, when clearly the best construction is to take all r-tuples with multiplicity t. The other is that equality holds in Claim 4.1, when we see from the start of the proof of that claim that there is a vertex x of degree $d(x) = k\binom{n-1}{r-1}$. Then all r-tuples containing x have multiplicity k and $\mathcal{G}'$ has size $\geqslant k\left(\binom{n-1}{r} - \binom{n-t}{r}\right)$. In addition, in both cases $k = k_c$ or $k = k_c + 1$ we have from the proof that $|\mathcal{G}'| > (t-1)\binom{n-1}{r}$. Hence $\mathcal{G}'$ has an r-tuple of multiplicity strictly larger than $t-1$. Since $\mathcal{G}'$ has no multicoloured matching of size t, by induction $\mathcal{G}'$ must be a maximum size r-uniform set system on $n-1$ vertices with no matching of size t taken with multiplicity k. Therefore every set in $\mathcal{G}$ has multiplicity k, and so these sets form a maximum size r-uniform set system with no matching of size $t+1$.

5. The multicoloured Sauer–Shelah theorem

Recall that a set $Y \subset X$ is *shattered* by a set system $\mathcal{A}$ if for every $Z \subset Y$ there is a set A in $\mathcal{A}$ such that $A \cap Y = Z$. The Sauer–Shelah theorem states that if $\mathcal{A}$ is a family that does not shatter any set of size r then $|\mathcal{A}| \leqslant \sum_{i=0}^{r-1} \binom{n}{i}$.

Say that a set Y is *multicolour shattered* by a k-family $\mathcal{G}$ if we can pick sets in $\mathcal{G}$ that achieve all possible intersections with Y in such a way that each set comes from a different colour. We will prove Theorem 1.6, which is the following multicoloured version of the Sauer–Shelah theorem. Let $\mathcal{G}$ be a k-family on a set X of size n with no multicolour shattered set of size r, and suppose $n > 10^{4r}$. Then $|\mathcal{G}| \leqslant \max\left\{k\sum_{i=0}^{r-1}\binom{n}{i}, (2^r - 1)2^n\right\}$, and equality can occur.

A useful tool for dealing with the shattering property is *compressing*. For any element x in X we define a compression operator C_x as follows. The family $C_x(\mathcal{A})$ is obtained from $\mathcal{A}$ by deleting the element x from any set $A \in \mathcal{A}$ that contains x, unless the set $A \backslash x$ is already present in $\mathcal{A}$. Note that $|C_x(\mathcal{A})| = |\mathcal{A}|$. For a k-family $\mathcal{G}$ we define a k-family $C_x(\mathcal{G})$ by $C_x(\mathcal{G})_i = C_x(\mathcal{G}_i)$, i.e. the ith colour of $C_x(\mathcal{G})$ is obtained by compressing the ith colour of $\mathcal{G}$. It is well-known (see, e.g., [1,9]) that if a set is shattered by a family $\mathcal{A}$ then it is also shattered by $C_x(\mathcal{A})$. We prove the following multicoloured version of this proposition.

Proposition 5.1. *If a set Y is multicolour shattered by $C_x(\mathcal{G})$ then it is also multicolour shattered by $\mathcal{G}$.*

Proof. Without loss of generality we can suppose that there are sets A_i, $1 \leqslant i \leqslant 2^{|Y|}$ that shatter Y such that $A_i \in C_x(\mathcal{G})_i$ for each i. For any i with $A_i \notin \mathcal{G}_i$ we have $A_i \cup x \in \mathcal{G}_i$ by definition of C_x. Thus we may suppose that $x \in Y$, or by replacing A_i by $A_i \cup x$ for all such i we find a shattering of Y that is multicoloured in $\mathcal{G}$. Consider any i for which $A_i \notin \mathcal{G}_i$. Then $A_i \cup x \in \mathcal{G}_i$ as noted before. By assumption there is j such that $A_j \cap Y = (A_i \cup x) \cap Y$. Since $x \in A_j \in C_x(\mathcal{G})_j$ we have both A_j and $A_j \backslash x$ belonging to $\mathcal{G}_j$. Replacing A_j by $A_j \backslash x$ and A_i by $A_i \cup x$ we still have a multicoloured shattering of Y. If repeat this process then we eventually find a multicolour shattering of Y that only uses sets from $\mathcal{G}$, as required. □

Next we need the following lemma.

