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Orthonormal Representations of *H*-Free Graphs

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Abstract

Let $x_1, \dots, x_n \in \mathbb{R}^d$ be unit vectors such that among any three there is an orthogonal pair. How large can n be as a function of d, and how large can the length of $x_1 + \dots + x_n$ be? The answers to these two celebrated questions, asked by Erdős and Lovász, are closely related to orthonormal representations of triangle-free graphs, in particular to their Lovász ϑ-function and minimum semidefinite rank. In this paper, we study these parameters for general H-free graphs. In particular, we show that for certain bipartite graphs H, there is a connection between the Turán number of H and the maximum of $\vartheta(\overline{G})$ over all H-free graphs G.

Keywords Lovász ϑ-function · Minrank · Orthonormal representation · Turán numbers

1 Introduction

Given a graph G, a map $f : V(G) \to \mathbb{R}^d$ is called an *orthonormal representation* of G (in $\mathbb{R}^d$) if $\|f(u)\| = 1$ for all $u \in V(G)$ and $\langle f(u), f(v)\rangle = 0$ for all distinct

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$u, v \in V(G)$ such that $uv \notin E(G)$. Note that every graph G on n vertices has an orthonormal representation, since we may assign each vector to a corresponding orthonormal basis vector in $\mathbb{R}^{|G|}$. Given an orthonormal representation f of a graph G with vertex set $[n]$, we define M_f to be the *Gram matrix* of the vectors $f(1), \ldots, f(n)$, so that $(M_f)_{i,j} = \langle f(i), f(j) \rangle$.

The concept of orthonormal representations goes back to a seminal paper of Lovász [26], who used them to define a graph parameter now known as the Lovász ϑ-function. The ϑ-function of a graph G has several equivalent definitions. Here we list the ones that we shall use later.

Definition 1.1 Let G be a graph with vertex set $[n]$. The *ϑ-function of G*, denoted $\vartheta(G)$, can be defined in the following ways, which are shown to be equivalent in [26].

1. $\vartheta(G)$ is the maximum, over all orthonormal representations f of the complement graph $\overline{G}$, of the largest eigenvalue of the Gram matrix M_f.
2. $\vartheta(G)$ is the maximum of $1 - \lambda_1(A)/\lambda_n(A)$, over all $n \times n$ real symmetric matrices A such that $A_{i,j} = 0$ if $ij \in E(G)$ or $i = j$.[1]
3. $\vartheta(G)$ is the minimum, over all orthonormal representations f of G and all unit vectors x, of $\max_{v \in V(G)} \langle x, f(v) \rangle^{-2}$.
4. $\vartheta(G)$ is the maximum, over all orthonormal representations f of the complement graph $\overline{G}$ and all unit vectors x, of $\sum_{v \in V(G)} \langle x, f(v) \rangle^2$.

Lovász originally introduced the notion of the ϑ-function in order to bound the Shannon capacity of a graph, and since then, the combinatorial and algorithmic applications of the Lovász ϑ-function have been studied extensively, see e.g. Knuth [22].

Given a graph G, let us define the *minimum semidefinite rank* of G, denoted $\mathrm{msr}(G)$, to be the minimum d such that there exists an orthonormal representation of G in $\mathbb{R}^d$. Note that $\mathrm{msr}(G)$ can be seen as a vector generalization of the chromatic number of $\overline{G}$, see [20]. Indeed, by assigning a standard basis vector of $\mathbb{R}^{\chi(\overline{G})}$ to each vertex of a given color, one can see that $\mathrm{msr}(G) \le \chi(\overline{G})$. In the same paper where he introduced the ϑ-function, Lovász [26, Thm. 11] showed that

$$\vartheta(G) \le \mathrm{msr}(G).$$

Various notions of the minimum rank of a graph have been studied in the literature, see Fallat and Hogben [13] for a survey. Note that an equivalent way to define the minimum semidefinite rank of a graph G is as the minimum rank of a positive semidefinite matrix M such that $M_{i,i} = 1$ for all i and $M_{i,j} = 0$ if $ij \notin E(G)$. Dropping the positive semidefinite assumption, we arrive at the notion of minrank, which has applications in theoretical computer science, see Golovnev et al. [17] for references. In particular, it is related to important problems on the complexity of arithmetic circuits [9].

[1] In [26], Lovász forgets to include the assumption that A is symmetric and $A_{i,i} = 0$ for all i in his statement of Theorem 6, but it is clear that this is what he intended.

1.1 A Geometric Problem of Lovász

One very interesting application of the Lovász ϑ-function is to the following geometric problem posed by Lovász and first studied by Konyagin [23].

> What is the maximum Δ_n, of the length $\|\sum_{i=1}^n x_i\|$, over all d and all unit vectors $x_1, \ldots, x_n \in \mathbb{R}^d$ such that among any three, there is at least one pair of orthogonal vectors?

Konyagin [23] gave upper and lower bounds on Δ_n, in particular showing that $\Delta_n \le \frac{3}{2} n^{2/3}$. Then Kashin and Konyagin [21] improved the lower bound to within a logarithmic factor of the upper bound, and finally, Alon [1] was able to give an asymptotically tight construction showing that $\Delta_n = \Theta(n^{2/3})$. Note that if we define $L(G)$ to be the maximum of $\|\sum_{v \in V(G)} f(v)\|$ over all orthonormal representations f for G, then the above problem is equivalent to asking for the maximum of $L(G)$ over all triangle-free graphs G on n vertices. The following claim, whose proof we defer to Sect. 3, connects $L(G)$ to $\vartheta(G)$ and $\vartheta(\overline{G})$.

Claim 1.2 *For any graph G on n vertices, we have*

$$\frac{n}{\sqrt{\vartheta(G)}} \le L(G) \le \sqrt{n\vartheta(\overline{G})}.$$

Moreover, if G is vertex-transitive, then $L(G) = \sqrt{n\vartheta(\overline{G})}$.

For graphs G, H we say that G is *H-free* if G does not contain a copy of H as a subgraph. Generalizing from a triangle to an arbitrary H, let us now define $\lambda(n, H)$ to be the maximum value of $\vartheta(\overline{G})$ over all H-free graphs G on n vertices. Although in this paper we study only $\lambda(n, H)$, we remark that, roughly speaking, Claim 1.2 would allow one to translate these results to the corresponding geometric problem of finding the maximum of $L(G)$ over all H-free graphs G on n vertices, especially because the constructions we consider are either Cayley graphs, which are vertex-transitive, or are very similar to Cayley graphs. Indeed, for $H = K_3$, Konyagin's argument for the upper bound on Δ_n can be adapted to obtain $\lambda(n, K_3) \le O(n^{1/3})$, and since Alon's construction for the lower bound on Δ_n is vertex-transitive, Claim 1.2 implies that $\lambda(n, K_3) \ge \Omega(n^{1/3})$, so that we have $\lambda(n, K_3) = \Theta(n^{1/3})$. Generalizing to larger cliques, it is known that

$$\Omega(n^{1-O(1/\log t)}) \le \lambda(n, K_t) \le O(n^{1-2/t}),$$

where Alon and Kahale [3] proved the upper bound and Feige [14] proved the lower bound.

