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Multicolour Turán problems

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Abstract

A *simple k-colouring* of a multigraph G is a decomposition of the edge multiset as the sum of k simple graphs, called 'colours'. A copy of some fixed graph H in G is called *multicoloured* if its edges all have distinct colours. Recall that the Turán number $\mathrm{ex}(n, H)$ of H is the maximum number of edges in a graph on n vertices not containing a copy of H. We consider a multicolour generalisation $\mathrm{ex}_k(n, H)$, defined as the maximum number of edges in a multigraph on n vertices, that has a simple k-colouring not containing a multicoloured copy of H.

A natural construction of such a multigraph is k copies of a fixed extremal graph for H. We show that this is optimal for sufficiently large $k = k(n)$, i.e., $\mathrm{ex}_k(n, H) = k \cdot \mathrm{ex}(n, H)$, and moreover only this construction achieves equality. For $k \leqslant e(H) - 1$ one can take k copies of the complete graph without creating a multicoloured copy of H, so this is trivially the best possible construction. Even for $k \geqslant e(H)$, we should consider a competing construction along these lines, namely $e(H) - 1$ copies of the complete graph K_n. When $H = K_r$ and n is large, the optimal construction is always one of these two, i.e.,

$$\mathrm{ex}_k(n, K_r) = \begin{cases} k \cdot \mathrm{ex}(n, K_r) & \text{for } k \geqslant (r^2 - 1)/2, \\ \left(\binom{r}{2} - 1\right) \cdot \binom{n}{2} & \text{for } \binom{r}{2} \leqslant k < (r^2 - 1)/2. \end{cases}$$

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We prove a similar result for 3-colour-critical graphs. We also have some partial results for bipartite graphs. In particular, there are constants $c < C$ so that for infinitely many values of n

$$\mathrm{ex}_k(n, C_4) = \begin{cases} k \cdot \mathrm{ex}(n, C_4) & \text{for } k > C\sqrt{n}, \\ 3 \cdot \binom{n}{2} & \text{for } 4 \leqslant k < c\sqrt{n}. \end{cases}$$

1. Introduction

The Turán problem asks for the maximum number of edges in a graph on n vertices that contains no copy of some fixed non-empty graph H. A graph G that achieves the maximum is an *extremal H-free graph*, and $e(G) = \mathrm{ex}(n, H)$ is the *Turán number* of H. This extremal problem has a rich history in combinatorics, going back to 1907, when Mantel solved the case that H is a triangle. Its systematic study began with Turán [10], who considered the case that $H = K_r$ is a complete graph on r vertices. The *Turán graph* $T_{r-1}(n)$ is the complete $(r-1)$-partite graph with part sizes as equal as possible; we write $t_{r-1}(n)$ for the number of edges in $T_{r-1}(n)$. Then Turán's theorem states that $\mathrm{ex}(n, K_r) = t_{r-1}(n)$, and $T_{r-1}(n)$ is the unique extremal K_r-free graph. Erdös, Stone and Simonovits showed that the behaviour of the Turán number of a general graph H is determined by the chromatic number. They proved that if $\chi(H) = r$ then $\mathrm{ex}(n, H) = t_{r-1}(n) + o(n^2)$, which is an asymptotic result except when H is bipartite. For bipartite graphs, even the asymptotics for Turán numbers are only known in isolated cases, and there are many interesting open problems.

Speaking rather broadly, the essential feature of a 'Turán type' result is deducing a global fact from local considerations: a bound on the total number of edges from the behaviour of edges in small subgraphs. For example, a generalisation of Turán's problem introduced by Erdös in 1963 asks for the largest number of edges in a graph such that every r vertices span at most s edges (the case $s = \binom{r}{2} - 1$ being Turán's problem). A multigraph version of this problem was recently studied by Füredi and Kündgen [6]. In this paper we will be concerned with a Turán problem for coloured multigraphs, in which our local restriction is to forbid multicoloured copies of some fixed graph.

To state this precisely, we introduce the following definitions. A *simple k-colouring* of a multigraph G is a decomposition of the edge multiset as the sum of k simple graphs, called 'colours'. A copy of some fixed graph H in G is called *multicoloured* if its edges all have distinct colours. The *multicolour Turán number* $\mathrm{ex}_k(n, H)$ of H is the maximum number of edges in a multigraph on n vertices, that has a simple k-colouring not containing a multicoloured copy of H. A simply k-coloured multigraph that achieves this maximum is called *extremal*.

If $k \leqslant e(H) - 1$, then the multigraph consisting of k copies of the complete graph trivially contains no multicoloured copy of H, and is the unique extremal multigraph. Therefore, we may henceforth only consider the case $k \geqslant e(H)$. Even in this case, one possible construction is to take $e(H) - 1$ colours to be copies of the complete graph, and the remaining colours to be empty. An alternative construction to consider is k copies of a fixed extremal H-free graph. Our first theorem is that this second construction is always extremal for sufficiently large k.

Theorem 1.1. *Let H be a graph, let k and n be integers with $k \geqslant \binom{n}{2} - \mathrm{ex}(n, H) + e(H)$, and let G be a simply k-coloured multigraph containing no multicoloured H. Then G has size at most $k \cdot \mathrm{ex}(n, H)$, with equality if and only if all colours of G are identical extremal H-free graphs.*

In the light of this theorem and its preceding remarks we can define the following functions

$$\rho_H(n) = \max\left\{k\colon \mathrm{ex}_k(n, H) = \big(e(H) - 1\big) \cdot \binom{n}{2}\right\},$$
$$\sigma_H(n) = \min\big\{k\colon \mathrm{ex}_k(n, H) = k \cdot \mathrm{ex}(n, H)\big\}.$$

There are two natural questions to ask about these parameters. Firstly, what values do they take? Secondly, if $\rho_H(n) < \sigma_H(n) - 1$ then what constructions achieve $\mathrm{ex}_k(n, H)$ for $\rho_H(n) < k < \sigma_H(n)$?

Comparing the two constructions mentioned above gives the general bound

$$\rho_H(n) \leqslant \frac{(e(H) - 1)\binom{n}{2}}{\mathrm{ex}(n, H)} \leqslant \sigma_H(n). \tag{1}$$

These bounds are not tight in general, indeed we will later give an example of a graph H with $\sigma_H(n) \geqslant \binom{n}{2} - \mathrm{ex}(n, H)$, which is close to the maximum allowed by Theorem 1.1. On the other hand, for certain 'well-behaved' graphs it seems that these two constructions may give the whole picture. To support this assertion, we will now discuss some results for specific classes of graphs.

1.1. Complete graphs and colour-critical graphs

Perhaps the most natural starting point is to take H to be a complete graph. The following theorem completely solves the multicolour Turán problem in this case, for sufficiently large n. In particular, it shows that $\sigma_{K_r}(n) = \lceil (r^2 - 1)/2 \rceil$ and $\rho_{K_r}(n) = \sigma_{K_r}(n) - 1$, for large n.

Theorem 1.2. *Let $r \geqslant 2$, $k \geqslant \binom{r}{2}$, $n > 10^4 r^{34}$, and let G be an extremal simply k-coloured multigraph containing no multicoloured K_r. Then all colours of G are identical Turán graphs $T_{r-1}(n)$, or there are exactly $\binom{r}{2} - 1$ non-empty colours of G, all of which are complete graphs K_n. In particular,*

$$\mathrm{ex}_k(n, K_r) = \begin{cases} k \cdot t_{r-1}(n) & \text{for } k \geqslant \frac{1}{2}(r^2 - 1), \\ \big(\binom{r}{2} - 1\big)\binom{n}{2} & \text{for } \binom{r}{2} \leqslant k < \frac{1}{2}(r^2 - 1). \end{cases}$$

A graph H is *r-colour-critical* if it has chromatic number r, and there is some edge e such that $H - e$ has chromatic number $r - 1$. For such H, a result of Simonovits [9] shows that $\mathrm{ex}(n, H) = t_{r-1}(n)$ for sufficiently large n. It seems natural to believe the following extension of the previous theorem.

Conjecture 1.3. *Let $r \geqslant 3$ and H be an r-colour-critical graph with h edges. Suppose $k \geqslant h$, n is sufficiently large and G is an extremal simply k-coloured multigraph containing no multicoloured H. Then all colours of G are identical Turán graphs $T_{r-1}(n)$, or there are exactly $h - 1$ non-empty colours of G, all of which are complete graphs K_n.*

In support of this conjecture, we prove the case $r = 3$, which in particular solves the multicolour Turán problem for odd cycles.

Theorem 1.4. *Let H be a 3-colour-critical graph with h edges and $k \geqslant h$. Then, provided n is sufficiently large, all colours of an extremal simply k-coloured multigraph are $T_2(n)$, or there are exactly $h - 1$ non-empty colours, all of which are complete graphs K_n. In particular*

$$\mathrm{ex}_k(n, H) = \begin{cases} k \cdot \lfloor n^2/4 \rfloor & \text{for } k \geqslant 2(h-1), \\ (h-1)\binom{n}{2} & \text{for } h \leqslant k < 2(h-1). \end{cases}$$

1.2. Bipartite graphs

For bipartite graphs, the current state of knowledge of Turán numbers is sketchy, with even asymptotic results being rare. An example that is relatively well understood is the 4-cycle C_4. Let q be such that there is a projective plane with lines of size $q + 1$, e.g., q can be any prime power. For $n = q^2 + q + 1$ a result of Füredi [4] shows that $\mathrm{ex}(n, C_4) = q(q + 1)^2/2$. For these values of n we can prove the following.

