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Biased orientation games

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ABSTRACT

We study biased *orientation games*, in which the board is the complete graph K_n, and OMaker (oriented maker) and OBreaker (oriented breaker) take turns in directing previously undirected edges of K_n. At the end of the game, the obtained graph is a tournament. OMaker wins if the tournament has some property $\mathcal{P}$ and OBreaker wins otherwise.

We provide bounds on the bias that is required for OMaker's win and for OBreaker's win in three different games. In the first game OMaker wins if the obtained tournament has a cycle. The second game is Hamiltonicity, where OMaker wins if the obtained tournament contains a Hamilton cycle. Finally, we consider the H-creation game, where OMaker wins if the obtained tournament has a copy of some fixed digraph H.

1. Introduction

In this work, we study *orientation games*. The board consists of the edges of the complete graph K_n. In the $(p : q)$ game the two players, called *OMaker* and *OBreaker*, take turns in orienting (or directing) previously undirected edges. OMaker starts the game, and at each round OMaker directs at most p edges and then OBreaker directs at most q edges (usually, we consider the case where $p = 1$ and q is large). Both players have to direct at least one edge at each round. The game ends when all the edges are oriented, and then we obtain a *tournament*. OMaker then wins if the tournament has some fixed property $\mathcal{P}$, and OBreaker wins otherwise. Here we focus on the $1 : b$ game, which is referred to as the b-biased game. Since at each round each player has to orient at least one edge, the number of rounds is clearly bounded. It is easy to verify that the game is *bias monotone*, meaning that increasing b can only help OBreaker. Hence, every property $\mathcal{P}$ admits some threshold $t(n, \mathcal{P})$ so that OMaker wins the b-biased game if $b < t(n, \mathcal{P})$ and OBreaker wins the b-biased game if $b \geq t(n, \mathcal{P})$. We stress that in all these games OMaker wins if the obtained tournament has the desired property, no matter who directs each edge of a winning directed subgraph.

This game is an alteration of the well studied classical Maker–Breaker game, which is defined by a hypergraph $(X, \mathcal{F})$ and bias $(p : q)$. In that game, at each round Maker claims p elements of X, and Breaker claims q elements of X. Maker wins if by the end of the game he claimed all the elements of some hyperedge $A \in \mathcal{F}$, and Breaker wins otherwise. Usually, a typical problem goes as follows. Given a game hypergraph $H = (X, \mathcal{F})$, determine or estimate the threshold function t_H such that if $b > t_H$ then Breaker wins in a $(1 : b)$ game, and if $b \leq t_H$ then Maker wins in a $(1 : b)$ game. There has been a long line of research that studies the bias threshold of various games (see, e.g., [3,5,9,11–13] and their references).

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Here we study orientation games for the following three properties.

Creating a cycle. OMaker wins if the obtained tournament contains a cycle, and OBreaker wins otherwise. It is well known that a tournament contains a cycle if and only if it contains a cyclic triangle (cycle of length 3). This is a problem which has already been studied by Alon (unpublished result) and by Bollobás and Szabó [7], and here we improve their results.

Creating a Hamilton cycle. Here OMaker wins if the final tournament contains a Hamilton cycle, and OBreaker wins otherwise. Recently, the second author [12] solved a long standing question and provided tight bounds on the bias threshold for Maker's win in the classical Maker–Breaker Hamiltonicity game. We use a variant of his approach, together with a new application of the Gebauer–Szabó method [11] and give tight bounds in our case as well.

Creating a copy of H. Here we are given a fixed digraph H. OMaker wins if the obtained tournament contains a copy of H, and OBreaker wins otherwise. We provide both upper and lower bounds, and give some nearly tight bounds for specific cases. We conjecture that the correct threshold is closely related to the size of the *minimum feedback arc set* of H, and provide some results that support this conjecture.

Our first theorem considers the cycle creation game. It is easy to observe that if $b \geq n-2$ then OBreaker has a winning strategy (for completeness, we give a detailed proof in Section 3). Bollobás and Szabó [7] proved that if $b = (2-\sqrt{3})n$, then OMaker wins the game, and conjectured that the correct threshold is $b = n-2$.

In this work we provide a simple argument that improves their result.

Theorem 1 (*The Cycle Game*)**.** *For every $b \leq n/2 - 2$, OMaker has a strategy guaranteeing a cycle in the b-biased orientation game.*

The second game we consider is the Hamiltonicity game, where OMaker wins if and only if the obtained tournament contains a Hamilton cycle. Here we apply techniques from [9,11,12] to get tight bounds on the bias threshold for a win of OBreaker.

Theorem 2 (*The Hamiltonicity Game*)**.**

(i) *If $b \geq \frac{n(1+o(1))}{\ln n}$, OBreaker has a strategy to guarantee that in the b-biased orientation game the obtained tournament has a vertex of in-degree* 0*, and in particular to win the Hamiltonicity game.*

(ii) *If $b \leq \frac{n(1+o(1))}{\ln n}$, OMaker has a strategy guaranteeing a Hamilton cycle in the b-biased orientation game.*

In the H-creation game we have some partial results. We conjecture that the bias that guarantees OMaker's win depends on the minimum feedback arc set of H, and support this result for graphs with a small feedback arc set. We will present and discuss corresponding notions and results in Section 5.

2. Preliminaries

Let K_n be the complete graph on n vertices, a tournament is an orientation of K_n. A directed graph is called *oriented* if it contains neither loops nor cycles of length 2. Every oriented graph is a subgraph of a tournament. A directed graph is *strongly connected* if for every two vertices u, v there is a directed path from u to v and a directed path from v to u. All directed graphs we consider here are *oriented*, i.e., do not have parallel or opposite edges.

All log signs are in base 2, except ln signs which are in the natural base.

The well known results of Erdős and Selfridge [10] and Beck [4] give a sufficient condition for Breaker's win in biased Maker–Breaker games.

Theorem 3. *Suppose that Maker and Breaker play a $(p : q)$-game on a hypergraph $H = (V, \mathcal{F})$, with Maker starting. If*

$$\sum_{A\in\mathcal{F}} (q+1)^{-\frac{|A|}{p}} < \frac{1}{q+1},$$

then Breaker has a winning strategy.

An orientation game is defined by a series of moves by OMaker and OBreaker. In round t, OMaker orients $1 \leq m_t \leq p = p(n)$ edges (usually in our settings $p = 1$) and OBreaker orients $1 \leq b_t \leq q = q(n)$ edges. The game ends when all the edges are oriented, so the obtained graph is a tournament. OMaker wins if the tournament has some predetermined property $\mathcal{P}$, otherwise OBreaker wins.