Another way to generalize forbidding a triangle is to forbid longer cycles. Indeed, Alon and Kahale [3] also showed that for any t, if G is a graph on n vertices having no odd cycle of length at most $2t+1$, then $\vartheta(\overline{G}) \le 1 + (n-1)^{1/(2t+1)}$. Our first contribution is a generalization of this upper bound to graphs that have no cycle of length exactly $2t+1$.

Theorem 1.3 *For all $n, t \geq 1$ we have $\lambda(n, C_{2t+1}) \leq O(tn^{1/(2t+1)})$.*

We say that a graph G has an *optimal spectral gap* if $|\lambda_i(A)| \leq O\big(\sqrt{\lambda_1(A)}\big)$ for $2 \leq i \leq n$, where A is the adjacency matrix of G and $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$ are its eigenvalues. Alon and Kahale also noted that their bound for graphs having no odd cycles of length at most $2t + 1$, is tight via a modification (see, e.g., [25, Sect. 3, Exam. 10]) of the construction of Alon [1]. The key properties that make such a construction useful are that it is regular, dense, and has an optimal spectral gap. Indeed, a dense graph with an optimal spectral gap has an adjacency matrix with a large ratio of $|\lambda_1(A)/\lambda_n(A)|$, which by the second definition of the ϑ-function (Definition 1.1) leads to a good lower bound for $\vartheta(\overline{G})$. For a graph H, the Turán number $\mathrm{ex}(n, H)$ is the maximum number of edges in an H-free graph G on n vertices. For bipartite H such as $C_4, C_6, C_{10}, K_{2,t}$ and $K_{t,(t-1)!+1}$, there are known constructions of H-free graphs G that attain good lower bounds for the Turán number, i.e., $|E(G)| \geq \Omega(\mathrm{ex}(n, H))$. Interestingly, most of these constructions are also known to have optimal spectral gaps (see [25, Sect. 3, Exams. 6, 7, 12]). Since they are regular with degree on the order of $\mathrm{ex}(n, H)/n$, it follows from the previous discussion that in such cases

$$\lambda(n, H) \geq \vartheta(\overline{G}) \geq \Omega\left(\sqrt{\frac{\mathrm{ex}(n, H)}{n}}\right).$$

In Sect. 3, we prove the following theorem by showing that the graphs discussed above have optimal spectral gaps.

Theorem 1.4 *Let $n \geq 1$.*

1. *For all $t \in \{4, 6, 10\} \cup (2\mathbb{N} + 1)$, we have $\lambda(n, C_t) \geq \Omega(n^{1/t})$.*
2. *For all $t \geq 2$, we have $\lambda(n, K_{2,t}) \geq \Omega(t^{1/4}n^{1/4})$.*
3. *For all $t \geq 3$, we have $\lambda(n, K_{t,(t-1)!+1}) \geq \Omega(n^{(1-1/t)/2})$.*

Since the Turán number can sometimes provide a lower bound for $\lambda(n, H)$, one might wonder if it can also provide an upper bound. If H is a graph such that we can remove a vertex to obtain a tree and we have $\mathrm{ex}(n, H) \leq O(n^{1+\alpha})$ for some $\alpha > 0$, then we are able to obtain such an upper bound.

Theorem 1.5 *Let $h \geq 1$ and let H be a connected graph on h vertices, containing a vertex v with $H \backslash v$ being a tree. Furthermore, suppose that there exist c, α with $0 < \alpha \leq 1$ and $c \geq 1$ such that $\mathrm{ex}(n, H) \leq cn^{1+\alpha}$ for all $n \geq 1$. Then for all $n \geq 1$, it holds that*

$$\lambda(n, H) \leq 20 \cdot \frac{\sqrt{ch}}{\alpha} \cdot n^{\alpha/2}.$$

Now define $\theta_{t,s}$ to be the graph consisting of s internally disjoint paths of length t between a pair of vertices, and note that in particular $\theta_{t,2} = C_{2t}$ and $\theta_{2,s} = K_{2,s}$. Since $\theta_{t,s}$ consists of a tree together with an additional vertex, we will use Theorem 1.5 together with known upper bounds on Turán numbers to obtain the following corollary.

Corollary 1.6 *Let $n \geq 1$. For all $t, s \geq 2$, we have $\lambda(n, \theta_{t,s}) \leq O(t^2 s^{1-1/(2t)} n^{1/(2t)})$. In particular, for all $s, t \geq 2$, we have*

$$\lambda(n, C_{2t}) \leq O\big(t^2 n^{1/(2t)}\big), \qquad \lambda(n, K_{2,s}) \leq O\big(s^{3/4} n^{1/4}\big).$$

Remark The upper bound for $\lambda(n, C_{2t})$ can be improved to $O(t\, n^{1/(2t)})$ using the proof technique from Theorem 1.3, see the appendix in the arXiv version of this paper (arXiv:1905.01539) for details.

Note that the lower bounds for C_t and $K_{2,t}$ given in Theorem 1.4 have corresponding upper bounds via Theorem 1.3 and Corollary 1.6, which are tight up to the constants depending on t. Unfortunately, since $K_{t,(t-1)!+1}$ for $t \geq 3$ is not a tree together with a vertex, we are able to obtain only a weak upper bound in this case.

Theorem 1.7 *For all $s \geq t \geq 2$, there exists a constant $c_{t,s}$ such that*

$$\lambda(n, K_{t,s}) \leq c_{t,s} n^{1-2/t+1/(t2^{t-1})}.$$

1.2 Almost Orthogonal Vectors

Upon hearing about the results of Kashin and Konyagin [21] towards Lovász's problem, Erdős asked the following related question (see Nešetřil and Rosenfeld [28] for a historical summary):

> What is the maximum, $\alpha(d)$, of the number of vectors in $\mathbb{R}^d$ such that among any three distinct vectors there is at least one pair of orthogonal vectors?

Rosenfeld [30] called such vectors *almost orthogonal*. By taking two copies of each of the vectors from a basis in $\mathbb{R}^d$, we obtain $2d$ almost orthogonal vectors. Erdős believed that a construction with more than $2d$ vectors does not exist, and indeed Rosenfeld showed that $\alpha(d) = 2d$ (see Deaett [10] for a short and nice proof which is slightly more general).