Lemma 5.2. *Suppose X is a set of size $n > 10^{4r}$ and $\mathcal{P} \subset X^{(r)}$ has size at least $10^{-(r+1)}\binom{n}{r}$. Then there are at most $2^n/n^r$ subsets of X that do not contain an element of $\mathcal{P}$.*

Proof. For a family $\mathcal{A}$, let $\partial^r \mathcal{A}$ denote the sets of size r that are contained in some set of $\mathcal{A}$. We use the following version of the Kruskal–Katona theorem, due to Lovász [13, Exercise 13.31]: if $\mathcal{A} \subset X^{(s)}$ with $|\mathcal{A}| \geqslant \binom{a}{s}$ then $|\partial^r \mathcal{A}| \geqslant \binom{a}{r}$.

Let $\mathcal{Q}$ be the subsets of X that do not contain an element of $\mathcal{P}$. Suppose for a contradiction that $|\mathcal{Q}| > n^{-r}2^n$. Let $\mathcal{R}$ be the subsets of X with size between $n/3$ and $2n/3$. Using the estimate $\binom{n}{\theta n} \leqslant 2^{H(\theta)n}$, where $H(\theta) = -\theta \log_2 \theta - (1-\theta)\log_2(1-\theta)$ is the binary entropy function, we get $|\mathcal{R}| > (1 - 2^{-n/20})2^n$. Then $|\mathcal{Q} \cap \mathcal{R}| > (n^{-r} - 2^{-n/20})2^n > \frac{1}{2}n^{-r}2^n$. Let $\mathcal{Q}_m = \mathcal{Q} \cap X^{(m)}$. Then by averaging there is some m with $n/3 \leqslant m \leqslant 2n/3$ such that $|\mathcal{Q}_m| \geqslant \frac{1}{2}n^{-r}\binom{n}{m} > \binom{(n^{-r}/2)^{1/m}n}{m}$. By the Kruskal–Katona theorem $|\partial^r \mathcal{Q}_m| > \binom{(n^{-r}/2)^{1/m}n}{r} > n^{-10r^2/n}\binom{n}{r}$. But $\partial^r \mathcal{Q}_m$ is disjoint from $\mathcal{P}$, so we must have $n^{-10r^2/n} + 10^{-(r+1)} < 1$. This contradicts our assumption that $n > 10^{4r}$, so we are done. $\square$

We are now ready to prove the multicoloured version of the Sauer–Shelah theorem.

Proof of Theorem 1.6. For notational convenience we set $N = \sum_{i=0}^{r-1}\binom{n}{i}$. Suppose that $\mathcal{G}$ is a k-family with no multicoloured collection of sets that shatter a set of size r and that $|\mathcal{G}| > \max\{kN, (2^r - 1)2^n\}$. As noted in Section 2, we can assume that k is equal to k_c or $k_c + 1$, where $k_c = \lfloor (2^r - 1)2^n/N \rfloor$, that the colours are nested and all sets either have multiplicity equal to k or at most $2^r - 1$. By Proposition 5.1, we can repeatedly apply compression operators to each colour until it is an ideal, i.e. if a colour contains a set Y then it contains all subsets of Y.

Let $\mathcal{A}$ be the sets with multiplicity k and $\mathcal{B}$ those with multiplicity at most $2^r - 1$. Since $\mathcal{A}$ is an ideal it cannot contain a set of size $\geqslant r$, so $|\mathcal{A}| \leqslant N$. If equality holds here then $\mathcal{B}$ must be empty, and then $|\mathcal{G}| \leqslant kN$, which is contrary to assumption. Therefore $|\mathcal{A}| \leqslant N - 1$.