Theorem 1.5. *There are constants $c < C$ so that, for infinitely many values of n*

$$\mathrm{ex}_k(n, C_4) = \begin{cases} k \cdot \mathrm{ex}(n, C_4) & \text{for } k > C\sqrt{n}, \\ 3 \cdot \binom{n}{2} & \text{for } 4 \leqslant k < c\sqrt{n}. \end{cases}$$

Moreover, for $4 \leqslant k < c\sqrt{n}$ an extremal simply k-coloured multigraph containing no multicoloured C_4 has exactly 3 non-empty colours, all of which are complete graphs K_n, and for $k > C\sqrt{n}$ all the colours of an extremal simply k-coloured multigraph are identical extremal C_4-free graphs.

For complete bipartite graphs $K_{r,s}$ with $s > (r - 1)!$, by the construction in [1] (modifying that of [7]), and by the result of Kövari, Sós and Turán [8], it is known that $\mathrm{ex}(n, K_{r,s}) = \Theta(n^{2-1/r})$. Thus, the lower bound from Eq. (1) gives $\sigma_{K_{r,s}}(n) = \Omega(n^{1/r})$. The following theorem strengthens this observation by solving the multicolour Turán problem for $K_{r,s}$ when $k = O(n^{1/r})$ and proving that $\rho_{K_{r,s}}(n) = \Omega(n^{1/r})$.

Theorem 1.6. *For any $s \geqslant r$ there is a constant c, such that if $rs \leqslant k < cn^{1/r}$ and G is an extremal simply k-coloured multigraph containing no multicoloured $K_{r,s}$, then there are exactly $rs - 1$ non-empty colours of G, all of which are complete graphs K_n.*

By analogy with the behaviour for C_4, it seems plausible to make the following conjecture.

Conjecture 1.7. *If H is either a complete bipartite graph or an even cycle then there are constants $c \leqslant C$ such that*

$$cn^2/\operatorname{ex}(n, H) \leqslant \rho_H(n) \leqslant \sigma_H(n) \leqslant Cn^2/\operatorname{ex}(n, H).$$

The rest of this paper is organised as follows. In the next section we present some general observations on the multicolour Turán problem and prove Theorem 1.1. Our results for complete graphs, bipartite graphs and 3-colour-critical graphs appear in Sections 3, 4 and 5 respectively. The final section contains some concluding remarks.

Notation. If G is a multigraph, $E(G)$ denotes its edge multiset and $V(G)$ denotes its vertex set. We generally use G to denote a simply k-coloured multigraph with colours $G_1, G_2, \dots, G_k$. The multiplicity of an edge $e \in G$ is written $w(e)$. The *degree* $d(v)$ of a vertex v is the number of edges incident with v. We write $d_T(v)$ for the number of edges between a vertex $v \in G$ and a set of vertices $T \subset V(G)$, and $e(S, T)$ for the number of edges between sets $S, T \subset V(G)$.

2. The multicolour Turán problem for general graphs

The main result proved in this section is Theorem 1.1. The key step is the following lemma, which will be used throughout the paper.

Lemma 2.1. *Suppose G is a simply k-coloured multigraph with colours $G_1, G_2, \dots, G_k$, and G does not contain a multicoloured copy of H. Then there exists a simply k-coloured multigraph F on the same vertex set as G and with colours $F_1, \dots, F_k$ satisfying*

(1) *F and G have the same edge set as multigraphs.*
(2) *$F_1 \subset F_2 \subset \cdots \subset F_k$.*
(3) *F contains no multicoloured copy of H.*

Proof. If $G_i = G_j$ for all i, j, then we are done. Suppose $G_i \neq G_j$ for some i, j. Consider the simply k-coloured multigraph F with the same colours as G, except that G_i is replaced by $G_i \cap G_j$ and G_j is replaced by $G_i \cup G_j$. Clearly (1) holds for this F. Suppose, for a contradiction, that F contains a multicoloured copy of H. This copy of H is not multicoloured in G, so must contain an edge $e \in G_i \cup G_j$ and an edge $f \in G_i \cap G_j$. We may assume $e \in G_i$. Then in G we can colour e with colour i and f with colour j, so this H is in fact multicoloured in G, a contradiction. This proves condition (3). Finally, by repeatedly applying the above transformation to pairs of colours which are not nested, after a finite number of steps, we obtain a simply k-coloured multigraph F in which (2) is satisfied. This completes the proof. □

Proof of Theorem 1.1. Suppose H has h edges ($h \geqslant 2$), and let G be a simply k-coloured multigraph with colours $G_1, \dots, G_k$ that does not contain a multicoloured copy of H. By Lemma 2.1 we can assume that $G_1 \subset \cdots \subset G_k$. Then we may also assume $e(G_k) \geqslant \mathrm{ex}(n,H)+1$, or we are done. Suppose, for the sake of contradiction, that $e(G_{k-i}) \geqslant \mathrm{ex}(n,H) - i + 1$ for all $i \leqslant h-1$. Fix a subgraph G^* of G_{k-h+1} with exactly $\mathrm{ex}(n,H) - h + 2$ edges. Then by definition $e(G_{k-h+1+i}) - e(G^*) \geqslant i$, so we can successively pick edges $e_1, \dots, e_{h-1}$ such that

$$e_i \in G_{k-h+1+i} \setminus \big(G^* \cup \{e_j \colon j < i\}\big).$$

Then $G^* \cup \{e_1, \dots, e_{h-1}\}$ is a graph with $\mathrm{ex}(n,H)+1$ edges, so contains a copy of H. To see that this H can be multicoloured, suppose it contains the edges $\{e_i \colon i \in I\}$ for some $I \subset \{1, \dots, h-1\}$, and its other edges belong to G^*. For each $i \in I$ we colour e_i with colour $k-h+1+i$. The remaining $h - |I|$ edges of H all belong to every G_i with $i \geqslant k-h+1$ so can be coloured by colours $k-h+1$ and $k-h+1+i$ for $i \notin I$. This contradiction shows that there is some $t \leqslant h-1$ such that $e(G_{k-t}) \leqslant \mathrm{ex}(n,H) - t$.

Fix such a t. Then

$$e(G) = \sum_{i=1}^{k} e(G_i) \leqslant (k-t)\big(\mathrm{ex}(n,H) - t\big) + t\binom{n}{2} < k \cdot \mathrm{ex}(n,H),$$

where we have used the fact that $k \geqslant \binom{n}{2} - \mathrm{ex}(n,H) + h > \binom{n}{2} - \mathrm{ex}(n,H) + t$. We conclude that $\mathrm{ex}_k(n,H) \leqslant k \cdot \mathrm{ex}(n,H)$, which proves the first part of the theorem. In addition, for multigraphs $G_1 + \cdots + G_k$ satisfying $G_1 \subset \cdots \subset G_k$ we have shown that equality can only occur when $e(G_k) \leqslant \mathrm{ex}(n,H)$, i.e., when all the colours G_i are equal to some fixed extremal H-free graph. Now consider any extremal simply k-coloured multigraph G. Applying the intersection/union transformation in the proof of Lemma 2.1, we reach a simply k-coloured multigraph F in which all colours are equal to some fixed extremal H-free graph. But clearly these k colours cannot be obtained by the above transformations from any k-coloured multigraph other than F, so $G = F$, i.e., the only case of equality is when all colours are equal to some fixed extremal H-free graph. □

With a similar argument, we can prove the following proposition, which gives the asymptotics of multicolour Turán numbers for certain values of k.

Proposition 2.2. $\mathrm{ex}_k(n,H) < k \cdot \mathrm{ex}(n,H) + e(H)\binom{n}{2}$. *In particular, whenever* $k \cdot \mathrm{ex}(n,H)/n^2 \to \infty$ *as* $n \to \infty$, *then*

$$\mathrm{ex}_k(n,H) = \big(1 + o(1)\big)k \cdot \mathrm{ex}(n,H).$$

Proof. Suppose H has h edges, and let G be a simply k-coloured multigraph with colours $G_1, \ldots, G_k$ that does not contain a multicoloured copy of H. By Lemma 2.1 we can assume that $G_1 \subset \cdots \subset G_k$. Since edges of G_{k-h+1} have multiplicity h we see if G_{k-h+1} contains a copy of H then it is multicoloured. Therefore G_{k-h+1} has at most $\mathrm{ex}(n, H)$ edges. Even if the last $h - 1$ graphs are complete, we get that $\mathrm{ex}_k(n, H) < k \cdot \mathrm{ex}(n, H) + e(H)\binom{n}{2}$. □

Now we give an example to show that both Theorem 1.1 and the proposition cannot be improved in general. Let H be the graph consisting of two edge-disjoint triangles intersecting in exactly one vertex (also called a 'bowtie'). It is known that $\mathrm{ex}(n, H) = \lfloor n^2/4 \rfloor + 1$ (see, e.g., [2, Exercise IV.20]), and moreover any extremal H-free graph consists of a complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ together with a single edge in one of its parts. We can construct a simply k-coloured multigraph G which does not contain a multicoloured copy of H, by taking all colours but one equal to some fixed $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, and the final colour G_k equal to K_n. Indeed, any copy of H contains at least two edges that only belong to G_k, so is not multicoloured. Now G has $(k-1)\lfloor n^2/4 \rfloor + \binom{n}{2}$ edges, which is larger than $k(\lfloor n^2/4 \rfloor + 1)$ whenever $k < \binom{n}{2} - \lfloor n^2/4 \rfloor$. We deduce that $\sigma_H(n) > \binom{n}{2} - \mathrm{ex}(n, H)$, which is close to the upper bound given by Theorem 1.1. Furthermore, if $k \cdot \mathrm{ex}(n, H)/n^2$ does not tend to infinity then there is a constant C such that $k \leqslant C$ for all n. Hence, the conclusion of the Proposition 2.2 does not hold for such k, as

$$e(G) = (k-1)\big\lfloor n^2/4 \big\rfloor + \binom{n}{2} > \left(1 + \frac{1}{2C}\right) \cdot k\big(\big\lfloor n^2/4 \big\rfloor + 1\big).$$

We have a few remarks to make about the use of Lemma 2.1 throughout this paper. First of all, we note that the argument at the end of the proof of Theorem 1.1 applies in general. Whenever we can show that the only case of equality for $\mathrm{ex}_k(n, H)$ for a simply k-coloured multigraph with nested colours $G_1 \subset \cdots \subset G_k$ is when $G_1 = \cdots = G_k$, it follows that this is also the only case of equality for any simply k-coloured multigraph. We will use this observation without further comment in the future.