After his initial question was resolved, Erdős further asked what happens if we replace 3 by a larger integer t. Füredi and Stanley [16] defined $\alpha(d, t)$ to be the maximum number of vectors in $\mathbb{R}^d$ such that, among any $t + 1$ distinct vectors, some pair is orthogonal. By considering t orthogonal bases in $\mathbb{R}^d$, we obtain $\alpha(d, t) \geq dt$, and Erdős asked whether equality holds. Füredi and Stanley proved that it does not always hold by showing that $\alpha(4, 5) \geq 24$, and conjectured that there exists a constant c such that $\alpha(d, t) < (dt)^c$. This conjecture was later also proven to be false by Alon and Szegedy [4], who showed that for some constant $\delta > 0$ and t large enough, $\alpha(d, t) \geq d^{\frac{\delta \log t}{\log \log t}}$.

One can see that Erdős's question is almost equivalent to asking for the minimum of $\mathrm{msr}(G)$ over all K_{t+1}-free graphs G on n vertices. The difference is that Erdős was asking for the vectors to be distinct, while an orthonormal representation of a graph may label multiple vertices with the same vector. Nonetheless, we define $\rho(n, H)$ to be the minimum of $\mathrm{msr}(G)$ over all H-free graphs G on n vertices. Some further motivation for studying $\rho(n, H)$ comes from Pudlák [29], who, inspired by questions in circuit complexity, studied the minrank and minimum semidefinite rank of graphs without a cycle of given length. More recently, Haviv [18,19] studied the minrank and Lovász ϑ-function, in particular using the probabilistic method, in order to construct graphs with large minrank and whose complements are H-free.

We note that the aforementioned results now take the form $\rho(n, K_3) = \lceil n/2 \rceil$, and

$$\rho(n, K_{t+1}) \leq n^{\frac{\log \log t}{\delta \log t}}$$

for some constant $\delta > 0$, t sufficiently large, and an infinite number of values of n. Surprisingly, for t fixed and n large, it seems that the best known lower bound on $\rho(n, K_{t+1})$ is just what one gets from Ramsey theory: if $n > \binom{d+t}{t} \geq R(d+1, t+1)$ then any K_{t+1}-free graph on n vertices has an independent set of size $d+1$, and therefore cannot have an orthonormal representation in $\mathbb{R}^d$. Since $\binom{d+t}{t} = O(d^t)$, we may conclude that $\rho(n, K_{t+1}) \geq \Omega(n^{1/t})$. Making use of Alon and Kahale's result [3] that $\lambda(n, K_k) \leq O(n^{1-2/k})$, we give a small improvement to this lower bound.

Theorem 1.8 *There exists a constant $\delta > 0$ such that for all $t \geq 3$ and $n \geq 1$, $\rho(n, K_t) \geq \delta n^{3/t}$.*

In the previous section, we saw that another way to generalize a question for triangle-free graphs is to forbid a longer cycle. Pudlák [29, Theorem 10] gave a case-based proof showing that there exists $c > 0$ such that $\rho(n, C_5) \geq cn$. Taking $t-1$ copies of each vector of an orthonormal basis in $\mathbb{R}^d$ gives an orthonormal representation of the graph consisting of d cliques of size $t-1$, which implies

$$\rho(n, C_t) \leq \lceil n/(t-1) \rceil.$$

Inspired by Erdős, we may ask if equality holds. Unlike before, we show that the answer turns out to be yes, in particular improving and generalizing Pudlák's result.

Theorem 1.9 *For all $t \geq 3$, $n \in \mathbb{N}$, we have $\rho(n, C_t) = \lceil n/(t-1) \rceil$.*

Indeed, this follows from the following more general result, which holds for all connected graphs H containing a vertex whose removal leaves a tree.

Theorem 1.10 *Let $t \geq 1$ and let H be a connected graph such that $V(H) = T \cup \{v\}$ where $H[T]$ is a tree on t vertices. Then for all $n \geq 1$, $\rho(n, H) = \lceil n/t \rceil$.*

Remark Our definition of $\mathrm{msr}(G)$ differs from the minimum semidefinite rank defined by Deaett [10]. Indeed, the representations $f : V(G) \to \mathbb{C}^d$ that he considers map into complex d-dimensional space, are allowed to map vertices to the 0 vector, and most importantly, must satisfy that $\langle f(u), f(v) \rangle \neq 0$ if and only if $uv \in E(G)$. The last condition defines a *faithful* representation, as studied by Lovász et al. [27]. Nevertheless, Theorems 1.8–1.10 may be adapted to work with these alternate assumptions.

We prove our results in the next two sections. We first prove Theorems 1.8–1.10 in Sect. 2, and then proceed to prove the remaining results in Sect. 3. The final section of the paper contains some concluding remarks.

2 Minimum Semidefinite Rank for *H*-Free Graphs

To study the minimum semidefinite rank of a graph, we will need the following useful inequality. It goes back to [5, p. 138] and its proof is based on a trick employed by Schnirelman in his work on Goldbach's conjecture [31]. For various combinatorial applications of this inequality, see, for instance, the survey by Alon [2].

Lemma 2.1 *Let M be a symmetric real matrix. Then* $\mathrm{tr}(M)^2 \le \mathrm{rk}(M)\,\mathrm{tr}(M^2)$.

Proof Let r denote the rank of M. Since M is a symmetric real matrix, M has precisely r non-zero real eigenvalues $\lambda_1, \dots, \lambda_r$. Note that $\mathrm{tr}(M) = \sum_{i=1}^r \lambda_i$ and $\mathrm{tr}(M^2) = \sum_{i=1}^r \lambda_i^2$. An application of Cauchy–Schwarz yields the desired $(\sum_{i=1}^r \lambda_i)^2 \le r \sum_{i=1}^r \lambda_i^2$. □

Now we are ready to prove Theorems 1.8 and 1.10. Theorem 1.9 follows immediately from Theorem 1.10.

Proof of Theorem 1.8 Let δ be a sufficiently small constant. We proceed by induction on t. For $t = 3$ we know that $\rho(n, K_3) = \lceil n/2 \rceil \ge \delta n$.