We denote by H_t the graph containing the edges oriented after t rounds. Clearly, this graph has at most $(p+q)\cdot t$ edges.

Given a directed graph $G = (V, E)$, we write $(u, v) \in E$ if there is an edge from u to v. Given a set $A \subseteq V$, we let

$$N^+(A) = \{u \in V \setminus A : \exists v \in A, (v, u) \in E\},$$

and

$$N^-(A) = \{u \in V \setminus A : \exists v \in A, (u, v) \in E\}.$$

A tournament T on n vertices is *transitive* if there is a bijection $\sigma : V(T) \to [n]$ such that for every edge $(u, v) \in E(T)$, $\sigma(u) < \sigma(v)$. A tournament $T = (V, E)$ is k-colorable if there is a partition of V into k sets $V_1, \ldots, V_k$ such that the induced tournament on each V_i is transitive. Thus, a transitive tournament is 1-colorable.

3. The cycle game

In this section we prove Theorem 1. Namely, we show that in the $(n/2-2)$-biased game OMaker can create a cycle. For the sake of completeness we also prove that in the $(n-2)$-biased game OBreaker can force an acyclic tournament.

OBreaker's strategy. Suppose that $b \geq n-2$, we show that OBreaker can block all cycles in the graph as follows. Whenever OMaker orients an edge from u to v, OBreaker responds by orienting all edges from u to every vertex $w \in V(K_n)$ such that the edge uw has not been oriented yet. Clearly, OBreaker in his turn has to orient at most $n-2$ edges.

We proceed by proving that no cycle is created when OBreaker applies this strategy. Indeed, suppose that a cycle C is created and let (u, v) be the first edge in C that was oriented (by either OMaker or OBreaker), and suppose also that $(w, u) \in C$. If OMaker oriented the edge from u to v, by the strategy above OBreaker responded by orienting the edge from u to w, and thus $(w, u) \notin C$. If OBreaker orients the edge from u to v, he did it because OMaker oriented some other edge from u to some vertex z. In this case, again OBreaker will also orient the edge from u to w, and therefore again $(w, u) \notin C$. We conclude that no cycle is created.

OMaker's strategy. Our main lemma states that OMaker has a strategy so H_t contains a directed path of length t throughout the first $n-1$ rounds of the game.

Lemma 3.1. *In the b-biased game, OMaker has a strategy such that for every $t \leq n-1$, the graph H_t obtained after t rounds contains a directed path of length t.*

Proof. We prove by induction that at each round OMaker can extend a longest path by one, no matter how OBreaker plays, unless the board already contains a Hamilton path. Clearly OMaker can create a path of length 1 at the first round. Suppose that a longest path in $E(H_t)$ is $P_t = u_1, u_2, \ldots, u_r$, where $r < t$. Let v be a vertex not in the path. Let k be the maximal index such that there is no edge from v to u_k. This is well defined as if there is an edge from v to u_1 then $v, u_1, \ldots, u_r$ is a longer path, contradicting the maximality of P_t.

Observe first that if there is an edge in the opposite direction from u_k to v then $u_1, \ldots, u_r$ is not a maximal path. Indeed, if $k = r$ then $u_1, \ldots, u_r, v$ is a longer path; otherwise by the definition of k there is an edge from v to u_{k+1} and therefore by using the vertices $u_1, \ldots, u_k, v, u_{k+1}, \ldots, r$ we can find a longer path, and in both cases this contradicts the maximality of P_t.

Therefore OMaker in his turn orients the edge from u_k to v and creates a path of length at least $r+1$, and the result follows. $\square$

Proof of Theorem 1. The strategy of OMaker is as follows. At each round, if he can close a cycle he does so and wins. Otherwise, he increases the length of a longest directed path. We next show that after a large enough number of rounds, OBreaker cannot block all possible cycles.

As long as OMaker cannot orient an edge such that a cycle is created, OMaker can extend a longest directed path by 1 by Lemma 3.1. After t rounds, there is a path P_t of length at least t. Let $V_t = V(P_t)$. There are $\binom{t}{2} - t$ potential edges in $G[V_t]$ opposite to the direction of the path P_t, and orienting any of them creates a cycle.

Consider the graph H_{t-1} just before OMaker starts round t. There are $\binom{t-1}{2} - (t-1)$ edges that may close a cycle, of which at most $(b+1)(t-1)-(t-1)$ have been oriented in previous rounds. If $(b+1)(t-1)-(t-1) < \binom{t-1}{2} - (t-1)$ at the beginning of round t then OMaker wins. Unless OMaker wins before that, the game lasts at least $\frac{\binom{n}{2}}{b+1}$ rounds, and therefore by taking $t = \lfloor \frac{\binom{n}{2}}{b+1} \rfloor$ we get that if $b \leq n/2-2$ then OMaker surely wins. $\square$

4. The Hamiltonicity game

In this game OMaker wins if the obtained tournament contains a Hamilton cycle, and OBreaker wins otherwise. We start with recalling a well known and theorem due to Camion (see, e.g. Theorem 19.7 of [8]).

Lemma 4.1. *Let T be a strongly connected tournament on at least three vertices. Then T contains a Hamilton cycle.*

We conclude that if OMaker constructs a strongly connected graph from the edges he orients then he wins the game.

OBreaker's strategy. Assuming that the bias is sufficiently large, OBreaker has a strategy to guarantee that the obtained tournament T contains a vertex with in-degree 0. In this case clearly T does not contain a Hamilton cycle. To this end, we reduce this problem to a *box game*, similarly to the treatment in [9].

Let K_n be the complete graph on n vertices, and consider the b-biased game, where $b \geq \frac{(1+o(1))n}{\ln n}$. Recall that H_t is the oriented graph obtained after t rounds. Fix a partition $V(K_n) = A \cup B$, where A and B are disjoint sets, $|A| = b$, $|B| = n-b$. Throughout the game, OBreaker orients the edges from A to B until after some round t there are two vertices $u, u' \in A$ such that for every vertex $w \in B$, both $(u, w), (u', w) \in H_t$, and the in-degree of both u, u' is 0. Then in the last turn he orients edges within A so that either u or u' will have in-degree 0.

Chvátal and Erdős [9] studied the *box* game, which is a special case of a Maker–Breaker game where all the winning sets are disjoint. Their main result can be summarized as follows.

Theorem 4. *Suppose that there are r disjoint sets (or boxes) $B_1, \ldots, B_r$, each box B_i containing k elements. At each round, Box-Maker claims b elements and then Box-Breaker claims a single element. If*

$$k \le b \sum_{i=1}^{r} \frac{1}{i},$$

then Box-Maker has a strategy to occupy all the elements of a single box.