Next we note that there is a unique simple k-colouring of a multigraph G in which the colours are nested: if the colours are $G_1 \subset \cdots \subset G_k$ then G_i consists of all edges of multiplicity at least $k + 1 - i$. This will often allow us to simplify our discussion by looking only at the multigraph structure of G. We say that G contains a multicoloured copy of H if its nested simple k-colouring does. The following Hall-type condition characterises this property by reference only to the multigraph structure.

Proposition 2.3. *Let G be a simply k-coloured multigraph with nested colours and let H be a simple subgraph of G. Then H is not multicoloured if and only if there is some integer w, for which at least $w + 1$ edges of H have multiplicity at most w in G.*

Proof. This is immediate from Hall's theorem. □

We conclude this section with some conditions that should be satisfied by a simply k-coloured multigraph achieving $\mathrm{ex}_k(n, H)$.

Proposition 2.4. *Suppose G is a multigraph with maximum multiplicity k, where $k \geqslant e(H)$, and that the nested simple k-colouring of G contains no multicoloured copy of H. Then there is a multigraph F such that*

(1) $e(F) \geqslant e(G)$.
(2) *F contains no multicoloured copy H.*
(3) *Every edge of F either has multiplicity at most $e(H) - 1$ or exactly equal to k.*
(4) *The set of edges of F of multiplicity k form a graph with no subgraph isomorphic to H.*

Proof. Form F from G by the following rule: increase any edge of multiplicity at least $e(H)$ to multiplicity k. Then (1) and (3) hold by construction. Consider a copy of H in G. It is not multicoloured, so by Proposition 2.3, there is an integer w and a set of $w + 1$ edges $W \subset H$ so that each edge of W has multiplicity at most w in G. Since $W \subset H$ we have $w \leqslant e(H) - 1$, so the above rule has no effect on edges of W, i.e., they have the same multiplicities in F. It follows that F contains no multicoloured copy H, proving (2). As $k \geqslant e(H)$, (4) is immediate. □

Proposition 2.5. *Suppose that for every extremal simply k-coloured multigraph G for H, $e(G) = k \cdot \mathrm{ex}(n, H)$ and all colours of G are identical copies of some extremal H-free graph. Then the same holds for every $\ell \geqslant k$.*

Proof. We argue by induction on ℓ, the base case $\ell = k$ being true by assumption. Let G be an extremal simply ℓ-coloured multigraph for H. As before we can assume its colours are nested as $G_1 \subset \cdots \subset G_\ell$. The $(\ell - 1)$-coloured multigraph consisting of colours $G_2, \ldots, G_\ell$ contains no multicoloured copy of H, and therefore $\sum_{i=2}^{\ell} e(G_i) \leqslant (\ell - 1)\,\mathrm{ex}(n, H)$, with equality if and only if $G_2 = G_3 = \cdots = G_\ell = F$ for some Turán graph F for H. Therefore

$$\sum_{i=1}^{\ell} e(G_i) \leqslant \frac{\ell}{\ell - 1} \sum_{i=2}^{\ell} e(G_i) \leqslant \ell \cdot \mathrm{ex}(n, H).$$

If equality occurs then $G_2 = G_3 = \cdots = G_\ell = F$, and therefore $G_1 = F$. □

3. Complete graphs

In this section, we determine the multicolour Turán numbers for complete graphs. This is perhaps the most natural starting point, since the ordinary Turán numbers for complete graphs are well known. Consider a simply k-coloured multigraph G on n vertices, with colours $G_1, G_2, \ldots, G_k$. For $k \leqslant \binom{r}{2} - 1$, we can set $G_i = K_n$ for all i, so $\mathrm{ex}_k(n, K_r) = k\binom{n}{2}$ in this case. For $k \geqslant \binom{r}{2}$, there are two natural constructions to consider, namely G_i is the Turán graph $T_{r-1}(n)$ for all i, or $G_i = K_n$ for $i \leqslant \binom{r}{2} - 1$ and $G_i = \emptyset$ otherwise. When n is sufficiently large, we will show that one of these constructions is always the unique extremal multigraph. The first construction has $\bigl(\binom{r}{2}-1\bigr)\binom{n}{2}$ edges and the second has $kt_{r-1}(n)$ edges, which is larger if

$$k > \frac{\bigl(\binom{r}{2}-1\bigr)\binom{n}{2}}{t_{r-1}(n)}.$$

As $t_{r-1}(n) \sim (r-2)\binom{n}{2}/(r-1)$, this occurs when k is about $(r^2-1)/2$. For large n, we will now see that $\lceil (r^2-1)/2\rceil$ is the critical value for k, in other words,

$$\sigma_{K_r}(n) = \left\lceil \frac{1}{2}\bigl(r^2-1\bigr)\right\rceil, \qquad \rho_{K_r}(n) = \sigma_{K_r}(n) - 1.$$

We will split the proof of Theorem 1.2 into two separate theorems. The first deals with the case $k < (r^2-1)/2$. In what follows, we make no attempt to find the smallest n for which this holds.

Theorem 3.1. *If $\binom{r}{2} \leqslant k < (r^2-1)/2$ and $n > r^8$, then*

$$\mathrm{ex}_k(n, K_r) = \left(\binom{r}{2} - 1\right)\binom{n}{2}.$$

Furthermore, if G is an extremal simply k-coloured multigraph, then exactly $\binom{r}{2}-1$ colours of G are non-empty, and all these are complete graphs K_n.

Proof. First we claim that it suffices to prove Theorem 3.1 for multigraphs that have $n > r^3$ vertices and minimum degree at least $\bigl(\binom{r}{2}-1\bigr)(n-1)$. Indeed, suppose we have done this, and let G be a simply k-coloured multigraph with $n > r^8$ vertices, $e(G) \geqslant \bigl(\binom{r}{2}-1\bigr)\binom{n}{2}$ and no multicoloured K_r. If the minimum degree of G is at least $\bigl(\binom{r}{2}-1\bigr)(n-1)$, then we are done. Otherwise we obtain a contradiction as follows. Let $G = G(n), G(n-1), \ldots$ be a sequence of multigraphs where $G(m)$ has m vertices and is obtained from $G(m+1)$ by deleting a vertex of degree strictly less than $\bigl(\binom{r}{2}-1\bigr)m$. Setting $f(m) = e(G(m)) - \bigl(\binom{r}{2}-1\bigr)\binom{m}{2}$ we have $f(n) \geqslant 0$ and $f(m) \geqslant f(m+1)+1$. If we can continue this process to obtain a multigraph $G(r^3)$, then

$$n - r^3 \leqslant \sum_{m=r^3}^{n-1} \bigl(f(m) - f(m+1)\bigr) \leqslant f\bigl(r^3\bigr) < k\binom{r^3}{2} < \frac{r^2}{2}\binom{r^3}{2},$$

which is a contradiction for $n > r^8$. Otherwise we obtain a multigraph $G(n')$ with $n > n' > r^3$ having minimal degree at least $\bigl(\binom{r}{2}-1\bigr)(n'-1)$, no multicoloured K_r and $e(G(n')) > \bigl(\binom{r}{2}-1\bigr)\binom{n'}{2}$, which contradicts our assumption.

Hence, from now on, we can assume that G has $n > r^3$ vertices and minimum degree at least $\bigl(\binom{r}{2}-1\bigr)(n-1)$. Let T be a set of $t \leqslant r-1$ vertices. Then, by the minimum degree assumption,

$$t\left(\binom{r}{2}-1\right)(n-1) \leqslant \sum_{x\in T} d(x) = 2e(T) + e\bigl(T, V(G)-T\bigr)$$
$$\leqslant 2k\binom{t}{2} + \sum_{v\notin T} d_T(v).$$

As $n > r^3$, $t < r$, and $k < (r^2-1)/2$, this inequality shows that for any choice of the set T, there exists a vertex $v \in V(G) - T$ such that $d_T(v) \geqslant t\bigl(\binom{r}{2}-1\bigr)$.