Now let $t \ge 3$ and let G be a K_{t+1}-free graph on n vertices. Let $f : V(G) \to \mathbb{R}^d$ be an orthonormal representation of G in $\mathbb{R}^d$ with $M = M_f$ being the corresponding Gram matrix. We will make use of Lemma 2.1. To this end, we shall upper bound $\mathrm{tr}(M^2)$. We have

$$\begin{aligned}\mathrm{tr}(M^2) &= \sum_{u \in V(G)} \Bigl(\langle f(u), f(u)\rangle^2 + \sum_{w \in N(u)} \langle f(u), f(w)\rangle^2 + \sum_{w \notin N(u)\cup\{u\}} \langle f(u), f(w)\rangle^2\Bigr)\\ &= \sum_{u \in V(G)} \Bigl(1 + \sum_{w \in N(u)} \langle f(u), f(w)\rangle^2\Bigr).\end{aligned}$$

Now fix $u \in V(G)$ and note that $G[N(u)]$ is K_t-free. Thus by the induction hypothesis, we have $d \ge \rho(|N(u)|, K_t) \ge \delta |N(u)|^{3/t}$. Since Alon and Kahale [3] showed that there exists a constant c such that $\lambda(n, K_t) \le c\, n^{1-2/t}$, we have via the fourth definition of the ϑ-function (Definition 1.1) that

$$\begin{aligned}\sum_{w \in N(u)} \langle f(u), f(w)\rangle^2 &\le \vartheta\bigl(\overline{G[N(u)]}\bigr) \le \lambda(|N(u)|, K_t) \le \lambda((d/\delta)^{t/3}, K_t)\\ &\le c\bigl((d/\delta)^{t/3}\bigr)^{1-2/t} = c \cdot (d/\delta)^{(t-2)/3}.\end{aligned}$$

Therefore, we conclude that $\mathrm{tr}(M^2) \le n(1 + c \cdot (d/\delta)^{(t-2)/3})$. Clearly $\mathrm{tr}(M) = n$ and $\mathrm{rk}(M) \le d$, so applying Lemma 2.1 and dividing by n yields

$$n \le d(1 + c \cdot (d/\delta)^{(t-2)/3}) = d + c \cdot \delta^{-(t-2)/3} d^{(t+1)/3} \le (d/\delta)^{(t+1)/3}$$

for δ a bit smaller than $1/c$. Thus $d \ge \delta n^{3/(t+1)}$ and since G and f were arbitrary, we conclude

$$\rho(n, K_{t+1}) \ge \delta n^{3/(t+1)}.$$ □

Proof of Theorem 1.10 Let $d = \lceil n/t \rceil$ and let G be a graph consisting of d cliques of size t. Since H is connected and has $t+1$ vertices, G is clearly H-free. By assigning the standard basis vector $e_i \in \mathbb{R}^d$ to each vertex in the i-th clique for $i \in [d]$, we obtain an orthonormal representation of G in $\mathbb{R}^d$, so that we conclude $\rho(n, H) \leq d = \lceil n/t \rceil$.

For the lower bound, let $d = \rho(n, H)$ and let G be an H-free graph on n vertices that has an orthonormal representation f in $\mathbb{R}^d$ with corresponding Gram matrix $M = M_f$. Next, we will use Lemma 2.1. Note that, as in the proof of Theorem 1.8,

$$\operatorname{tr}(M^2) = \sum_{u \in V(G)} \Bigl(1 + \sum_{w \in N(u)} \langle f(u), f(w)\rangle^2\Bigr).$$

Now fix $u \in V(G)$ and observe that since G has no copy of H, $G[N(u)]$ has no copy of some tree on t vertices. It is well known that in this case, $\chi(G[N(u)]) \leq t-1$, see e.g. Corollaries 1.5.4 and 5.2.3 of Diestel [11]. Thus we can partition $N(u)$ into $t-1$ independent sets $B_1, \ldots, B_{t-1}$. Since $\{f(w) : w \in B_i\}$ is an orthonormal set of vectors, we have by Parseval's inequality that $\sum_{w \in B_i} \langle f(w), v\rangle^2 \leq \|v\|^2$ for any $v \in \mathbb{R}^d$. In particular for $v = f(u)$, we therefore have

$$\sum_{w \in N(u)} \langle f(u), f(w)\rangle^2 = \sum_{i=1}^{t-1} \sum_{w \in B_i} \langle f(u), f(w)\rangle^2 \leq \sum_{i=1}^{t-1} \|f(v)\|^2 = t-1,$$

and thus

$$\operatorname{tr}(M^2) \leq \sum_{v \in V(G)} (1 + t - 1) = nt.$$

Clearly we have $\operatorname{tr}(M) = n$ and $\operatorname{rk}(M) \leq d$, so that by Lemma 2.1 we obtain $n^2 \leq dnt$. Thus we conclude that $d \geq n/t$ and so $\rho(n, H) = d \geq \lceil n/t \rceil$, as desired. □

3 Lovász ϑ-Function for *H*-Free Graphs

Proof of Claim 1.2 Let f be an orthonormal representation of G that attains the maximum in the definition of $L(G)$, and denote its Gram matrix by M_f. Let $\mathbb{1}$ denote the all 1's column vector (here and later all of our vectors will be column vectors). We have that

$$L(G)^2 = \Bigl\| \sum_{v \in V(G)} f(v) \Bigr\|^2 = \sum_{u,v \in V(G)} \langle f(u), f(v)\rangle = \mathbb{1}^\intercal M_f \mathbb{1} \leq n\vartheta(\overline{G}),$$

where the last inequality follows from the first definition of the ϑ-function.

For the other direction, let f^* be an orthonormal representation of G and x be a unit vector that together attain the minimum in the third definition of $\vartheta(G)$ (Definition 1.1). We therefore have that $\vartheta(G) \geq \langle x, f^*(v)\rangle^{-2}$ for all $v \in V(G)$. By changing the sign of $f^*(v)$ if necessary, we can ensure that $\langle x, f^*(v)\rangle \geq \vartheta(G)^{-1/2}$ for all $v \in V(G)$, so that by Cauchy–Schwarz we obtain

$$L(G) \geq \|x\| \Big\| \sum_{v\in V(G)} f^*(v) \Big\| \geq \Big\langle x, \sum_{v\in V(G)} f^*(v) \Big\rangle \geq \frac{n}{\sqrt{\vartheta(G)}}.$$

Moreover, Lovász [26, Thm. 8] showed that every vertex-transitive graph G satisfies $\vartheta(\overline{G})\vartheta(G) = n$, in which case the upper and lower bounds for $L(G)$ coincide. Thus if G is vertex-transitive, we conclude

$$L(G) = \sqrt{n\vartheta(\overline{G})}.$$

□

In order to prove Theorem 1.3 about C_{2t+1}-free graphs, we will need the following result proven implicitly by Erdős et al. [12]. It allows us to bound the chromatic number of the set of vertices at a fixed distance from a given vertex, for any graph without a cycle of prescribed length.