Note that in each round, Box-Breaker *destroys* a single box (i.e., puts his element in this box thus preventing Box-Maker from winning using that specific box), and so throughout the game Box-Maker tries to claim all elements of a single box before it is destroyed by Box-Breaker.

Here we need a variant of this theorem, for the case that Box-Maker actually has to complete two boxes.

Claim 4.2. *Suppose that there are r disjoint sets, $B_1, \ldots, B_r$, each B_i containing k elements. At each round, Box-Breaker destroys one set and then Box-Maker claims b elements. If*

$$k + b \le b \sum_{i=1}^{r} \frac{1}{i},$$

then Box-Maker has a strategy to occupy all the elements of two boxes.

Proof. For every box B_i we add a set B'_i of b virtual items. Consider a standard box game where the i'th box is $B_i \cup B'_i$, and suppose that Box-Maker always claims the elements of B_i before he claims the elements of B'_i, for every $1 \le i \le r$. If

$$k + b \le b \sum_{i=1}^{r} \frac{1}{i},$$

then by Theorem 4 Box-Maker has a strategy to win the game. Consider the last round before Box-Maker wins, when the next move should be taken by Box-Breaker. Since Box-Breaker cannot avoid Box-Maker's win there are at least two indices $i \ne j$ such that all but at most b elements of boxes i and j are already claimed by Box-Maker. Therefore we conclude that there are at least two indices $i \ne j$ such that B_i and B_j have been fully claimed. We conclude that Box-Maker claimed all the elements of two of the original boxes, no matter what Box-Breaker did. The claim follows. □

In our setting, OMaker and OBreaker switch their roles. That is, OBreaker assumes Box-Maker's role, and we define the boxes so that if OBreaker claims a full box the obtained tournament has a vertex of in-degree 0. For every vertex $v \in A$ we define a box X_v as $\{vw : w \in B\}$. Note that $|X_v| = |B| = n - b$. In every turn, OMaker (that is, Box-Breaker) can destroy one box X_v by directing an edge toward v, either from a vertex from A or from B. On the other hand, OBreaker (Box-Maker) can orient b edges from A to B, which is equivalent to taking b elements from the various boxes. By Claim 4.2, if

$$n = |X_v| + b \le b \sum_{i=1}^{|A|} \frac{1}{i}, \tag{4.1}$$

then OBreaker has a strategy to have two vertices u, u' from A for which all their incident edges that connect them to B are directed toward B, and none of the edges from A enters u or u'. Therefore, no matter what OMaker does, OBreaker can direct all the edges from either u or u', thus creating a vertex with in-degree 0 and destroying any chance for creating a Hamilton cycle in the final tournament. Taking $b \ge \frac{n(1+o(1))}{\ln n}$ satisfies (4.1) and thus OBreaker wins the game, and Item (i) in Theorem 2 follows.

OMaker's strategy. OMaker's strategy consists of two stages. His goal in the first stage is to create a graph with some expansion properties, so that all sufficiently small sets have at least one in-going edge and at least one out-going edge. (This is quite similar to the approach of [12] and is somewhat reminiscent of some of the arguments from [2].) To this end, he employs a randomized strategy that creates a graph with min in-degree and out-degree at least 3, playing against any fixed strategy by OBreaker. Moreover, we will show that with positive probability (and actually, with high probability) after this stage the graph has the desired expansion properties. Since the game considered is a perfect information game with no chance moves, we conclude that OMaker has a **deterministic** strategy that guarantees these properties after the first stage. Moreover, the first stage lasts at most $8n$ rounds in any case.

At the second stage, OMaker will ensure that for every large enough disjoint sets of vertices A, B there is at least one edge from A to B and at least one edge from B to A. We will show that if he succeeds at the first stage then after the second stage we will have a strongly connected graph and hence by the end of the game OMaker will win.

We say that a directed graph G is *k-expanding* if the following properties hold for $k > 0$.

- For every set A of size at most k, $|N^+(A)|, |N^-(A)| > 0$.
- For every two disjoint sets A, B of size at least k, there is an edge from A to B and there is an edge from B to A.

We have the following.

Lemma 4.3. *Let G be a directed graph, and suppose that G is k-expanding for some k. Then G is strongly connected.*

Proof. Let $A_1, A_2, \ldots, A_t$ be the strongly connected components of G, and suppose that $t > 1$. Let T be a graph where each A_i is represented by a vertex, and there is an edge from A_i to A_j if and only if there is a vertex $v_i \in A_i$ and a vertex $v_j \in A_j$ such that $(v_i, v_j) \in E(G)$. It is well known that T is a directed forest, and therefore contains a leaf, i.e., a set A_i with no outgoing edges. If $|A_i| < k$ then since $|N^+(A_i)| > 0$ we get a contradiction. If $|A_i| > n - k$, then since $|N^-(V \setminus A_i)| > 0$ we get a contradiction. Finally, if $k \le |A_i| \le n - k$, then by the second property there is an edge from A_i to $V \setminus A_i$. Therefore, we conclude that $t = 1$ and hence G is strongly connected. $\square$

We will show that after the first stage the obtained graph will have the first property in the definition of a k-expanding digraph with high probability, and after the second stage it will have the second property. More specifically, we will show that for $k = \frac{n}{(\ln n)^{2/5}}$, at the first stage OMaker ensures that for every set A of size at least k, $|N^+(A)|, |N^-(A)| > 0$, and at the second stage OMaker ensures that for every two sets A, B of size at least k, there is an edge from A to B. By Lemmas 4.1 and 4.3, after the second stage OMaker wins.

The first stage. At the first stage we adapt the techniques of Gebauer and Szabó [11] in a way similar to [12] and show that if $b = \frac{(1-o(1))n}{\ln n}$ then Maker has a strategy to achieve his goal at this stage. We start by reducing our game to an undirected game on the edges of a bipartite graph.

Suppose that OMaker and OBreaker play a biased orientation game on the edges of the complete graph $G = (V, E)$ on n vertices, and let $V = \{v_1, v_2, \ldots, v_n\}$. Let $H = (V_1, V_2, E')$ be the complete bipartite graph on $2n$ vertices, where $V_1 = \{v_{1,1}, v_{1,2}, \ldots, v_{1,n}\}$ and $V_2 = \{v_{2,1}, v_{2,2}, \ldots, v_{2,n}\}$. Throughout the game we maintain two subgraphs, H_M consisting of edges that are associated with OMaker and H_B consisting of edges that are associated with OBreaker. Initially both graphs are empty.