To finish the proof it is enough to show that every edge of G has multiplicity at most $\binom{r}{2}-1$. For the sake of contradiction suppose that there is some edge v_1v_2 of multiplicity at least $\binom{r}{2}$. Applying the above observation we can select vertices v_i, $3 \leqslant i \leqslant r$, so that

$$\sum_{j<i} w(v_iv_j) \geqslant (i-1)\left(\binom{r}{2}-1\right).$$

Now we claim that $K = \{v_1, \ldots, v_r\}$ spans a multicoloured copy of K_r. If not, then Hall's condition must fail, i.e., for some number w at least $w+1$ edges in K have multiplicity at most w. By construction, the total weight of edges in K is larger than

$$\sum_{i=1}^{r}(i-1)\left(\binom{r}{2}-1\right) = \binom{r}{2}\left(\binom{r}{2}-1\right).$$

On the other hand, it is at most $w(w+1) + \bigl(\binom{r}{2}-w-1\bigr)k$. This gives the inequality

$$w(w+1) + \left(\binom{r}{2}-w-1\right)k - \binom{r}{2}\left(\binom{r}{2}-1\right) > 0,$$

which factorises as $\bigl(w+\binom{r}{2}-k\bigr)\bigl(w+1-\binom{r}{2}\bigr) > 0$. Since clearly $w \leqslant \binom{r}{2}-1$, we deduce $w < k - \binom{r}{2}$. There is at least one edge with multiplicity at most w; let v_iv_j with $i > j$ be such an edge. Then, by definition

$$(i-1)\left(\binom{r}{2}-1\right) \leqslant \sum_{j=1}^{i-1} w(v_iv_j) \leqslant (i-2)k + w < (i-1)k - \binom{r}{2}.$$

Hence $\binom{r}{2} < (i-1)\bigl(k-\binom{r}{2}+1\bigr)$. Taking into account that $i \leqslant r$ and $k \leqslant (r^2-2)/2$, we deduce that $\binom{r}{2} < (i-1)(r/2) \leqslant \binom{r}{2}$, which is impossible. Therefore K spans a multicoloured copy of K_r. This contradicts the assumption of the theorem and completes the proof. □

To finish the proof of Theorem 1.2 we now consider the case $k \geqslant (r^2-1)/2$.

Theorem 3.2. *If $k \geqslant (r^2-1)/2$ and $n > 10^4r^{34}$ then $\mathrm{ex}_k(n, K_r) \leqslant k \cdot t_{r-1}(n)$. Furthermore, if G is an extremal simply k-coloured multigraph, then all colours of G are identical Turán graphs $T_{r-1}(n)$.*

Proof. By Proposition 2.5, we need only consider the case $k = (r^2 - 1)/2$, if r is odd, or $k = r^2/2$, if r is even. Suppose G is a simply k-coloured multigraph which contains no multicoloured K_r, and $e(G) \geqslant k \cdot t_{r-1}(n)$. Let $d(n) = n - \lceil n/(r-1) \rceil$ be the minimum degree in $T_{r-1}(n)$. Initially, we will proceed in the same way as in Theorem 3.1. By a vertex deletion argument, as in that proof, we may assume that G has minimum degree at least $kd(n)$ and $n > 100r^{16}$.

Let T be a set of vertices with $|T| = t \leqslant r - 1$. Note that, since $n > 100r^{16}$,

$$\begin{aligned} e\big(T, V(G) - T\big) &= \sum_{v \in T} d(v) - 2e(T) \geqslant tkd(n) - kt(t-1) \\ &> tk\left(\frac{r-2}{r-1}n - 1\right) - kt(t-1) = \frac{r-2}{r-1}ktn - kt^2 \\ &> (n-t)\left(\frac{r-2}{r-1}kt - \frac{1}{r-1}\right). \end{aligned} \tag{2}$$

Thus there is a vertex $v \in V - T$ with

$$d_T(v) > \frac{r-2}{r-1}kt - \frac{1}{r-1}.$$

Moreover, since $d_T(v)$ is an integer we conclude that for any choice of T, there is a $v \in V - T$ with

$$d_T(v) \geqslant \frac{r-2}{r-1}kt.$$

There exists some edge of multiplicity k in G, otherwise Theorem 3.1 shows $e(G) \leqslant \left(\binom{r}{2} - 1\right)\binom{n}{2} < kt_{r-1}(n)$, a contradiction. Let v_1v_2 be an edge of multiplicity k. Applying the above observation, we can successively select vertices $v_3, \ldots, v_{r-1}$ so that for all $3 \leqslant i \leqslant r-1$,

$$\sum_{j<i} w(v_iv_j) \geqslant \frac{r-2}{r-1}k(i-1).$$

Let $K = \{v_1, \ldots, v_{r-1}\}$. Consider a vertex $v \in V(G) - K$ for which $d_K(v) \geqslant k(r-2)$. Since there is no multicoloured K_r, we know that for some w at least $w+1$ edges of $K \cup v$ have multiplicity at most w. Then

$$w(w+1) + \left(\binom{r}{2} - w - 1\right)k \geqslant k + \sum_{i=3}^{r} \frac{r-2}{r-1}k(i-1) > \frac{r-2}{r-1}k\binom{r}{2}.$$

When r is odd we have $k = (r^2-1)/2$ and we can rewrite this inequality as

$$\left(\frac{r-1}{2} - w\right)\left(\binom{r}{2} - w - 1\right) > 0.$$

When r is even we have $k = r^2/2$ and

$$\left(\frac{r-1}{2} - w\right)\left(\binom{r}{2} - w - 1\right) > \frac{1}{2}\left(w + 1 - \frac{r}{2}\right).$$

Since $w \leqslant \binom{r}{2} - 1$, in both cases we see that $w < (r-1)/2$. We claim that such a small multiplicity cannot occur on edges within K. For suppose $w(v_i v_j) \leqslant w$ and $j < i \leqslant r-1$. Then

$$\frac{r-2}{r-1}k(i-1) \leqslant \sum_{j \leqslant i} w(v_i v_j) \leqslant (i-2)k + w,$$

which gives the contradiction $w \geqslant (r-i)k/(r-1) \geqslant (r+1)/2$. We deduce that the $w+1$ edges with multiplicity at most w must all occur on edges joining v to K. Then $k(r-2) \leqslant d_K(v) \leqslant w(w+1) + (r-2-w)k$, i.e., $wk \leqslant w(w+1)$, which is only possible for $w = 0$. Therefore, some edge from v to K has multiplicity 0. Even if the others all have maximum multiplicity k, the total weight is at most $k(r-2)$. We deduce that $d_K(v) \leqslant k(r-2)$ for any $v \in V - K$, and if equality holds then there is some i such that $w(vv_i) = 0$ and $w(vv_j) = k$ for all $j \neq i$.

Let S be the set of vertices v in $V - K$ for which $d_K(v) < k(r-2)$. First note that

$$e(K, V-K) \leqslant |S|\big(k(r-2) - 1\big) + \big(n - (r-1) - |S|\big)k(r-2),$$

and also by Eq. (2) we have

$$e(K, V-K) > k(r-2)n - k(r-1)^2.$$

Therefore $|S| < k(r-1)$. As noted above, we can partition the rest of the vertices $V - (S \cup K)$ as $V_1 \cup \cdots \cup V_{r-1}$, where $v \in V_i$ iff $w(vv_i) = 0$ and $w(vv_j) = k$ for all $j \neq i$.

Next we claim that each V_i is an independent set in G (i.e., all pairs of vertices in V_i have multiplicity 0). For suppose that $u, v \in V_i$ with $w(uv) \geqslant 1$. For each $j \neq i$ we have $w(uv_j) = k \geqslant w(v_i v_j)$, i.e., the degree sequence of the subgraph of G induced by $L = \big(K \setminus v_i\big) \cup u$ dominates that of K. However, $d_L(v) > k(r-2)$, so the argument we gave above for K shows that subgraph induced by $L \cup v$ spans a multicoloured K_r. This contradiction shows that each V_i is an independent set.

To finish the proof, it suffices to show that the remaining vertices of $S \cup K$, of which there are at most $(k+1)(r-1) < r^3$, can be distributed among the V_i so that they remain independent. This indeed suffices, as then the edges with positive multiplicity form an $(r-1)$-partite graph, so by definition of $T_{r-1}(n)$ there are at most $t_{r-1}(n)$ of them. Even if they all have maximum multiplicity, we have $e(G) \leqslant k \cdot t_{r-1}(n)$, with equality when the maximum multiplicity edges form a Turán graph. Therefore it is enough to prove the following claim.

Claim. *If $U_1, \dots, U_{r-1}$ are any disjoint independent subsets in G with $U = \bigcup_{i=1}^{r-1} U_i$ and $|V(G) - U| < r^3$, then for any vertex v in $V - U$, there is some index i for which $U_i \cup v$ is still independent.*

Proof. Suppose that the claim is not true, i.e., $d_{U_i}(v) > 0$ for each i. First note that $|U_i| > n/(r-1) - r^2\sqrt{n}$ for each i. For if $|U_i| \leqslant n/(r-1) - r^2\sqrt{n}$ for some i then

$$\begin{aligned}
e(G) &= e(U) + e(U, V-U) + e(V-U) \\
&\leqslant k\left(|U_i||U-U_i| + \binom{r-2}{2}\left(\frac{|U-U_i|}{r-2}\right)^2 + |U||V-U| + \binom{|V-U|}{2}\right) \\
&< k\left(\frac{r-2}{r-1}\cdot\frac{n^2}{2} + r^3 n + \frac{r^6}{2} - \frac{r-1}{r-2}r^4\cdot\frac{n}{2}\right) < k\, t_{r-1}(n) \leqslant e(G),
\end{aligned}$$

which is contrary to our hypothesis. Also, if there were some i for which $|U_i| > n/(r-1) + r^3\sqrt{n}$, then there would be have to be some j for which $|U_j| < n/(r-1) - r^2\sqrt{n}$. Therefore we deduce that for each i,

$$\left||U_i| - \frac{n}{r-1}\right| < r^3\sqrt{n}.$$

Without loss of generality $d_{U_1}(v)$ is the smallest among $\{d_{U_i}(v) \mid 1 \leqslant i \leqslant r-1\}$. Let $M_i \subset U_i$ be the vertices u in U_i such that $w(uv) \geqslant r-1$. Then for every $i \neq 1$ we have $|M_i| > n/(10(r-1))$. For otherwise we would have

$$\begin{aligned}
d_{U_1}(v) \leqslant d_{U_i}(v) &\leqslant k\cdot\frac{n}{10(r-1)} + (r-2)\left(\frac{n}{r-1} + r^3\sqrt{n} - \frac{n}{10(r-1)}\right) \\
&< \frac{kn}{10(r-1)} + \frac{r-2}{r-1}\,n
\end{aligned}$$

which yields a contradiction, as

$$\begin{aligned}
d(v) &< k\left((r-3)\left(\frac{n}{r-1} + r^3\sqrt{n}\right) + 2\cdot\frac{n}{10(r-1)} + r^3\right) + 2\cdot\frac{r-2}{r-1}\,n \\
&< kd(n) - \frac{4k/5 - 2(r-2)}{r-1}\,n + kr^4\sqrt{n} < kd(n).
\end{aligned}$$