Lemma 3.1 *Let G be a graph having no cycle of length k and let $i \leq \lfloor (k-1)/2 \rfloor$. Fix a vertex u_0 in G and define $A_i = \{u \in V(G) : d(u, u_0) = i\}$ to be the set of vertices at a distance of exactly i from u_0. Then the induced subgraph $G[A_i]$ satisfies $\chi(G[A_i]) \leq k-2$.*

Proof In the proof of Theorem 1 of [12], Erdős et al. show that if G does not contain a cycle of length k and $i \leq \lfloor (k-1)/2 \rfloor$, then one can assign $k-2$ labels to the vertices of A_i so that no two vertices having the same label are adjacent. Hence $\chi(G[A_i]) \leq k-2$. □

Proof of Theorem 1.3 Let f be an orthonormal representation of G maximizing the largest eigenvalue of the corresponding Gram matrix $M = M_f$. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of M and observe that by the first definition of the ϑ-function (Definition 1.1), $\vartheta(\overline{G}) = \lambda_1$. Now note that $\mathrm{tr}(M^{2t+1}) = \sum_{i=1}^n \lambda_i^{2t+1}$ and that $\lambda_i \geq 0$ for all i since M is positive semidefinite. Thus we have $\lambda_1^{2t+1} \leq \sum_{i=1}^n \lambda_i^{2t+1} = \mathrm{tr}(M^{2t+1})$, and hence $\vartheta(\overline{G}) \leq \mathrm{tr}(M^{2t+1})^{1/(2t+1)}$. Therefore it will be enough for us to show that $\mathrm{tr}(M^{2t+1}) \leq (6t)^{2t} n$.

For convenience, given vertices $u_0, u_1, \ldots, u_k$, we define

$$W(u_0, \ldots, u_k) = \prod_{i=1}^{k} \langle f(u_{i-1}), f(u_i)\rangle = f(u_0)^{\mathsf{T}} \Bigg(\prod_{i=1}^{k-1} f(u_i) f(u_i)^{\mathsf{T}} \Bigg) f(u_k)$$

and note that $W(u_0, u_1, \ldots, u_{2t}, u_0) = 0$ whenever $u_0u_1 \ldots u_{2t}u_0$ is not a closed walk in G, i.e., whenever one of the pairs $u_0u_1, \ldots, u_{2t}u_0$ is a non-edge in G. Moreover, if $u_0u_1 \ldots u_{2t}u_0$ form a closed walk in G, then $d(u_0, u_i) \leq t$ for all i, so if we define $N^t(u_0) = \{v \in G : d(v, u_0) \leq t\}$ to be the set of vertices at a distance of at most t from u_0, we obtain

$$\begin{aligned}\operatorname{tr}(M^{2t+1}) &= \sum_{u_0,u_1,\ldots,u_{2t}\in V(G)} W(u_0,u_1,\ldots,u_{2t},u_0)\\ &= \sum_{u_0\in V(G)}\ \sum_{u_1,\ldots,u_{2t}\in N^t(u_0)} W(u_0,u_1,\ldots,u_{2t},u_0).\end{aligned}$$

Thus if we define

$$Y(u_0) := \sum_{u_1,\ldots,u_{2t}\in N^t(u_0)} W(u_0,u_1,\ldots,u_{2t},u_0)$$

for $u_0 \in V(G)$, then it suffices for us to show that $Y(u_0) \le (6t)^{2t}$ for all u_0, since we may then conclude

$$\operatorname{tr}(M^{2t+1}) = \sum_{u_0\in V(G)} Y(u_0) \le (6t)^{2t} n.$$

To bound $Y(u_0)$, we use Lemma 3.1. For any $r \le t$, define $A_r = \{u \in V(G) : d(u,u_0) = r\}$ to be the set of vertices at a distance of exactly r from u_0. Since G has no cycle of length $2t+1$, we have by Lemma 3.1 that $\chi(G[A_r]) \le 2t$, and so we let $\{B(r,1),\ldots,B(r,2t)\}$ be a partition of A_r into $2t$ independent sets. Note that for every closed walk $u_0\ldots u_{2t}u_0$, if we let $d_i = d(u_0,u_i)$ denote the distance from u_0 to u_i, then $|d_{i+1}-d_i| \le 1$. Thus we obtain

$$Y(u_0) = \sum_{\substack{d_1,\ldots,d_{2t}:\\ d_1=1,\ |d_{i+1}-d_i|\le 1}}\ \sum_{a_1,\ldots,a_{2t}\in[2t]}\ \sum_{\substack{u_1,\ldots,u_{2t}:\\ u_i\in B(d_i,a_i)}} W(u_0,u_1,\ldots,u_{2t},u_0).$$

Now since each $B(r,s)$ is an independent set, it follows that $\{f(u) : u \in B(r,s)\}$ is an orthonormal set of vectors. Moreover, observe that $P_{r,s} := \sum_{u\in B(r,s)} f(u)f(u)^\intercal$ is precisely the orthogonal projection onto the subspace spanned by $\{f(u) : u \in B(r,s)\}$. Thus for any $d_1,\ldots,d_{2t}$ such that $d_1 = 1$, $|d_{i+1}-d_i| \le 1$ for all i and for any $a_1,\ldots,a_{2t} \in [2t]$, we have

$$\begin{aligned}\sum_{\substack{u_1,\ldots,u_{2t}:\\ u_i\in B(d_i,a_i)}} W(u_0,u_1,\ldots,u_{2t},u_0) &= \sum_{\substack{u_1,\ldots,u_{2t}:\\ u_i\in B(d_i,a_i)}} f(u_0)^\intercal\left(\prod_{i=1}^{2t} f(u_i)f(u_i)^\intercal\right)f(u_0)\\ &= f(u_0)^\intercal\left(\prod_{i=1}^{2t}\sum_{u_i\in B_{d_i,a_i}} f(u_i)f(u_i)^\intercal\right)f(u_0)\\ &= f(u_0)^\intercal\left(\prod_{i=1}^{2t} P_{d_i,a_i}\right)f(u_0),\end{aligned}$$

and since any orthogonal projection P satisfies $\|Pv\| \le \|v\|$, we may apply Cauchy–Schwarz to obtain

$$\begin{aligned} f(u_0)^{\mathsf{T}}\biggl(\prod_{i=1}^{2t} P_{d_i,a_i}\biggr) f(u_0) &= \biggl\langle f(u_0)\biggl(\prod_{i=1}^{2t} P_{d_i,a_i}\biggr) f(u_0)\biggr\rangle \\ &\le \|f(u_0)\|\biggl\|\biggl(\prod_{i=1}^{2t} P_{d_i,a_i}\biggr) f(u_0)\biggr\| \\ &\le \|f(u_0)\|\cdot\|f(u_0)\| = 1. \end{aligned}$$