If OBreaker orients a previously undirected edge from v_i to v_j in G, we add the edge between $v_{2,i}$ and $v_{1,j}$ to H_B.

OMaker, in his turn, would like to create a graph with a constant minimum degree in H_M. Whenever OMaker wants to add some edge $(v_{1,i}, v_{2,j})$ to H_M and $(v_{2,i}, v_{1,j})$ has not been taken yet, he does it and also orients v_i to v_j in G (note that in this case the edge between v_i and v_j is undirected before this step). In this case we also add the edge $(v_{2,i}, v_{1,j})$ to H_B. If, on the other hand, $(v_{2,i}, v_{1,j}) \in E(H_B)$, then he adds $(v_{1,i}, v_{2,j})$ to H_M, and then plays another turn by taking a free edge according to his strategy, and adding the opposite edge to OBreaker's graph. Finally, if OMaker has taken the edge $(v_{1,i}, v_{2,i})$ then he plays another turn. Since the classical Maker–Breaker game is bias-monotone, if OMaker takes more than one edge it can only help him. Also, note that edges between $v_{1,i}$ and $v_{2,i}$ are useless for OMaker in the real game. Therefore OMaker will have to construct a graph with minimum degree $c + 1$ so that every vertex has at least c neighbors other than itself.

Observe also that $(v_{2,i}, v_{1,j}) \notin E(H_M)$, as otherwise $(v_{1,i}, v_{2,j})$ would have been added to H_B in some previous step. The following proposition summarizes this reduction.

Proposition 4.4. *If at some step H_M has minimum degree $c + 1$ then at the same time every vertex in G has min in-degree and out-degree at least c.*

Gebauer–Szabó Proof. In [11], Gebauer and Szabó provided a strategy for Maker (in the classical Maker–Breaker setting) to construct a spanning tree, a graph with positive minimum degree, and a connected graph with high minimum degree while playing a $(1 : b)$ game on $E(K_n)$ when $b = \frac{(1+o(1))n}{\ln n}$. Here we summarize their method and highlight the slight differences between their strategy for the min-degree game and what we need in our case. We refer the reader to [11] for a complete proof. Their strategy is defined as follows. The goal of Maker is to construct a graph with min-degree c. Throughout the game, a vertex v is *dangerous* if $d_M(v) \le c - 1$. Define the danger value of v as $\mathtt{dang}(v) = d_B(v) - 2b \cdot d_M(v)$. Initially, the danger of all vertices is 0. At every round, Maker takes a vertex v with maximum danger value (ties are broken arbitrarily), and then takes an arbitrary unclaimed edge incident to v. The proof goes by assuming Breaker's win, and analyzing the change of danger value of the vertices for which Maker took incident edges in the game, and showing that the average danger value must be greater than 0. This in turn would lead to a contradiction. $\square$

In our case, our board consists of the edges of the complete bipartite graph $K_{n,n}$ instead of the edges of the complete graph K_n. Moreover, when OMaker claims an edge, OBreaker may get the opposite edge as well; we add this edge to the next move of OBreaker. Therefore, OMaker plays against OBreaker that claims at most $(b + 1)$ edges in his turn. The danger of a vertex is defined only with respect to edges (and degrees) that belong to the bipartite graph $K_{n,n}$, and hence at the beginning of the game the danger of every vertex is 0. The rest of the analysis is essentially the same as [11].

It was observed in [12] that, for any constant $c > 0$, Maker can achieve degree at least c at every vertex before Breaker claimed $(1 - \delta)n$ of its incident edges, for $\delta = \frac{15}{(\ln n)^{1/4}}$. Also, if OMaker is ordered by his strategy to claim an edge incident to a vertex v, he chooses one of the edges randomly and uniformly among the free incident edges. Note that in this case Breaker also gets only one new edge.

We will show that after the first stage, the obtained graph has typically some expanding properties. In our case after at most $8n$ rounds, H_M has min-degree at least 4, which results in an oriented graph with the property that every vertex has in-degree and out-degree at least 3. Observe that this stage lasts at most $8n$ moves as in every round OMaker increases the degree of one of the vertices in $K_{n,n}$ by at least one.

We conclude the description of this approach with the following proposition.

Proposition 4.5. *Suppose that $b = \frac{(1-o(1))n}{\ln n}$. Then OMaker has a strategy to construct after at most $8n$ turns a directed graph with min in-degree and min out-degree at least 3. Moreover, throughout the game, OMaker chooses at each turn a vertex v according to that strategy, and picks a random incident edge out of a set of at least δn choices, where $\delta = \frac{15}{(\ln n)^{1/4}}$.*

For completeness we provide the proof details in the Appendix.

Applying the Gebauer–Szabó approach. Let A be a set of vertices of size $O(\frac{n}{(\ln n)^{2/5}})$. We next prove that almost surely after the first stage A has at least one ingoing edge and at least one outgoing edge. We start by claiming that almost surely every such set has at least one ingoing edge. Observe first that the property trivially holds for every set with a single vertex, as every vertex has in-degree at least one. Consider a fixed set A of size i, and assume that A has no ingoing edges, then all edges that enter A have their other endpoint also in A, and there are at least $3i$ such edges. By Proposition 4.5, whenever OMaker chooses a dangerous vertex v from A, there are at least δn unclaimed edges incident to v. Therefore, the probability that OMaker chooses an edge between v and another vertex of A is at most $\frac{|A|-1}{\delta n-1}$. After the first stage there are $3i$ ingoing edges to vertices of A, hence the probability that A does not have even a single ingoing edge from a vertex outside A is at most $(\frac{|A|-1}{\delta n-1})^{3i}$. Therefore, by the union bound, the probability that there is set A of size i with no ingoing edge is at most

$$\binom{n}{i} \cdot \left(\frac{|A|-1}{\delta n-1}\right)^{3i} \leq \left(\frac{en}{i}\right)^{i} \cdot \left(\frac{2i}{\delta n}\right)^{3i} \leq \left(\frac{8ei^2}{\delta^3 n^2}\right)^{i}.$$

By considering the two cases when $i \leq n^{1/3}$ and $i \geq n^{1/3}$ it is easy to check that for every $2 \leq i \leq \frac{n}{(\ln n)^{2/5}}$ and $\delta = \frac{15}{(\ln n)^{1/4}}$, the last expression is bounded by $o(1/n)$. Therefore by the union bound every set of size at most $\frac{n}{(\ln n)^{2/5}}$ has at least one ingoing edge, assuming that n is sufficiently large. Essentially the same argument shows that almost surely every such set of that size contains at least one outgoing edge, as claimed.