The last inequality follows from the fact that $4k/5 - 2(r-2) > k/5$ for $r \geqslant 3$, and since $n > 100r^{16}$, then

$$\frac{4k/5 - 2(r-2)}{r-1}\,n > \frac{kn}{5r} > kr^4\sqrt{n}.$$

Select $u_1 \in U_1$ with $w(vu_1) > 0$. For each $i > 1$, remove from M_i any vertex u for which $w(u_1u) < k$. We remove at most $r^6\sqrt{n}$ points. Indeed, since $w(u_1u) = 0$ for every $u \in U_1$, we otherwise would have

$$d(u_1) < k\left((r-2)\left(\frac{n}{r-1} + r^3\sqrt{n}\right) + r^3\right) - r^6\sqrt{n} < kd(n),$$

which is a contradiction. Now we successively select $u_i \in M_i$, and, for each $j > i$, remove from M_j any vertex u for which $w(u_iu) < k$. By the same argument as for u_1, we see that at each stage we remove at most $r^6\sqrt{n}$ points from each M_j. Since

$$|M_i| > \frac{n}{10(r-1)} > r^7\sqrt{n}$$

for $n > 100r^{16}$, we never make any M_i empty. Thus we construct a set of r points $\{v, u_1, \ldots, u_{r-1}\}$ such that $w(vu_1) > 0$, $w(vu_i) \geqslant r-1$ for $2 \leqslant i \leqslant r-1$, and all other edges have multiplicity k. This set spans a multicoloured K_r, a contradiction that proves the claim, and thereby the theorem. $\square$

4. Bipartite graphs

Here we consider multicolour Turán numbers for bipartite graphs. For certain simple bipartite graphs it is not difficult to determine the extremal multigraphs $G = G_1 + G_2 + \cdots + G_k$. For example, when H is a path of length two, either $G_k = K_n$ and the remaining G_i are empty, or $G_1 = \cdots = G_k = M$, where M is a matching of size $\lfloor n/2 \rfloor$. Similarly, if H is a pair of disjoint edges, either $G_k = K_n$ and the remaining G_i are empty, or the G_i are all equal to some fixed star of size $n-1$.

The problem is more challenging for bipartite graphs that contain cycles. In this section we will be concerned with the case of complete bipartite graphs $K_{r,s}$. The following proposition of Kövari, Sós and Turán gives an upper bound for their Turán numbers.

Proposition 4.1. *If $r \leqslant s$ then there is a constant $\alpha(r,s)$ such that $\mathrm{ex}(n, K_{r,s}) < \alpha(r,s)n^{2-1/r}$.*

Since we make no attempt to optimise our constants, for simplicity we will formulate the next theorem in terms of this $\alpha(r,s)$. The interested reader can find the best known bound on this constant in [5].

Theorem 4.2. *If $rs \leqslant k < n^{1/r}/(24r\alpha(r,s))$ and n is sufficiently large then*

$$\mathrm{ex}_k(n, K_{r,s}) = (rs-1)\binom{n}{2}.$$

Furthermore, if G is an extremal simply k-coloured multigraph, then exactly $rs-1$ colours are non-empty, and all these colours are complete graphs K_n.

Proof. By Proposition 2.4, we can assume that the colours are nested and any edge multiplicity either lies between 1 and $rs-1$ or equals k. Note that if we can show that the only extremal simply k-coloured multigraph in which the colours are nested has exactly $rs-1$ colours non-empty, and all these colours are complete graphs K_n, then in fact this is the only extremal simply k-coloured multigraph, even without assuming that the colours are nested. This follows from the proof of Lemma 2.1. If there is a different extremal simply k-coloured multigraph, then after some sequence of intersection/union transformations we arrive at $rs-1$ complete graphs. One step before we have $rs-2$ colours equal to complete graphs, 2 colours equal to some graph H and its complement $\overline{H}$, and all others colours empty. Clearly we can choose a copy of $K_{r,s}$ that shares edges with both H and $\overline{H}$. The other edges of this copy can be assigned different colours arbitrarily from the $rs-2$ colours that are complete graphs, so it is multicoloured. This contradiction shows that there is no different extremal simply k-coloured multigraph.

Now we claim that it suffices to prove the theorem for simply k-coloured multigraphs with minimum degree at least $(rs-1-1/(5r))(n-1)$ and $k < n^{1/r}/(12r\alpha(r,s))$. Indeed, suppose we have done this, and let G be a multigraph with maximum multiplicity k, $e(G) \geqslant (rs-1)\binom{n}{2}$ and no multicoloured $K_{r,s}$. If the minimum degree of G is at least $(rs-1-1/(5r))(n-1)$, we are done. Otherwise we obtain a contradiction as follows. Form a sequence $G = G(n), G(n-1), \ldots,$ where $G(m)$ is a graph on m vertices, and whenever $G(m)$ has a vertex v with $d(v) < (rs-1-1/(5r))(m-1)$ we set $G(m-1) = G(m) - v$. If this sequence can be continued to reach a graph $G(n/2)$, then this graph has at least

$$(rs-1)\binom{n/2}{2} + \frac{1}{5r}\sum_{m=n/2+1}^{n}(m-1) > (rs-1)\binom{n/2}{2} + \frac{1}{20r}n^2$$

edges. This is only possible with at least

$$24r\alpha(r,s)\cdot\frac{1}{20r}n^{2-1/r} > \alpha(r,s)(n/2)^{2-1/r}$$

edges of multiplicity k, but by Proposition 4.1 these edges form a multicoloured $K_{r,s}$, which is a contradiction. Otherwise we obtain a multigraph $G(m)$ for some $m > n/2$ with minimum degree at least $(rs-1-1/(5r))(m-1)$, maximum multiplicity

$$k < \frac{1}{24r\alpha(r,s)}n^{1/r} < \frac{1}{12r\alpha(r,s)}m^{1/r},$$

no multicoloured $K_{r,s}$ and $e(G(m)) > (rs-1)\binom{m}{2}$, which contradicts our assumption.

Hence, from now on, we can assume that G has minimum degree at least $(rs-1-1/(5r))(n-1)$ and $k < n^{1/r}/(12r\alpha(r,s))$. Let H be the graph consisting of the edges of multiplicity k. We can assume H is non-empty. Let U be the vertices with degree at least

$3\alpha(r,s)n^{1-1/r}$ in H. For a vertex v let $F(v)$ consist of the vertices w for which vw has multiplicity exactly $rs-1$, and write $f(v) = |F(v)|$. If a vertex v is not in U then

$$\left(rs - 1 - \frac{1}{5r}\right)(n-1) < d(v) < k \cdot 3\alpha(r,s)n^{1-1/r} + (rs-2)(n-1) + f(v),$$

so

$$f(v) > \left(1 - \frac{1}{5r} - \frac{3\alpha(r,s)}{12r\alpha(r,s)}\right)(n-1) - 1 > \left(1 - \frac{19}{40r}\right)n.$$

Let uv be an edge of multiplicity k. First we show that one of u or v must belong to U. Otherwise $F(u)$ and $F(v)$ both contain at least $(1-1/(2r))n$ vertices, so $|F(u) \cap F(v)| \geqslant (1-1/r)n$. We bound the number of edges of G as follows. The edges of multiplicity k certainly contain no $K_{r,s}$, so there are at most $\alpha(r,s)n^{2-1/r}$ of these, and they contribute at most $k \cdot \alpha(r,s)n^{2-1/r} < n^2/(12r)$. The remaining edges of G have multiplicity at most $rs-1$. There can be no $K_{r-1,s-1}$ within $F(u) \cap F(v)$ with every edge of multiplicity at least $(r-1)(s-1)$, as this, together with uv, would create a multicoloured $K_{r,s}$. Then this set contains at most $O(n^{2-1/(r-1)})$ edges of multiplicity at least $(r-1)(s-1)$, so contains at least $\binom{(1-1/r)n}{2}/2$ edges of multiplicity at most $(r-1)(s-1) = rs-1-(r+s-2)$. Thus we conclude that

$$e(G) \leqslant (rs-1)\binom{n}{2} - (r+s-2)\frac{1}{2}\binom{(1-1/r)n}{2} + \frac{1}{12r}n^2 < (rs-1)\binom{n}{2}.$$

This contradiction shows that one of u or v must belong to U; in particular U is non-empty.

Now we claim that for u in U there cannot be $v_1, \dots, v_r$ in $V - U$ such that each uv_i has multiplicity k. If this happens, then by definition of U we have

$$\left|\bigcap_{i=1}^{r} F(v_i)\right| > n - r \cdot \frac{19}{40r}n > n/2 > s-1.$$

Taking S to be a set containing $s-1$ points from $\bigcap F(v_i)$ and u, and $R = \{v_1, \dots, v_r\}$, we see that the edges between R and S form a multicoloured $K_{r,s}$, which is a contradiction. This shows that each vertex in U is incident to at least $3\alpha(r,s)n^{1-1/r} - r$ vertices of U by edges of multiplicity k. Then there are at least $(3\alpha(r,s)n^{1-1/r} - r) \cdot |U|/2 > \alpha(r,s)|U|^{2-1/r}$ edges of H within U, so by Proposition 4.1 these edges form a multicoloured $K_{r,s}$. This contradiction completes the proof of the theorem. □

It seems plausible that $\rho_{K_{r,s}}(n)$ and $\sigma_{K_{r,s}}(n)$ are of order $n^{1/r}$ whenever $2 \leqslant r \leqslant s$, but we are unable to prove this here. The difficulty is that for general bipartite graphs, the known lower bounds for the Turán numbers of bipartite graphs are not of the same order of magnitude as the upper bounds. We will consider the specific case of the 4-cycle, which is better understood, and where such a matching lower bound is known.