Next we note that if $\mathcal{A}$ is empty then $|\mathcal{G}| \leqslant (2^r - 1)|\mathcal{B}| \leqslant (2^r - 1)2^n$, contradiction. So $\mathcal{A}$ is not empty, and then we must have $\emptyset \in \mathcal{A}$, as it is an ideal. Now $\mathcal{B}$ cannot contain a set Y of size r with multiplicity $2^r - 1$. For then all subsets of Y have multiplicity at least $2^r - 1$ and the empty set has multiplicity k, so Y is shattered by multicoloured sets: a contradiction. This shows that $\mathcal{B}$ has at most N sets of multiplicity $2^r - 1$. Now

$$(2^r - 1)2^n < |\mathcal{G}| \leqslant k|\mathcal{A}| + (2^r - 1)N + (2^r - 2)2^n$$

so

$$|\mathcal{A}| > (2^n - (2^r - 1)N)/k > 2^{n-1}/(2^{n+r}/N) = N/2^{r+1}.$$

Suppose that $\frac{i-1}{r} + 2^{-r-1} < |\mathcal{A}|/N \leqslant \frac{i}{r} + 2^{-r-1}$, where $1 \leqslant i \leqslant r$. Let $\mathcal{A}'$ be the subsets of $\mathcal{A}$ of size $r-1$. Since $\mathcal{A}$ contains no subsets of size $\geqslant r$ we have $|\mathcal{A}'| \geqslant |\mathcal{A}| - \sum_{i=0}^{r-2}\binom{n}{i} > \left(\frac{i-1}{r} + 2^{-r-2}\right)\binom{n}{r-1}$, since $n > 10^{4r}$. Let $\mathcal{D}$ consist of all r-sets that contain at least i elements of $\mathcal{A}'$. We count pairs consisting of an element of $\mathcal{A}'$ and an r-set containing it. Clearly there are $(n-r+1)|\mathcal{A}'|$ such pairs. On the other hand, by definition of $\mathcal{D}$ there are at most $r|\mathcal{D}| + (i-1)\left(\binom{n}{r} - |\mathcal{D}|\right)$ such pairs. Therefore

$$\begin{aligned}(r-i+1)|\mathcal{D}| &\geqslant (n-r+1)|\mathcal{A}'| - (i-1)\binom{n}{r}\\ &> \left(\frac{i-1}{r} + 2^{-r-2}\right)(n-r+1)\binom{n}{r-1} - (i-1)\binom{n}{r}\\ &= 2^{-r-2}r\binom{n}{r}\end{aligned}$$

which gives $|\mathcal{D}| > 2^{-r-2}\binom{n}{r}$.

Suppose that $\mathcal{B}$ contains a set D from $\mathcal{D}$. By definition there are elements $d_1, \ldots, d_i$ in D so that $D \backslash d_j \in \mathcal{A}$ for $1 \leqslant j \leqslant i$. Therefore $\mathcal{A}$ contains all subsets of D, except possibly those containing the set $\{d_1, \ldots, d_i\}$, i.e. all but 2^{r-i} subsets. It follows that D has multiplicity less than 2^{r-i}, or it would be shattered by multicoloured sets. Since each colour is an ideal, any element of $\mathcal{B}$ that contains a set from $\mathcal{D}$ has multiplicity less than 2^{r-i}. By Lemma 5.2 there are at most $n^{-r}2^n$ elements of $\mathcal{B}$ that do not contain an element of $\mathcal{D}$.

We can now finish the proof in the case when $r \geqslant 4$ and $1 \leqslant i \leqslant r-1$. Then we get

$$\begin{aligned}|\mathcal{G}| &\leqslant k|\mathcal{A}| + (2^r - 1)n^{-r}2^n + (2^{r-i} - 1)(1 - n^{-r})2^n\\ &\leqslant \left(\frac{i}{r} + 2^{-r-1} + 2^{-i} + n^{-r}\right)2^{n+r} < (2^r - 1)2^n,\end{aligned}$$

which gives a contradiction in this case.