Clearly, the first stage takes at most $8n$ rounds, so the total number of taken edges is at most $8n(b+1)$.

The second stage. Recall that at the second stage OMaker needs to connect in both directions every two disjoint sets A, B of size $\frac{(1-o(1))n}{(\ln n)^{2/5}}$.

Consider a random tournament obtained from K_n by directing each edge uniformly and independently of the other edges. For every two disjoint sets of vertices A, B, the number of edges from A to B is binomially distributed. Denote by $e(A, B)$ the number of edges from A to B and by $e(B, A)$ the number of edges from B to A. By the Chernoff bound (see, e.g., [1]), we have

$$Pr\left[\left|e(A,B) - \frac{|A \parallel B|}{2}\right| \geq \varepsilon |A \parallel B|\right] \leq e^{-\varepsilon^2 |A\|B|/2},$$

and similar inequality holds also for $e(B, A)$.

Therefore, if $|A| = |B| = \frac{(1-o(1))n}{(\ln n)^{2/5}}$ the probability that $e(A, B)$ or $e(B, A)$ is greater than $\frac{1}{2} \cdot |A \parallel B|(1 + n^{-1/2+o(1)})$ is 2^{-2n}, and hence by the union bound for every two such sets, there are at least $\frac{1}{2} \cdot |A \parallel B|(1 + n^{-1/2+o(1)}) \geq \frac{0.99n^2}{2(\ln n)^{4/5}}$ in each direction. Fix a tournament T^* with this property.

At the second stage, OMaker always directs edges according to T^*. That is, he can only direct an edge from u to v if $(u, v) \in E(T^*)$. For every two such sets, at most $8n(b+1) \leq \frac{12n^2}{\ln n}$ edges have been directed at the first stage of the game, and hence at the beginning of the second stage at least $\frac{0.99n^2}{2(\ln n)^{4/5}}$ edges that are directed from A to B in T^* are unclaimed.

Now OMaker and OBreaker switch roles. OMaker clearly wins if he prevents OBreaker from claiming all the edges from a set A to a set B, where $|A| = |B| = k = \frac{n}{(\ln n)^{2/5}}$. To this end, we apply the Beck–Erdős–Selfridge criterion (Theorem 3), with $p = b = \frac{n(1+o(1))}{\ln n}$, $q = 1$, the size of each hyperedge is at least $\frac{0.99n^2}{2(\ln n)^{4/5}}$ and the total number of sets is at most $\binom{n}{k}^2$. We have:

$$\begin{aligned}
\sum_{A \in \mathcal{F}} (q+1)^{-\frac{|A|}{p}} &< \binom{n}{k}^2 \cdot (q+1)^{-\frac{|A|}{p}} \\
&< \left(\frac{en}{k}\right)^{2k} \cdot 2^{-\frac{n \ln n}{3(\ln n)^{4/5}}} \\
&\leq 2^{\frac{4n \cdot \log\log n}{(\ln n)^{2/5}}} \cdot 2^{-\frac{n(\log n)^{1/5}}{6}} \ll 1.
\end{aligned}$$

Therefore in our case OMaker wins and hence every two sets of size k are connected in both ways.

We conclude that at the end of the second stage OMaker has a strongly connected graph, and hence by the end of the game the obtained tournament is strongly connected, and OMaker wins. This proves Item (ii) of Theorem 2. □

5. The *H*-creation game

In this game, a fixed oriented graph H is given. OMaker wins if the obtained tournament contains a copy of H, and OBreaker wins otherwise. Note that if H does not contain a directed cycle then OMaker surely wins for large enough n, as every tournament of size n contains a transitive tournament of size $\log n$, which contains H as a subgraph.

Our starting point is an upper bound on the bias threshold for a general fixed graph H. Given a directed graph H, and a bijection $\sigma : V(H) \to |V(H)|$, we define the *feedback arc set* of H with respect to σ as

$$\mathrm{FAS}(H, \sigma) = |\{(u, v) \in E(H) : \sigma(u) > \sigma(v)\}|.$$

In words, this parameter measures the number of edges that are going in the wrong direction with respect to σ. Let

$$\mathrm{FAS}(H) = \min_{\sigma}\{\mathrm{FAS}(H, \sigma)\}.$$

This is the minimal number of edges of H that has to be deleted in order to make H an acyclic graph. If, for example, H is a random tournament on t vertices, then it is easy to show that with high probability $\mathrm{FAS}(H)$ is close to $t(t-1)/4$.

We have the following upper bound.

Lemma 5.1. *Let H be a graph on t vertices, and let $r = \mathrm{FAS}(H)$. Suppose that OMaker and OBreaker play an orientation game on K_n. Then if $b > c(H) \cdot n^{t/r}$ then OBreaker has a strategy guaranteeing that the obtained tournament does not contain a copy of H, where $c(H) > 0$ depends only on H.*

Proof. The proof follows by a simple application of the Beck–Erdős–Selfridge theorem (Theorem 3). OBreaker chooses an arbitrary bijection σ of the vertices, and at every turn directs the edges according to σ. That is, whenever he chooses to direct an edge uv, and $\sigma(v) > \sigma(u)$ the edge will be directed from u to v. Hence, if OMaker creates a copy of H, by definition he orients at least $\mathrm{FAS}(H)$ edges in the opposite direction with respect to σ. We can thus reduce the game to the classical Maker–Breaker game as follows. In every set of t vertices, OMaker can win only if he claims at least r edges that are induced by this set, and OBreaker wins if he prevents OMaker from doing so. The total number of winning sets for OMaker is at most $\binom{n}{t} \cdot \binom{\binom{t}{2}}{r}$. Therefore, if $(q+1)^r = \Omega(\binom{n}{t} \cdot \binom{\binom{t}{2}}{r})$ then by Theorem 3, OBreaker has a winning strategy. This is the case if $b > c(t, r) \cdot n^{t/r}$, and hence the lemma holds. $\square$

It is worth noting that following the methods of Bednarska and Łuczak [5], one can prove that if $b = O(n^{\frac{|V(H)|-2}{|E(H)|-1}})$ then (assuming some balanceness conditions for H) Maker has a winning strategy as follows. Maker chooses at each round a random undirected edge and orients it randomly, independently of the other choices. Roughly speaking, one can show that by the end of the game the obtained graph has some properties of a random graph, and hence if the bias is not too large then with high probability the graph contains a copy of H.