Let q be such that there is a projective plane with lines of size $q+1$, e.g., q can be any prime power. For $n_0 = q^2+q+1$ and $e_0 = q(q+1)^2/2$, a result of Füredi [4] shows that

$\mathrm{ex}(n_0, C_4) = e_0$. We will prove a multicoloured version of this result. We make no attempt to optimize our constants, and will assume in the rest of this section that q is sufficiently large.

Theorem 4.3. *For $k > 10^3\sqrt{n_0}$, $\mathrm{ex}_k(C_4, n_0) = ke_0$. Furthermore, in any extremal simply k-coloured multigraph, the colours are identical extremal C_4-free graphs.*

To prove this theorem we first need the following three lemmas.

Lemma 4.4. *Let $t < q^2/3$ and let G be a C_4-free graph on $n = n_0 - t$ vertices. Suppose G has $e \geqslant e_0 - 5qt/6 - i$ edges for some $i \geqslant 0$ and minimum degree $5q/6$. Let S be a set of edges that can be added to G, so that there is no C_4 using one edge from S and 3 edges from G. Then*

$$|S| < 20\big((t+1)q + i + t^2/q\big).$$

Proof. Consider a vertex v. Let $N(v)$ be its neighbourhood, $N_2(v)$ its second neighbourhood and $W(v) = V(G) - v - N(v) - N_2(v)$. By our assumption $|N(v)| \geqslant 5q/6$, and since G is C_4-free

$$\big|N_2(v)\big| \geqslant \sum_{u\sim v}\big(d(u) - 2\big) \geqslant \frac{5}{6}q\left(\frac{5}{6}q - 2\right).$$

Therefore $|W(v)| < 11q^2/36 + 2q$. We also have

$$\begin{aligned}\sum_v\big|N_2(v)\big| &\geqslant \sum_v\sum_{u\sim v}\big(d(u)-2\big) = \sum_v\big(d(v)^2 - 2\big|N(v)\big|\big)\\ &\geqslant n\left(\frac{\sum_v d(v)}{n}\right)^2 - 4e = 4e^2/n - 4e\end{aligned}$$

and so

$$\sum_v\big|W(v)\big| \leqslant n(n-1) - \sum_v\big|N(v)\big| - \sum_v\big|N_2(v)\big| \leqslant n(n-1) - 4e^2/n + 2e.$$

Let $S(v)$ be those u such that uv is in S. By definition of S, any vertex u in $S(v)$ has no neighbour in $N_2(v) \cup v$. Since G is C_4-free u can have at most one neighbour in $N(v)$, so it has at least $d(u) - 1$ neighbours in $W(v)$. Consider any $X \subset S(v)$ with $|X| = x$ and the edges of G with one endpoint in X and the other in $W(v)$. A pair of vertices in X have at most one common neighbour, so by a Bonferroni inequality we get

$$\frac{11}{36}q^2 + 2q > \big|W(v)\big| \geqslant \sum_{u\in X}\big(d(u) - 1\big) - \binom{x}{2} \geqslant x\left(\frac{5}{6}q - 1 - (x-1)/2\right).$$

Note that since q is large, it is impossible to set $x = 5q/6 - 1$. Therefore $s_v = |S(v)| < 5q/6 - 1$, and so $|W(v)| \geqslant s_v(5q/6 - 1 - (s_v - 1)/2) > 5qs_v/12$. Thus

$$|S| = \frac{1}{2}\sum_v s(v) < \frac{6}{5q}\sum\bigl|W(v)\bigr| < \frac{2}{q}\bigl(n(n-1) - 4e^2/n + 2e\bigr).$$

Substituting $n = q^2 + q + 1 - t$ and $e \geqslant q(q+1)^2/2 - 5qt/6 - i$ we have a crude estimate

$$nq|S| < 2\bigl(n^2(n-1) - 4e^2 + 2en\bigr) < 10\bigl((t+1)q^4 + iq^3 + t^2q^2\bigr).$$

Since $t < q^2/3$, we have $2n - q^2 = q^2 + 2q + 2 - 2t > 0$. Therefore

$$\begin{aligned}&20nq\bigl((t+1)q + i + t^2/q\bigr) - 10\bigl((t+1)q^4 + iq^3 + t^2q^2\bigr)\\ &\quad = 10\bigl(2n - q^2\bigr)\bigl((t+1)q^2 + t^2 + iq\bigr) > 0,\end{aligned}$$

so $|S| < 20((t+1)q + i + t^2/q)$, as required. □

The next two lemmas are the most technical part of the proof. They contain algebraic manipulations that are quite involved, so we checked them with Mathematica to ensure their correctness.

Lemma 4.5. *A C_4-free graph G on n_0 vertices with $t < q^2/3$ vertices of degree at most $5q/6 + 1$ and some vertex of degree at least $q + 2$ has at most $e_0 - tq/150$ edges.*

Proof. The case $t = 0$ is a lemma of Füredi in [3]. We will extend his argument to deal with the case $t > 0$. Let v be a vertex of degree $q + 2 + x$, with $x \geqslant 0$. We count paths of length 2 that join two points of $V - N(v)$. Since G is C_4-free there is at most one such path between each pair, giving at most $\binom{n_0-(q+2+x)}{2}$ paths. On the other hand any vertex $u \neq v$ has at most one neighbour in $N(v)$, so contributes at least $\binom{d(u)-1}{2}$ such paths. Therefore

$$\binom{n_0 - (q+2+x)}{2} \geqslant \sum_{u\neq v}\binom{d(u)-1}{2}. \tag{3}$$

Note that $\sum_{u\neq v} d(u) - 1 = 2e - (q + 2 + x) - (n_0 - 1) = 2e - n_0 - q - 1 - x$, where $e = e(G)$. Since $\binom{y}{2}$ is a convex function of y, the right hand side of Eq. (3) is minimised by taking t terms equal to $5q/6$, leaving $n_0 - t - 1$ terms with total $2e - n_0 - q - 1 - x - 5qt/6$. This gives

$$\binom{n_0 - (q+2+x)}{2} \geqslant t\binom{5q/6}{2} + (n_0 - 1 - t)\binom{\frac{2e-n_0-q-1-x-5qt/6}{n_0-1-t}}{2}.$$

Multiplying both sides by $n_0 - 1 - t$, substituting $n_0 = q^2 + q + 1$ and expanding gives $2e^2 - \alpha e - \beta \leqslant 0$, where

$$\alpha = 3q^2 + 5(1+t/3)q - t + 4 + 2x,$$
$$\beta = \frac{1}{2}q^6 + \frac{1}{2}q^5 - \left(\frac{61t}{72} + \frac{5}{2} + x\right)q^4 - \left(\frac{85t}{72} + 5 + x\right)q^3 + \left(\frac{t}{3} - 5 + tx + \frac{x^2}{2}\right)q^2 + \left(\frac{x^2}{2} - \frac{5tx}{6} - x - \frac{2t}{3} - 4\right)q - \left(2 + (t+2)x + \frac{(t+1)x^2}{2}\right).$$

Then we have $e \leqslant (\alpha + \sqrt{\alpha^2 + 8\beta})/4$. We want to show that this is at most $e_0 - qt/150$, i.e., that $\sqrt{\alpha^2+8\beta} < 4(e_0 - qt/150) - \alpha$. Substituting our expressions for e_0, α and β we can write

$$\big(4(e_0 - qt/150) - \alpha\big)^2 - (\alpha^2 + 8\beta) = A + Bx$$

where

$$A = \left(\frac{56q^2}{625} - \frac{4q}{75}\right)t^2 + \left(\frac{q^4}{225} + \frac{13q^3}{225} - \frac{88q^2}{75} + \frac{716q}{75}\right)t - 12q^3 - 8q^2 + 16q + 16,$$
$$B = 8q^4 - 8q^2(t+2) + \frac{508qt}{75} + 8t + 16 - 4\big(q^2 + q - t - 1\big)x.$$

Since $x \leqslant n_0 - (q+2) < q^2$ and $t < q^2/3$, for large q we have

$$B > 8q^4 - 8q^2(t+2) - 4\big(q^2+q\big)x > q^4 - 4q^3 - 16q^2 > 0.$$

Since $t \geqslant 1$, we can estimate crudely

$$A > \frac{1}{20}q^2t^2 + \frac{1}{250}q^4t - 20q^3 > 0.$$

Therefore $(4(e_0 - qt/150) - \alpha)^2 - (\alpha^2 + 8\beta) > 0$, as required. □

Lemma 4.6. *A C_4-free graph G on n_0 vertices with at least $q^2/3$ vertices of degree at most $5q/6$ has at most $e_0 - q^3/500$ edges.*

Proof. We count the number of paths of length two in G. Since G is C_4-free there is at most one such path between each pair of vertices. Therefore $\binom{n_0}{2} \geqslant \sum_v \binom{d(v)}{2}$. The right hand side of this inequality is clearly minimised by taking $q^2/3$ terms equal to $5q/6$, leaving $n_0 - q^2/3$ terms with total $2e - 5q^3/18$. This gives

$$\binom{n_0}{2} \geqslant \frac{q^2}{3}\binom{5q/6}{2} + (n_0 - q^2/3)\binom{\frac{2e-5q^3/18}{n_0-q^2/3}}{2}.$$