Since there are at most 3^{2t} sequences of integers $(d_1, \ldots, d_{2t})$ such that $d_1 = 1$ and $|d_{i+1} - d_i| \le 1$ for all i, we therefore conclude

$$\begin{aligned} Y(u_0) &= \sum_{\substack{d_1,\ldots,d_{2t}:\\ d_1=1,\ |d_{i+1}-d_i|\le 1}} \sum_{a_1,\ldots,a_{2t}\in[2t]} \sum_{\substack{u_1,\ldots,u_{2t}:\\ u_i\in B(d_i,a_i)}} W(u_0,u_1,\ldots,u_{2t},u_0) \\ &\le \sum_{\substack{d_1,\ldots,d_{2t}:\\ d_1=1,\ |d_{i+1}-d_i|\le 1}} \sum_{a_1,\ldots,a_{2t}\in[2t]} 1 \le 3^{2t}(2t)^{2t} = (6t)^{2t}. \end{aligned}$$

□

We now cite known constructions of C_t-free, $K_{2,t}$-free, or $K_{t,(t-1)!+1}$-free graphs with many edges and optimal spectral gaps, in order to obtain Theorem 1.4. Note that some of the graphs described below have loops on some of their vertices, so to get a simple graph these loops should be removed. Since this only affects the adjacency matrix, by subtracting a diagonal matrix with 1s and 0s on the diagonal, one can deduce from Weyl's interlacing inequality that the eigenvalues only change by at most 1, not affecting the asymptotic bounds obtained below.

Proof of Theorem 1.4 For the C_{2t+1}, C_4, C_6, C_{10}, and $K_{t,(t-1)!+1}$-free graph constructions and their spectral properties discussed below, see Sect. 3 of the survey on pseudo-random graphs [25] by Krivelevich and Sudakov.

As previously mentioned, Alon and Kahale [3] note that a modification of Alon's construction [1] gives a graph with an optimal spectral gap which is, in particular, C_{2t+1}-free for any fixed $t \ge 1$. For more details, see [25, Sect. 3, Exam. 10]. Indeed, for any k such that 2^k-1 is not divisible by $4t+3$, the construction yields a $2^{k-1}(2^{k-1}-1)$-regular graph G on $n = 2^{(2t+1)k}$ vertices which is C_{2t+1}-free, such that all eigenvalues of its adjacency matrix except the largest one are bounded in absolute value by $O(2^k)$. The adjacency matrix A of such a graph therefore has largest eigenvalue $\lambda_1(A) = 2^{k-1}(2^{k-1}-1)$ and all other eigenvalues bounded in absolute value by $O(2^k)$. Applying the second definition of $\vartheta(\overline{G})$ to the adjacency matrix of G, and using the fact that the smallest eigenvalue of G is negative (as the trace of the adjacency matrix is 0), we thus conclude

$$\lambda(n, C_{2t+1}) \ge \vartheta(\overline{G}) \ge 1 + \frac{2^{k-1}(2^{k-1}-1)}{O(2^k)} = \Omega(n^{1/(2t+1)}).$$

The construction of a C_4-free graph G with an optimal spectral gap and many edges comes from the projective space over a finite field of order $q = p^\alpha$ where p is a prime and α is an integer (see [25, Sect. 3, Exam. 6]). It has $n = q^2 + q + 1$ vertices, is $(q + 1)$-regular (so $\lambda_1 = q + 1$), and all of its eigenvalues except for the largest one are in absolute value equal to $\sqrt{q}$. Therefore, we obtain as above that

$$\lambda(n, C_4) \geq \vartheta(\overline{G}) \geq 1 + \frac{q+1}{\sqrt{q}} = \Omega(n^{1/4}).$$

The C_6-free graph and the C_{10}-free graph with optimal spectral gaps and many edges are the polarity graphs of a generalized 4-gon and 6-gon respectively, (see [25, Sect. 3, Exam. 7]). As above, these graphs yield the bounds $\lambda(n, C_6) \geq \Omega(n^{1/6})$ and $\lambda(n, C_{10}) \geq \Omega(n^{1/10})$.

The $K_{t,(t-1)!+1}$-free graph G with an optimal spectral gap and many edges is called a projective norm graph, see [25, Sect. 3, Exam. 12.]. For a prime p, G has $p^t - p^{t-1}$ vertices, is $(p^{t-1} - 1)$-regular, and all eigenvalues except for the largest one are in absolute value at most $p^{(t-1)/2}$. Thus we obtain

$$\lambda(n, K_{t,(t-1)!+1}) \geq \vartheta(\overline{G}) \geq 1 + \frac{p^{t-1} - 1}{p^{(t-1)/2}} = \Omega(n^{(1-1/t)/2}).$$

The following construction of a $K_{2,t+1}$-free graph with many edges is due to Füredi [15]. As he did not show that this construction has an optimal spectral gap, we prove it below. Let q be a prime power such that t divides $q - 1$ and let $\mathbb{F}$ be a finite field of order q. Let $h \in \mathbb{F}$ be an element of order t and let $H = \{1, h, h^2, \ldots, h^{t-1}\}$. Define an equivalence relation on $\mathbb{F} \times \mathbb{F} \setminus \{(0, 0)\}$ by $(a, b) \sim (a', b')$ iff there exists $c \in H$ such that $(a', b') = c \cdot (a, b)$. Let $\langle a, b\rangle$ denote the equivalence class of (a, b) under the relation $\sim$. Now define G to be the graph whose vertices are the equivalence classes of $(\mathbb{F} \times \mathbb{F} \setminus \{(0, 0)\})/\sim$, such that there is an edge between $\langle a, b\rangle$ and $\langle a', b'\rangle$ iff $aa' + bb' \in H$.

Each equivalence class has t elements, and therefore G has $n = (q^2 - 1)/t$ vertices. Moreover, for each vertex $(a, b) \in \mathbb{F} \times \mathbb{F} \setminus \{(0, 0)\}$, there are q solutions (x, y) to the equation $ax + by = c$ for any $c \in H$, and therefore $\langle a, b\rangle$ has degree $tq/t = q$. Now let $\langle a, b\rangle$, $\langle a', b'\rangle$ be a pair of distinct vertices and consider their common neighborhood. To determine its size, we must determine the number of solutions (x, y) to the equations

$$\begin{cases} ax + by = d, \\ a'x + b'y = e, \end{cases}$$

where $d, e \in H$. If there exists c such that $a' = ca$, $b' = cb$, then the equations have no solutions, since otherwise we would have $e = cax + cby = cd$, which would imply that $c \in H$, contradicting the fact that $\langle a, b\rangle \neq \langle a', b'\rangle$. Thus $\langle a, b\rangle$ and $\langle a', b'\rangle$ have no common neighbors in this case. Otherwise, if there does not exist c such that $a' = ca$, $b' = cb$, then the matrix $\begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$ is invertible and hence the system of equations has a unique solution (x, y) for each choice of $d, e \in H$. As there are t^2 choices for d and e, we obtain a total of t^2 solutions, which implies that there are

$t^2/t = t$ vertices in the common neighborhood of $\langle a, b\rangle$ and $\langle a', b'\rangle$. Thus G has no copy of $K_{2,t+1}$.