However, following their approach does not give sharp bounds in our case. To see this, observe that their results give a sharp bound of $b = \Theta(\sqrt{n})$ for the triangle creation game in the classical Maker–Breaker setting, while in orientation games the correct bias for creating a cyclic triangle is $b = \Theta(n)$.

We next generalize the result of Section 3 and show that in the case that H is a fixed cycle, OMaker wins even if $b = \Omega(n)$.

Proposition 5.2. *For every constant $k \geq 3$ there is a constant $\gamma(k) > 0$ such that if $b < \gamma(k) \cdot n$ then OMaker wins the b-biased C_k-creation game.*

Proof. We first observe that if a tournament T contains a cycle of length $k + (k-2)r$ for some $r \in \mathbb{N}$ then it also contains a cycle of length k. The proof of this observation is by induction. It is trivially true for $r = 0$. Suppose this is true for all values smaller than some fixed r, and let $v_1, v_2, \ldots, v_{k+(k-2)r}, v_1$ a cycle of length $k + (k-2)r$. Consider the edge between v_k and v_1. If the edge is directed from v_k to v_1 there is a cycle of length k and we are done. Otherwise, $v_k, v_{k+1}, \ldots, v_{k+(k-2)r}, v_1, v_k$ is a cycle of length $k + (k-2)(r-1)$ and therefore by the induction hypothesis T contains a cycle of length k, as required.

Therefore, in order to create a cycle C_k, OMaker needs to create some cycle of length $k + (k-2)r$. By Lemma 3.1, at each round OMaker can extend a longest directed path by 1. OMaker's strategy is to close a cycle of length $k+(k-2)r$ whenever it is possible, and to extend a longest directed path by 1 if it is not possible. After t rounds, there is a path of length t, and we denote it by $x_1, \ldots, x_t, x_{t+1}$. For every $i \geq k$, the number of edges from x_i to x_j, $j < i$, that may close a cycle of length $k + (k-2)r$ is at least $\frac{i}{k-2} - 2$. Hence the total number of edges that may close a cycle of length $k + (k-2)r$ for some r is at least

$$\sum_{i=k}^{t+1}\left(\frac{i}{k-2} - 2\right) > \frac{(t+k)(t-k)}{2(k-2)} - 2t.$$

Therefore, the number of such edges is at least $\frac{\binom{t}{2}}{k}$ for $t = \Omega(k^2)$, that is at least $(1/k)$-fraction of the edges for such t. Among these $\frac{\binom{t}{2}}{k}$ edges, at most $(b+1)t$ have been oriented by either OMaker or OBreaker in previous rounds. Note that as long as the bias b is smaller than $n/2$, the game lasts at least $t = n$ rounds, and results in a path of length $n-1$, unless OMaker wins before. Therefore if $b < \frac{n-1-2k}{2k}$ then OMaker wins the game, as required. □

Recall that a tournament T is k-colorable if its edges can be partitioned into k transitive tournaments. Berger et al. [6] studied the class of tournaments H with the property that there is a constant $c(H)$ such that every H-free tournament T is $c(H)$-colorable. They called every such tournament H a *hero*, and characterized the set of such tournaments. We next show that for every $k > 0$ OMaker has a strategy to create a non k-colorable tournament as long as the bias is a sufficiently small linear function of n.

Lemma 5.3. *Let $k > 1$, and suppose that $b = \frac{cn}{k \log k}$ for some sufficiently small constant $c > 0$. Then OMaker has a strategy to create a non k-colorable tournament.*

Proof. It is rather easy to see using a Chernoff bound (as it was done in Section 4) that a random tournament obtained by directing each edge uniformly and independently of the other choices has the following property almost surely: for every ordered pair of disjoint sets A, B of sets of size $n/2k$, there are $\Theta(\frac{n^2}{k^2})$ edges in each direction between A and B. Fix a tournament T^* with this property.

Define a hypergraph H whose vertices are the edges of T^* and whose edges are all the edges from A to B in T^* for every ordered pair A, B of sets of size $n/2k$. OMaker will win the game by orienting one edge from every hyperedge in H according to T^*. To this end, OMaker will play to prevent OBreaker from orienting all the edges in some hyperedge from H. By the end of the game, there is an edge between every two sets of size $n/2k$ and hence the obtained tournament does not contain an acyclic set of size n/k, and therefore is not k-colorable.

There are $\binom{n}{n/2k}^2 \leq (2ek)^{n/k}$ choices of ordered pairs (A, B), each corresponding to a hyperedge of H. The size of each hyperedge is $\Theta(\frac{n^2}{k^2})$. By applying the Beck–Erdős–Selfridge criterion (Theorem 3, with OMaker assuming the role of Breaker, $p = b$ and $q = 1$), if $b = \frac{cn}{k \log k}$ then OMaker has a winning strategy, as required. □

A simple consequence of Lemma 5.3 is the following generalization of Lemma 3.1. Berger et al. [6] provided a list of five minimal tournaments $H_1, H_2, \ldots, H_5$, and proved (Theorem 5.1 in [6]) that every non-hero tournament must contain at least one of $H_1, \ldots, H_5$ as a sub-tournament. For every $1 \leq i \leq 5$, one can check that $\mathrm{FAS}(H_i) \geq 2$.

Consider any oriented graph H with $\mathrm{FAS}(H) = 1$. Let σ be an ordering of $V(H)$ with a single edge that does not agree with σ. Let H' be a tournament on $V(H)$ that contains H as subgraph and is defined as follows. For every two vertices $u, v \in V(H)$, if $(u, v) \in E(H)$ we let $(u, v) \in E(H')$. If $(u, v), (v, u) \notin E(H)$, we let $(u, v) \in E(H')$ if $\sigma(v) > \sigma(u)$ and $(v, u) \in E(H')$ otherwise. We get that $\mathrm{FAS}(H') = 1$ as well.

Clearly, it is sufficient to construct a copy of H' for OMaker's win. The result of Berger et al. [6] can be applied only for tournaments, and hence we will use it to show that OMaker can construct a copy of H'.

Since $\mathrm{FAS}(H') = 1$ the tournament H' is a hero, and therefore every tournament that does not contain a copy of H' is $c(H')$-colorable, where $c(H')$ is a constant that depends only on H'. By Lemma 5.3, if $b = \Theta(n)$ then OMaker has a strategy so the obtained tournament is not $c(H')$-colorable. We therefore have the following.

Proposition 5.4. *For every oriented graph H with $\mathrm{FAS}(H) = 1$ there is a constant $\gamma(H) > 0$ such that OMaker wins the γn-biased H creation game.*

We conjecture that the bias threshold that guarantees OMaker's win strongly depends on $\mathrm{FAS}(H)$. It will be interesting to find further quantitative results in this direction.