We can also finish the proof for any $r > 1$ if $i = r$. Then we have $\left(\frac{r-1}{r} + 2^{-r-1}\right)N < |\mathcal{A}| < N$. Now there are at most $n^{-r}2^n$ elements of $\mathcal{B}$ with non-zero multiplicity. Since $(2^r - 1)n^{-r} < k/2^n$, we have

$$|\mathcal{G}| \leqslant k(N-1) + (2^r - 1)n^{-r}2^n < kN,$$

which also gives a contradiction.

Finally, we need to deal with the case $r = 1$, and the cases $r = 2, 3$ and $2^{-r-1} < |\mathcal{A}|/N \leqslant \frac{r-1}{r} + 2^{-r-1}$. The case $r = 1$ is easy to see directly. If there is any set of multiplicity more than 1 then it must the only set of positive multiplicity, so $|\mathcal{G}| \leqslant \max\{k, 2^n\}$, as required. In the case $r = 2$, let Y be the set of x such that $\mathcal{A}$ contains the singleton set $\{x\}$. Let $y = |Y|$. Any set that contains a point of Y has multiplicity at most 1, and no set of positive multiplicity contains 2 points of Y, so

$$3 \cdot 2^n < |\mathcal{G}| \leqslant k|\mathcal{A}| + 3 \cdot 2^{n-y} + y \cdot 2^{n-y} = k(y+1) + (3+y)2^{n-y}.$$

Since $y < \frac{5}{8}(n+1)$ we get $k(y+1) < 2^{n+1}$ and so $3 + y > 2^y$, which is a contradiction, as $y > n/8$.

Now consider the case $r = 3$. Recall that $\mathcal{A}'$ consists of the sets in $\mathcal{A}$ of size 2, which we can think of as a graph. We know that $\frac{1}{32}\binom{n}{2} < |\mathcal{A}'| \leqslant |\mathcal{A}| < \left(\frac{2}{3} + \frac{1}{16}\right)N = 35N/48$. Let $\mathcal{D}$ be the triples of vertices that contain at least 2 edges from $\mathcal{A}$. Let d be the average degree in $\mathcal{A}'$. Then $d > \frac{n-1}{32}$. By Cauchy-Schwartz the number of paths of length 2 is at least $n\binom{d}{2}$. A triple can contain at most 3 such paths, so

$|\mathcal{D}| \geqslant \frac{1}{3} n \binom{d}{2} > 10^{-4} \binom{n}{3}$. By Lemma 5.2, there are at most $n^{-3}2^n$ elements of $\mathcal{B}$ that do not contain an element of $\mathcal{D}$, and any other elements have multiplicity at most 1. Since $kN \leqslant 7 \cdot 2^n + N$ this gives

$$7 \cdot 2^n < |\mathcal{G}| \leqslant k|\mathcal{A}| + 7n^{-3}2^n + (1 - n^{-3})2^n \leqslant (35/48)kN + (1 + 6n^{-3})2^n$$
$$< (7 \cdot 35/48 + 1 + 10/n)2^n.$$

This gives $10/n > 6 - 245/48 = 43/48$, i.e. $n < 480/43 < 12$, a contradiction that completes the proof. □

On examining the proof, we see that there are only two circumstances in which equality can be achieved: one of $\mathcal{A}$ or $\mathcal{B}$ must be empty and the other as large as possible. This gives two possible constructions. One is to take all sets with multiplicity $2^r - 1$; the other is a maximum size set system that does not shatter a set of size r taken with multiplicity k.

6. Concluding remarks

- There are many extremal problems that we have not mentioned in this paper for which one might consider a multicoloured version. For all those that we considered we found that the size of the largest k-family without the forbidden configuration exhibits only two regimes of different behaviour as the multiplicity varies. It would be interesting to characterise the extremal problems for which this phenomenon occurs. (It was noted in [12] that it is not universal.) Recently, the second and third authors have studied the multicoloured versions of the Frankl–Ray–Chaudhuri–Wilson restricted intersection theorems [11], which also appear to have only two regimes of different behaviour.
- For most of our results, we have made no effort to obtain a good bound for the smallest size of ground set for which the result is true. Even with careful analysis it seems that our methods will not determine this, so it may be interesting to find the smallest size by other means.

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