Now let A be the adjacency matrix of G, indexed by the vertices $\langle a, b\rangle$, and consider A^2. Since G is q-regular, the diagonal entries of A^2 will all be q. The off-diagonal entry $A^2_{\langle a,b\rangle,\langle a',b'\rangle}$ is the number of common neighbors of $\langle a, b\rangle$ and $\langle a', b'\rangle$, which by the previous discussion is either 0 or t depending on whether or not there exists c such that $a' = ca$, $b' = cb$. Thus if we let Q be the $\{0, 1\}$ matrix indexed by the vertices of G so that $Q_{\langle a,b\rangle,\langle a',b'\rangle} = 1$ iff $\langle a, b\rangle$ and $\langle a', b'\rangle$ have no common neighbors, then we have

$$A^2 = (q - t)I + tJ - tQ,$$

where I is the identity matrix and J is the all-ones matrix. Now for any given $\langle a, b\rangle$, observe that we must have $c \in (\mathbb{F} \setminus \{0\}) \setminus H$ in order for $a' = ca$, $b' = cb$ to yield $(a', b') \neq (0, 0)$ such that $\langle a', b'\rangle \neq \langle a, b\rangle$. This gives $q - 1 - t$ choices for c and therefore there are exactly $(q - 1 - t)/t$ many vertices $\langle a', b'\rangle$ that have no common neighbors with $\langle a, b\rangle$, so that Q is a matrix with $(q - 1 - t)/t$ ones in each row. By the Perron–Frobenius theorem, the largest eigenvalue of Q is $\lambda_1(Q) = (q - 1 - t)/t$ with eigenvector $\mathbb{1}$, and all other eigenvalues satisfy $|\lambda_i(Q)| \leq \lambda_1(Q)$ and have eigenvectors which are orthogonal to $\mathbb{1}$. J has largest eigenvalue $\lambda_1(J) = n$ also with the eigenvector $\mathbb{1}$ and any vector orthogonal to $\mathbb{1}$ is an eigenvector of J with eigenvalue 0. Therefore, any eigenvector of Q is also an eigenvector of A^2 which implies that for all $i \geq 2$,

$$|\lambda_i(A^2)| \leq q - t + t \cdot \frac{q - 1 - t}{t} = 2q - 2t - 1.$$

Now since G is q-regular, the largest eigenvalue of A is q, and all other eigenvalues are square roots of eigenvalues of A^2. Thus we conclude

$$\max_{i\geq 2} |\lambda_i(A)| \leq \sqrt{2q - 2t - 1}\,.$$

Finally, applying the second definition of $\vartheta(\overline{G})$ (Definition 1.1) with the matrix A, we obtain

$$\lambda(n, K_{2,t+1}) \geq \vartheta(\overline{G}) \geq 1 - \frac{\lambda_1(A)}{\lambda_n(A)} \geq 1 + \frac{q}{\sqrt{2q - 2t - 1}} = \Omega(t^{1/4}n^{1/4}). \qquad \square$$

We now give a proof of Theorem 1.5, using an approach similar to that which was used by Alon and Kahale to prove $\lambda(n, K_t) \leq O(n^{1-2/t})$ in [3].

Proof of Theorem 1.5 We proceed by induction on n. For $n = 1$ the claim holds trivially. Now suppose $n \geq 2$ and let G be an H-free graph on n vertices. Define $U = \{v \in V(G) : d(v) \leq 4c \cdot n^{\alpha}\}$ and $W = V(G) \setminus U$. It follows from the fourth definition of the ϑ-function that $\vartheta(\overline{G}) \leq \vartheta(\overline{G[U]}) + \vartheta(\overline{G[W]})$. Moreover, observe that

$$4c \cdot n^{\alpha}|W| \leq \sum_{v\in W} d(v) \leq 2\,\mathrm{ex}(n, H) \leq 2c \cdot n^{1+\alpha},$$

so $|W| \le n/2$, and hence by the induction hypothesis

$$\vartheta(\overline{G[W]}) \le \lambda(n/2, H) \le 20 \cdot \frac{\sqrt{ch}}{\alpha} \cdot \left(\frac{n}{2}\right)^{\alpha/2}.$$

It remains to bound $\vartheta(\overline{G[U]})$. To this end let f be an orthonormal representation of $G[U]$ maximizing the largest eigenvalue $\lambda_1(M)$ of the corresponding Gram matrix $M = M_f$. By the first definition of the ϑ-function, we have $\vartheta(\overline{G[U]}) = \lambda_1(M)$. Now fix $u \in U$ and define $N'(u) = \{w \in U : uw \in E(G)\}$ to be the neighborhood of u in $G[U]$. Since $G[U]$ has no copy of H, we have that $N'(u)$ induces no copy of the tree T. Therefore, by the same argument as in the proof of Theorem 1.10, $N'(u)$ can be partitioned into at most h independent sets, each corresponding to a set of orthonormal vectors. Thus by Parseval's inequality, $\sum_{w\in N'(u)} \langle f(u), f(w)\rangle^2 \le h$. Since $|N'(u)| \le d(u) \le 4c \cdot n^\alpha$, we conclude via Cauchy–Schwarz that $\sum_{w\in N'(u)} |\langle f(u), f(w)\rangle| \le \sqrt{4c \cdot n^\alpha h}$. Note that $\lambda_1(M) \le \max_{u\in U} \sum_{w\in U} |\langle f(u), f(w)\rangle|$, and thus

$$\begin{aligned}\vartheta(\overline{G[U]}) &\le \max_{u\in U} \sum_{w\in U} |\langle f(u), f(w)\rangle| = \max_{u\in U} \Bigg(1 + \sum_{w\in N'(u)} |\langle f(u), f(w)\rangle|\Bigg)\\ &\le 1 + \sqrt{4c \cdot n^\alpha h} \le 3\sqrt{c \cdot n^\alpha h}.\end{aligned}$$