Multiplying both sides by $n_0 - q^2/3$ and substituting $n_0 = q^2 + q + 1$ gives $2e^2 - \alpha e - \beta \leqslant 0$, where

$$\alpha = \frac{5q^3}{9} + \frac{2q^2}{3} + q + 1, \qquad \beta = \frac{47q^6}{216} + \frac{227q^5}{216} + \frac{443q^4}{216} + \frac{7q^3}{3} + \frac{3q^2}{2} + \frac{q}{2}.$$

Then, since q is large, we have

$$e \leqslant \frac{1}{4}\left(\alpha + \sqrt{\alpha^2 + 8\beta}\right) = \frac{1}{4}\left(5/9 + \sqrt{(5/9)^2 + 8(47/216)}\right)q^3 + O\left(q^{5/2}\right) < 0.497q^3.$$

Therefore $e < e_0 - q^3/500$, as required. $\square$

Proof of Theorem 4.3. Let G be a simply k-coloured multigraph on n_0 vertices with colours $G_1, \dots, G_k$, containing no multicoloured C_4. By Proposition 2.4 we can assume that the colours are nested as $G_1 \subset \cdots \subset G_k$ with $G_1, \dots, G_{k-3}$ all equal to some fixed C_4-free graph. If $e(G_1) = e_0$, then it is a maximal C_4-free graph and adding a new edge to it from one of G_{k-2}, G_{k-1} or G_k will create a C_4 which is multicoloured in G. Therefore, in this case $G_1 = \cdots = G_k$, $e(G) = ke_0$ and we are done.

Now suppose $e(G_1) < e_0$. To finish the proof we show that $e(G) < ke_0$. Let T be the vertices of G_1 of degree less than $5q/6$ and let $t = |T|$. If $t \geqslant q^2/3$, then by Lemma 4.6 we have $e(G_1) < e_0 - q^3/500$. Since $k > 10^3 q$, this implies that

$$\sum e(G_i) < (k-3)e(G_1) + 3\binom{n_0}{2} < ke_0 - \left(\frac{(k-3)q^3}{500} - 3\binom{n_0}{2}\right) < ke_0.$$

Therefore we can assume that $t < q^2/3$.

Next suppose that $e(G_1) \leqslant e_0 - tq/150$. Since $e(G_1) < e_0$, this in particular covers the case when $t = 0$. Note that an edge from $S = E(G_k) - E(G_1)$ can not form a C_4 together with any three edges of G_1, since this cycle will be multicoloured. Consider the subgraph of G_1 induced by the set $V(G_1) - T$. It has $n = n_0 - t$ vertices and at least $e = e(G_1) - 5qt/6 = e_0 - 5qt/6 - (e_0 - e(G_1))$ edges. Therefore applying Lemma 4.4 with $i = e_0 - e(G_1)$ to this subgraph, we conclude that S has at most $20((t+1)q + e_0 - e(G_1) + t^2/q)$ edges in $V(G_1) - T$. Clearly the number of edges from S incident with T is at most tn_0. Since $k > 10^3 q$, we get

$$\begin{aligned} |S| &< t\left(q^2 + q + 1\right) + 20\left((t+1)q + e_0 - e(G_1) + t^2/q\right) < 20\left(e_0 - e(G_1)\right) + 2tq^2 \\ &< \frac{1}{3}k\left(e_0 - e(G_1)\right) \end{aligned}$$

and therefore $\sum e(G_i) < ke(G_1) + 3|S| < ke_0$.

Now we can assume that $e(G_1) > e_0 - tq/150$ and $t \geqslant 1$. Then by Lemma 4.5, all vertices of G_1 have degree at most $q + 1$. We must have $t \leqslant 6$, since if $t \geqslant 7$ we get the contradiction

$$e(G_1) < \frac{1}{2}\left(n_0(q+1) - tq/6\right) < e_0 - tq/100.$$

Note that no edge $s \in S$ can be incident with a vertex of degree $q+1$ in G_1. For then, by definition of S, $G_1 + s$ is C_4-free, has a vertex of degree $q+2$ and t vertices of degree at most $5q/6+1$. Thus by Lemma 4.5 it has at most $e_0 - tq/150$ edges, contrary to assumption. If there is a vertex in G_1 incident to at least $q+1$ edges of S, then the endpoints of these edges are vertices of degree at most q in G_1. Thus G_1 will have at least $q+1$ vertices of degree at most q, and hence at most q^2 vertices of degree $q+1$. This contradicts our assumption, as $t \leqslant 6$, so

$$e(G_1) < \frac{1}{2}\bigl(q^2(q+1) + q \cdot q + 5q/6\bigr) < e_0 - tq/150.$$

Therefore every vertex of G_1 is incident with at most q edges of S. In particular, the number of edges of S incident to vertices in T is at most $tq \leqslant 6q$. On the other hand, as we already mentioned above, S has at most

$$20\bigl((t+1)q + \bigl(e_0 - e(G_1)\bigr) + t^2/q\bigr) < 20\bigl((t+1)q + tq/150 + t^2/q\bigr) < 150q$$

edges in $V(G_1) - T$. This gives $|S| < 156q$ and so $\sum e(G_i) < k(e_0 - 1) + 3|S| < ke_0$, as required. □

5. 3-colour-critical graphs

A graph H is 3-*colour-critical* if it has chromatic number 3, and there is some edge e such that $H - e$ is bipartite. An example of such a graph is an odd cycle. In this section we will determine the multicolour Turán numbers of such graphs. A result of Simonovits [9] shows that, for sufficiently large n, the Turán numbers for these graphs are the same as for triangles, i.e., $\mathrm{ex}(n, H) = \lfloor n^2/4 \rfloor$. Suppose that H has p vertices and q edges. Note that H is a subgraph of the graph obtained by adding an edge in one of the classes of the complete bipartite graph $K_{p,p}$.

Now we describe the extremal simply k-coloured multigraphs G. For $k \leqslant q-1$ we can set $w(e) = k$ for every e, so $\mathrm{ex}_k(n, H) = k\binom{n}{2}$. Similarly to the case of complete graphs, there are two natural constructions for larger k: either exactly $q-1$ colours of G are taken to be the complete graph, or all colours of G are identical Turán graphs $T_2(n)$. For large n, we will show that one of these constructions is always the unique extremal solution for the problem. Note that the first has $(q-1)\binom{n}{2}$ edges and the second has $k\lfloor n^2/4 \rfloor$, which is better than the first for $k \geqslant 2(q-1)$. First we need the following lemma.

Lemma 5.1. *Let H be a graph with p vertices and q edges. Pick*

$$\varepsilon < \min\left(\frac{1}{32q^4}, \frac{1}{64pq^3}\right).$$

Let G be a simply k-coloured multigraph, containing no multicoloured copy of H, with $q \leqslant k \leqslant 2(q-1)$. Suppose n is sufficiently large, G has minimum degree at least

$(q-1)(n-1)$ and at most εn^2 edges with multiplicity at least q. Then $e(G)=(q-1)\binom{n}{2}$, exactly $q-1$ colours of G are complete graphs, and the rest are empty.

Proof. Let E_1 be the edges of G with multiplicity at least q, E_2 be the edges with multiplicity exactly $q-1$, and $E_3=E_1\cup E_2$. By hypothesis we have $|E_1|\leqslant \varepsilon n^2$. It suffices to show that E_1 is empty. Then we have $e(G)\leqslant (q-1)\binom{n}{2}$, and the minimum degree condition implies the assertion of the theorem. Suppose not, and consider some edge ab of E_1. For any vertex v we write $N_i(v)$ for the vertices u for which uv is an edge of E_i. Note that

$$(q-1)(n-1)\leqslant d(a)\leqslant 2(q-1)\big|N_1(a)\big|+(q-1)\big|N_2(a)\big| \\ +(q-2)\big(n-1-\big|N_1(a)\big|-\big|N_2(a)\big|\big),$$

so $|N_2(a)|\geqslant n-1-q|N_1(a)|$. Also, we have

$$(q-1)(n-1)\leqslant d(b)\leqslant 2(q-1)\big|N_3(b)\big|+(q-2)\big(n-1-\big|N_3(b)\big|\big),$$

so $|N_3(b)|\geqslant (n-1)/q$.

Suppose that $|N_1(a)|<n/4q^2$. Then $|N_2(a)|>(1-1/(4q))n$, so the set $S=N_2(a)\cap N_3(b)$ contains at least $n/4q$ vertices. Note that the subgraph induced by E_3 on S contains no K_{p-2}. Otherwise, together with $\{a,b\}$ we would have a K_p with every edge of multiplicity at least $q-1$, and at least one edge with multiplicity at least q. Such a K_p clearly contains a multicoloured copy of H, so this is impossible. Now, by Turán's theorem, E_3 has at most $(p-4)|S|^2/(2(p-3))$ edges in S. So at least $|S|^2/(2(p-3))+O(|S|)>|S|^2/(2p)$ edges in S have multiplicity at most $q-2$. Thus, using that $\varepsilon<1/(64pq^3)$, we obtain

$$e(G)<(q-1)\left(\binom{n}{2}-\frac{1}{p}|S|^2/2\right)+(q-2)\frac{1}{p}|S|^2/2+k\varepsilon n^2 \\ \leqslant (q-1)\binom{n}{2}-\left(\frac{1}{32pq^2}-2(q-1)\varepsilon\right)n^2<(q-1)\binom{n}{2},$$

which is contrary to hypothesis. Therefore $|N_1(a)|\geqslant n/(4q^2)$.