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Appendix. Proof of Proposition 4.5

Here we provide the complete details of Gebauer–Szabó approach and show that Maker can win the min-degree game if $b = \frac{(1-o(1))n}{\ln n}$ where the base graph is the complete bipartite graph $K_{n,n}$. We give the proof for every min-degree c, though we need only the case $c = 4$.

We assume for simplicity that Breaker starts the game, this does not change the asymptotic threshold of this game. We say that the game ends when either all vertices have degree at least c in Maker's graph (and Maker won) or one vertex has degree at least $n-c+1$ in Breaker's graph (and Breaker won). With $\deg_M(v)$ and $\deg_B(v)$ we denote the degree of a vertex v in Maker's graph and in Breaker's graph, respectively. A vertex v is called *dangerous* if $\deg_M(v) \leq c-1$. To establish Maker's strategy we define the *danger value* of a vertex v as $\mathtt{dang}(v) := \deg_B(v) - 2b \cdot \deg_M(v)$.

Maker's strategy S_M. Before his ith move Maker identifies a dangerous vertex v_i with the largest danger value, ties are broken arbitrarily. Then, as his ith move Maker claims an edge incident to v_i. We refer to this step as "easing v_i".

Observe that Maker can always make a move according to his strategy unless no vertex is dangerous (thus he won) or Breaker occupied at least $n-c+1$ edges incident to a vertex (and Breaker won).

Also, a vertex v_i was dangerous any time before Maker's ith move.

Suppose, for a contradiction, that Breaker, playing with bias b, has a strategy S_B to win the min-degree-c game against Maker who plays with bias 1. Let B_i and M_i denote the ith move of Breaker and Maker, respectively, in the game where they play against each other using their respective strategies S_B and S_M. Let g be the length of this game, i.e., the maximum degree of Breaker's graph becomes larger than $n-c$ in move B_g. We call this the end of the game.

For a set $I \subseteq V$ of vertices we let $\overline{\mathsf{dang}}(I)$ denote the average danger value $\frac{\sum_{v\in I}\mathsf{dang}(v)}{|I|}$ of the vertices of I. When there is risk of confusion we add an index and write $\mathsf{dang}_{B_i}(v)$ or $\mathsf{dang}_{M_i}(v)$ to emphasize that we mean the danger-value of v *directly before B_i* or M_i, respectively.

In his last move Breaker takes b edges to increase the maximum Breaker-degree of his graph to at least $n-c$ (in fact, at least $n-c+1$). In order to be able to do that, directly before Breaker's last move B_g there must be a dangerous vertex v_g whose Breaker-degree is at least $n-c-b$. Thus $\mathsf{dang}_{B_g}(v_g) \geq n-c-b-2b(c-1)$.

Recall that $v_1, \ldots, v_{g-1}$ were defined during the game. For $0 \leq i \leq g-1$, we define the set I_i as $I_i = \{v_{g-i}, \ldots, v_g\}$.

The following lemma estimates the change in the average danger during Maker's move.

Lemma A.1. *Let i, $1 \leq i \leq g-1$,*

(i) *if $I_i \neq I_{i-1}$, then* $\overline{\mathsf{dang}}_{M_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1}) \geq 0$.
(ii) *if $I_i = I_{i-1}$, then* $\overline{\mathsf{dang}}_{M_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1}) \geq \frac{2b}{|I_i|}$.

Proof. For part (i), we have that $v_{g-i} \notin I_{i-1}$. Since danger values do not increase during Maker's move we have $\overline{\mathsf{dang}}_{M_{g-i}}(I_{i-1}) \geq \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1})$. Before M_{g-i} Maker selected to ease vertex v_{g-i} because its danger was highest among dangerous vertices. Since all vertices of I_{i-1} are dangerous before M_{g-i} we have that $\mathsf{dang}(v_{g-i}) \geq \max(\mathsf{dang}(v_{g-i+1}), \ldots, \mathsf{dang}(v_g))$, which implies $\overline{\mathsf{dang}}_{M_{g-i}}(I_i) \geq \overline{\mathsf{dang}}_{M_{g-i}}(I_{i-1})$. Combining the two inequalities establishes part (i).

For part (ii), we have that $v_{g-i} \in I_{i-1}$. In M_{g-i} $\deg_M(v_{g-i})$ increases by 1 and $\deg_M(v)$ does not decrease for any other $v \in I_i$. Besides, the degrees in Breaker's graph do not change during Maker's move. So $\mathsf{dang}(v_{g-i})$ decreases by $2b$, whereas $\mathsf{dang}(v)$ do not increase for any other vertex $v \in I_i$. Hence $\overline{\mathsf{dang}}(I_i)$ decreases by at least $\frac{2b}{|I_i|}$, which implies (ii). □

The next lemma bounds the change of the danger value during Breaker's moves.

Lemma A.2. *Let i be an integer, $1 \leq i \leq g-1$.*

(i) $\overline{\mathsf{dang}}_{M_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i}}(I_i) \leq \frac{2b}{|I_i|}$
(ii) $\overline{\mathsf{dang}}_{M_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i}}(I_i) \leq \frac{b+|I_i|-1+a(i-1)-a(i)}{|I_i|}$, *where $a(i)$ denotes the number of edges spanned by I_i which Breaker took in the first $g-i-1$ rounds.*

Proof. Let e_{double} denote the number of those edges with both endpoints in I_i which are occupied by Breaker in B_{g-i}. Then the increase of $\sum_{v\in I_i}\deg_B(v)$ during B_{g-i} is at most $b+e_{\text{double}}$. Since the degrees in Maker's graph do not change during Breaker's move the increase of $\overline{\mathsf{dang}}(I_i)$ (during B_{g-i}) is at most $\frac{b+e_{\text{double}}}{|I_i|}$.

Part (i) is then immediate after noting that $e_{\text{double}} \leq b$.

For (ii), we bound e_{double} more carefully. By definition, Breaker occupied $a(i)$ edges spanned by I_i in his first $g-i-1$ moves. So, all in all, Breaker occupied $a(i)+e_{\text{double}}$ edges spanned by I_i in his first $g-i$ moves. On the other hand, we know that among these edges exactly $a(i-1)$ are spanned by $I_{i-1} \supseteq I_i \setminus \{v_{g-i}\}$ and there are at most $|I_i|-1$ edges in I_i incident to v_{g-i}. Hence $a(i)+e_{\text{double}} \leq a(i-1)+|I_i|-1$, giving us $e_{\text{double}} \leq |I_i|-1+a(i-1)-a(i)$. □

The following estimates for the change of average danger during one full round are immediate corollaries of the previous two lemmas.