Let T be the points that are incident to at least one edge of E_1. For each $a\in T$ we have $|N_1(a)|\geqslant n/4q^2$. So if T is non-empty, we have $|T|\geqslant n/4q^2$, and

$$|E_1|\geqslant \big(|T|\cdot n/4q^2\big)/2\geqslant \frac{n^2}{32q^4}>\varepsilon n^2.$$

This contradicts our hypothesis, unless T and E_1 are empty. □

Theorem 5.2. *Let H be a 3-colour-critical graph with p vertices and q edges. If $q \leqslant k < 2(q-1)$ and n is sufficiently large then*

$$\mathrm{ex}_k(n,H) = (q-1)\binom{n}{2}.$$

Furthermore, in any extremal simply k-coloured multigraph, exactly $q-1$ colours are non-empty and all of these $q-1$ colours are complete graphs K_n.

Proof. Suppose G is a simply k-coloured multigraph containing no multicoloured copy of H, and $e(G) \geqslant (q-1)\binom{n}{2}$. By a similar vertex deletion argument to that in Theorem 3.1, we can assume G has minimum degree at least $(q-1)(n-1)$. By Lemma 5.1 we can assume that there are at least εn^2 edges with multiplicity at least q, where ε is as defined in the lemma. Therefore, by the well known bounds on bipartite Turán numbers (see, e.g., Proposition 4.1), we can find a complete bipartite graph with bipartition (B_1, B_2) and $|B_1| = |B_2| = 2pq$, in which each edge has multiplicity at least q. Let $B = B_1 \cup B_2$. Note that all pairs within B_1 or B_2 must have multiplicity 0, or we find a multicoloured copy of H.

Consider a vertex v not in B. If there are p vertices $P \subset B_i$ in one part, all joined to v by edges of multiplicity at least $q-1$, then v cannot have any neighbours in the other part B_{3-i}. For suppose $w(uv) > 0$, for some $u \in B_{3-i}$, and choose a set of p points $Q \subset B_{3-i} \cup v$ that contains both u and v. Then (P, Q) is a copy of $K_{p,p}$+edge, so contains a copy of H, in which exactly one edge goes from v to $Q - \{v\}$, and the rest go between Q and P. We see that all but one of the edges of this copy of H have multiplicity at least $q-1$, and all edges not incident with v have multiplicity q. Since H clearly must contain an edge not incident with v, this copy of H is multicoloured—a contradiction.

It follows that the maximum degree of v in B is achieved by one of the following two options. Either v should be joined to one part with edges of maximum multiplicity, with no edges going to the other part, or v should be joined to $p-1$ vertices in each part with edges of maximum multiplicity, and to the remaining vertices in B with edges of multiplicity $q-2$. It is easy to check that in both cases $d_B(v) \leqslant (2q-3)2pq$. Therefore

$$\begin{aligned} 4pq(q-1)(n-1) = (q-1)(n-1)|B| &\leqslant \sum_{x \in B} d(x) = 2e(B) + e(B, V-B) \\ &\leqslant 8p^2q^2(2q-3) + (2q-3)2pq \cdot (n-4pq) = 2pq(2q-3)n. \end{aligned}$$

This is a contradiction for large n, which proves the result. □

Theorem 5.3. *Let H be a 3-colour-critical graph with p vertices and q edges. If $k \geqslant 2(q-1)$ and n is sufficiently large then $\mathrm{ex}_k(n,H) \leqslant k\lfloor n^2/4 \rfloor$. Furthermore, in any extremal simply k-coloured multigraph, all colours are identical Turán graphs $T_2(n)$.*

Proof. By Proposition 2.5 it is enough to consider only the case $k = 2(q-1)$. Suppose G is a simply k-coloured multigraph containing no multicoloured copy of H, and that $e(G) \geqslant k\lfloor n^2/4 \rfloor$. Again, by vertex deletion we can assume G has minimum degree at least

$k\lfloor n/2\rfloor \geqslant (q-1)(n-1)$. Hence, we can apply Lemma 5.1 as above, to find a bipartite graph (B_1, B_2) with $|B_1| = |B_2| = 2pq$ in which each edge has multiplicity at least q. Let $B = B_1 \cup B_2$. Similarly to the proof of the previous theorem, we see that B_1 and B_2 contain no edges, and any $v \notin B$ has maximum degree $4pq(q-1)$ in B, which can only be achieved by joining v to one part with edges of maximum multiplicity $2(q-1)$, with no edges going to the other part.

When n is even it is now easy to finish the argument. We claim that all vertices $v \notin B$ must have $d_B(v) = 4pq(q-1)$. Otherwise, if even one vertex has degree $d_B(v) \leqslant 4pq(q-1) - 1$, we have

$$\begin{aligned}\sum_{x\in B} d(x) &= 2e(B) + e(B, V-B)\\ &\leqslant 2\cdot 2(q-1)\cdot(2pq)^2 + 4pq(q-1)(n-4pq-1) + \big(4pq(q-1)-1\big)\\ &= 4pq(q-1)n - 1 < 4pqk\lfloor n/2\rfloor,\end{aligned}$$

which is a contradiction to the minimum degree assumption. This gives a partition $V - B = V_1 \cup V_2$, where each $v \in V_i$ has no edges joining it to B_i, and is joined by edges of maximum multiplicity to every vertex in B_{3-i}. It follows easily that both $V_i \cup B_i$, for $i = 1, 2$, are independent sets. This is sufficient to prove the result, as then there can be at most $\lfloor n^2/4\rfloor$ edges with positive multiplicity, with equality when they form a copy of $T_2(n)$ and all have multiplicity k. When n is odd there is rather more work to do, but we will merely sketch the argument, as it is similar to the one at the end of Theorem 3.2. In this case, the counting argument for $\sum_{x\in B} d(x)$ shows that all but some constant C vertices $v \notin B$ have $d_B(v) = 4pq(q-1)$, so we get two independent sets V_1, V_2 with $|V_1| + |V_2| \geqslant n - C$. Now it suffices to show the following claim.

Claim 5.4. *Let U_1, U_2 be disjoint independent subsets of V. Let $U = U_1 \cup U_2$ and suppose $|V(G) - U| < C$. Then for any v in $V(G) - U$, there is some i for which $U_i \cup v$ is independent.*

The proof of this is similar to the analogous part of Theorem 3.2, so we omit it. □

As we have remarked, an odd cycle C_{2t+1} is 3-colour-critical, so we have the following corollary.

Corollary 5.5. *For n sufficiently large, $\mathrm{ex}_k(n, C_{2t+1})$ equals $2t\binom{n}{2}$ for $2t < k < 4t$, and equals $k\lfloor n^2/4\rfloor$ for $k \geqslant 4t$.*

6. Concluding remarks

- There are two natural constructions of a simply k-coloured multigraph not containing a multicoloured copy of H. The first construction is to take all colours equal to some fixed extremal graph for H, the second is to take up to $e(H) - 1$ copies of a complete graph. When $H = K_r$ is a complete graph and n is sufficiently large, we showed that one of these two constructions is always extremal, and the transition occurs when $k = \sigma_{K_r}(n) = \lceil (r^2 - 1)/2 \rceil$. This probably remains true even for $n \geqslant Cr^2$, for some constant C. On the other hand, the picture seems less clear for smaller values of n, as the approximation $t_{r-1}(n)/\binom{n}{2} \sim (r-2)/(r-1)$ is no longer accurate, so $\sigma_{K_r}(n)$ will depend on n.

- It would be interesting to classify the graphs H that have only these two extremal constructions. Perhaps this class includes colour-critical graphs, complete bipartite graphs and even cycles? Our results for 3-colour-critical graphs and C_4 provide some preliminary evidence for this conjecture. However, as we saw with the example of the bowtie, there are graphs with other extremal constructions.
- Finally, we remark that Theorem 1.1 can be extended to a more general class of objects than graphs. For a set X, we write 2^X for the set of subsets of X. We refer to a subset of 2^X as a *family* and to a collection of families as a *property*. Motivated by the intersection/union transformation from Lemma 2.1, we call a property $\mathcal{P}$ a *lattice property* if whenever $P \in \mathcal{P}$ and $Q \in \mathcal{P}$ we have $P \cup Q \in \mathcal{P}$ and $P \cap Q \in \mathcal{P}$. Note that the set of all graphs is a lattice property, indeed, so is the set of all r-uniform hypergraphs for any r.
  For a family $F \subset 2^X$, we write $\mathrm{ex}(\mathcal{P}, F) = \max\{|P|\colon\ P \in \mathcal{P}, F \not\subset P\}$. A family in $\mathcal{P}$ achieving this maximum is called *extremal* for F. A *simply* $(k, \mathcal{P})$*-coloured family* is the multiset sum of k families $P_1, P_2, \dots, P_k \in \mathcal{P}$, called *colours*. Write $\mathrm{ex}_k(\mathcal{P}, F)$ for the maximum size of a simply $(k, \mathcal{P})$-coloured family G not containing a multicoloured copy of F. Using the same arguments as in the proof of Theorem 1.1 we can obtain the following generalization.

  Theorem 6.1. *Let $\mathcal{P}$ be a lattice property and $k > \max\{|P|\colon\ P \in \mathcal{P}\} - \mathrm{ex}(\mathcal{P}, F) + |F|$. Then $\mathrm{ex}_k(\mathcal{P}, F) = k \cdot \mathrm{ex}(\mathcal{P}, F)$. Furthermore, in an extremal simply k-coloured family for F, every colour is an identical extremal family for F.*

  This theorem can provide a basis for studying multicoloured versions of various other problems in extremal set theory.

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