Corollary A.3. *Let i be an integer, $1 \leq i \leq g-1$.*

(i) *if $I_i = I_{i-1}$, then* $\overline{\mathsf{dang}}_{B_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1}) \geq 0$.
(ii) *if $I_i \neq I_{i-1}$, then* $\overline{\mathsf{dang}}_{B_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1}) \geq -\frac{2b}{|I_i|}$
(iii) *if $I_i \neq I_{i-1}$, then* $\overline{\mathsf{dang}}_{B_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1}) \geq -\frac{b+|I_i|-1+a(i-1)-a(i)}{|I_i|}$, *where $a(i)$ denotes the number of edges spanned by I_i which Breaker took in the first $g-i-1$ rounds.*

Using Corollary A.3 we derive that before B_1, $\overline{\mathsf{dang}}(I_{g-1}) > 0$, which contradicts the fact that at the beginning of the game every vertex has danger value 0.

Let $k := \lfloor \frac{n}{\ln n} \rfloor$. For the analysis, we split the game into two parts: The main game, and the end game which starts when $|I_i| \le k$.

Let $|I_g| = r$. Let $i_1 < \cdots < i_{r-1}$ be those indices for which $I_{i_j} \neq I_{i_j - 1}$. Note that $|I_{i_j}| = j + 1$. Observe that by definition $a(i_{j-1}) \ge a(i_j - 1)$.

Recall that the danger value of v_g directly before B_g is at least $n - c - b(2c - 1)$.

Assume first that $k > r$.

$$\begin{aligned}
\overline{\mathsf{dang}}_{B_1}(I_{g-1}) &= \overline{\mathsf{dang}}_{B_g}(I_0) + \sum_{i=1}^{g-1} (\overline{\mathsf{dang}}_{B_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1})) \\
&\ge \overline{\mathsf{dang}}_{B_g}(I_0) + \sum_{j=1}^{r-1} (\overline{\mathsf{dang}}_{B_{g-i_j}}(I_{i_j}) - \overline{\mathsf{dang}}_{B_{g-i_j+1}}(I_{i_j-1})) \quad \text{[by Corollary A.3(i)]} \\
&\ge \overline{\mathsf{dang}}_{B_g}(I_0) - \sum_{j=1}^{r-1} \frac{b + j + a(i_j - 1) - a(i_j)}{j+1} \quad \text{[by Corollary A.3(iii)]} \\
&\ge \overline{\mathsf{dang}}_{B_g}(I_0) - bH_r - r - \frac{a(0)}{2} + \sum_{j=2}^{r-1} \frac{a(i_{j-1})}{(j+1)j} + \frac{a(i_{r-1})}{r} \quad \text{[since } a(i_{j-1}) \ge a(i_j - 1)\text{]} \\
&\ge \overline{\mathsf{dang}}_{B_g}(I_0) - bH_k - k \quad \text{[since } a(0) = 0 \text{ and } r \le k\text{]} \\
&\ge n - c - b(2c + \ln k) - k \\
&\ge n - \frac{n}{\ln n}(2c + \ln n - \ln\ln n) - \frac{n}{\ln n} - c \quad \left[\text{since } b \le \frac{n}{\ln n}\right] \\
&\ge \frac{n \ln\ln n}{3 \ln n} - \frac{n}{\ln n} - c \\
&> 0. \quad \text{[for large } n\text{]}
\end{aligned} \tag{A.1}$$

Assume now that $k \le r$.

$$\begin{aligned}
\overline{\mathsf{dang}}_{B_1}(I_{g-1}) &= \overline{\mathsf{dang}}_{B_g}(I_0) + \sum_{i=1}^{g-1} (\overline{\mathsf{dang}}_{B_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1})) \\
&\ge \overline{\mathsf{dang}}_{B_g}(I_0) + \sum_{j=1}^{r-1} (\overline{\mathsf{dang}}_{B_{g-i_j}}(I_{i_j}) - \overline{\mathsf{dang}}_{B_{g-i_j+1}}(I_{i_j-1})) \quad \text{[by Corollary A.3(i)]} \\
&= \overline{\mathsf{dang}}_{B_g}(I_0) + \sum_{j=1}^{k-1} (\overline{\mathsf{dang}}_{B_{g-i_j}}(I_{i_j}) - \overline{\mathsf{dang}}_{B_{g-i_j+1}}(I_{i_j-1})) \\
&\quad + \sum_{j=k}^{r-1} (\overline{\mathsf{dang}}_{B_{g-i_j}}(I_{i_j}) - \overline{\mathsf{dang}}_{B_{g-i_j+1}}(I_{i_j-1})) \\
&\ge \overline{\mathsf{dang}}_{B_g}(I_0) - \sum_{j=1}^{k-1} \frac{b + j + a(i_j - 1) - a(i_j)}{j+1} - \sum_{j=k}^{r-1} \frac{2b}{j+1} \quad \text{[by Corollary A.3(iii) and (ii)]} \\
&\ge \overline{\mathsf{dang}}_{B_g}(I_0) - b(2H_r - H_k) - k - \frac{a(0)}{2} + \sum_{j=2}^{k-1} \frac{a(i_{j-1})}{(j+1)j} + \frac{a(i_{k-1})}{k} \\
&\ge n - c - b(2c - 1 + 2H_{2n} - H_k) - k \quad \text{[since } 2n \ge r \text{ and } a(0) = 0\text{]} \\
&\ge n - c - \left(\frac{n}{\ln n} - \frac{n \ln\ln n}{\ln^2 n} - (2c+3)\frac{n}{\ln^2 n}\right)(\ln n + \ln\ln n + 2c + 2) - \frac{n}{\ln n} \\
&\ge \frac{n(\ln\ln n)^2}{\ln^2 n} \quad \text{[for } n \text{ large enough]} \\
&> 0.
\end{aligned} \tag{A.2}$$

Observe that in our proof we need Maker to have min-degree c for every vertex v before Breaker claims $(1 - \delta)n$ edges incident to v (for $\delta = O(\frac{1}{(\ln n)^{1/4}})$). The same analysis essentially holds, with the following differences. Assume that Breaker wins, then before his last move the vertex v has degree $(1-\delta)n-c-1$ (instead of $n-c-1$). All other calculations are essentially the same by taking $b=\frac{n}{\ln n}(1-\frac{1}{\ln\ln n})$.

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