Putting everything together, we have

$$\begin{aligned}\vartheta(\overline{G}) &\le \vartheta(\overline{G[U]}) + \vartheta(\overline{G[W]}) \le 3\sqrt{c \cdot n^\alpha h} + 20 \cdot \frac{\sqrt{ch}}{\alpha} \cdot \left(\frac{n}{2}\right)^{\alpha/2}\\ &= \sqrt{c \cdot n^\alpha h} \cdot \left(3 + \frac{20}{\alpha} \cdot \left(\frac{1}{2}\right)^{\alpha/2}\right).\end{aligned}$$

Now to complete the proof, we use the fact that $e^{-x} \le 1 - x/2$ for $0 \le x \le 1$ to conclude

$$3 + \frac{20}{\alpha} \cdot \left(\frac{1}{2}\right)^{\alpha/2} \le 3 + \frac{20}{\alpha} \cdot \left(1 - \frac{\alpha \ln 2}{4}\right) \le \frac{20}{\alpha}. \qquad \square$$

Corollary 1.6 will now follow from known upper bounds on Turán numbers.

Proof of Corollary 1.6 Recently, Bukh and Tait [8] showed that $\mathrm{ex}(n, \theta_{t,s}) \le O(ts^{1-1/t} n^{1+1/t})$, generalizing the well-known upper bounds $\mathrm{ex}(n, C_{2t}) \le O(tn^{1+1/t})$ due to Bondy and Simonovits [6], and $\mathrm{ex}(n, K_{2,t}) \le O(tn^{1+1/t})$ due to Füredi [15]. Since $\theta_{t,s}$ consists of a tree together with an additional vertex, we may apply Theorem 1.5 to obtain the desired upper bounds on $\lambda(n, H)$. □

Remark Bukh and Jiang [7] recently improved the upper bound on $\mathrm{ex}(n, C_{2t})$ to $O\big(n^{1+1/t}\sqrt{t}\log t\big)$ for n sufficiently large relative to t. Using Theorem 1.5, this implies $\lambda(n, C_{2t}) \le O\big(n^{1/(2t)} t^{7/4} \sqrt{\log t}\big)$. Nonetheless, in the appendix we show how to obtain the better bound $\lambda(n, C_{2t}) \le O(tn^{1/(2t)})$ via a different argument.

Theorem 1.7 will follow from an argument similar to that of Theorem 1.5, except that we will have to replace the result that the chromatic number of a neighborhood is bounded, with a bound on the ϑ-function of a neighborhood which will be obtained inductively.

Proof of Theorem 1.7 We proceed by induction on n and t, where $c_{t,s}$ will be defined recursively. For $s \geq t = 2$, let $c_{2,s}$ be the constant such that $\lambda(n, K_{2,s}) \leq c_{2,s} s^{3/4} n^{1/4}$ as given by Corollary 1.6. Now suppose $s \geq t \geq 3$. For $n = 1$, the claim trivially holds for $c_{t,s} \geq 1$. Now let $n \geq 2$.

Kövàri, Sós, and Turán [24] showed that there exists a constant $a_{t,s}$ such that $\mathrm{ex}(n, K_{t,s}) \leq a_{t,s} n^{2-1/t}$. As in the proof of Theorem 1.5, define $U = \{v \in V(G) : d(v) \leq 4a_{t,s} n^{1-1/t}\}$, $W = V(G) \setminus U$, and observe that by the fourth definition of the ϑ-function, $\vartheta(\overline{G}) \leq \vartheta(\overline{G[U]}) + \vartheta(\overline{G[W]})$. Moreover, observe that

$$4a_{t,s} n^{1-1/t} |W| \leq \sum_{v \in W} d(v) \leq 2\,\mathrm{ex}(n, K_{t,s}) \leq 2a_{t,s} n^{2-1/t},$$

so $|W| \leq n/2$, and hence by the induction hypothesis

$$\vartheta(\overline{G[W]}) \leq \lambda(n/2, K_{s,t}) \leq c_{t,s} \left(\frac{n}{2}\right)^{1-2/t+1/(t2^{t-1})}.$$

To bound $\vartheta(\overline{G[U]})$, let f be an orthonormal representation of $G[U]$ maximizing the largest eigenvalue $\lambda_1(M)$ of the corresponding Gram matrix $M = M_f$, so that we have $\vartheta(\overline{G[U]}) = \lambda_1(M)$. Now fix $u \in U$ and let $N'(u) = \{w \in U : uw \in E(G)\}$ be the neighborhood of u in $G[U]$. Note that $G[U]$ has no copy of $K_{t-1,s}$, so that via the fourth definition of the ϑ-function and induction, we have

$$\begin{aligned}\sum_{w \in N'(u)} \langle f(u), f(w)\rangle^2 &\leq \vartheta(\overline{G[N'(u)]}) \leq \lambda(|N'(u)|, K_{t-1,s}) \\ &\leq c_{t-1,s} |N'(u)|^{1-2/(t-1)+1/((t-1)2^{t-2})}.\end{aligned}$$

Thus using the fact that $|N'(u)| \leq 4a_{t,s} n^{1-1/t}$ and applying Cauchy–Schwarz, we conclude

$$\begin{aligned}\sum_{w \in N'(u)} |\langle f(u), f(w)\rangle| &\leq \sqrt{|N'(u)| \cdot c_{t-1,s} \cdot |N'(u)|^{1-2/(t-1)+1/((t-1)2^{t-2})}} \\ &\leq 4\sqrt{c_{t-1,s}} \cdot a_{t,s}^{1-1/(t-1)+1/((t-1)2^{t-1})} \cdot n^{1-2/t+1/(t2^{t-1})}.\end{aligned}$$

As in the proof of Theorem 1.5, we therefore obtain

$$\begin{aligned}\vartheta(\overline{G[U]}) &\leq \max_{u \in U} \sum_{w \in U} |\langle f(u), f(w)\rangle| = \max_{u \in U} \Bigl(1 + \sum_{w \in N'(u)} |\langle f(u), f(w)\rangle|\Bigr) \\ &\leq \Bigl(1 + 4\sqrt{c_{t-1,s}} \cdot a_{t,s}^{1-1/(t-1)+1/((t-1)2^{t-1})}\Bigr) n^{1-2/t+1/(t2^{t-1})}.\end{aligned}$$

Thus if we set

$$c_{t,s} = \frac{1 + 4\sqrt{c_{t-1,s}} \cdot a_{t,s}^{1-1/(t-1)+1/((t-1)2^{t-1})}}{1 - (1/2)^{1-2/t+1/(t2^{t-1})}},$$

then we conclude the desired result

$$\vartheta(\overline{G}) \leq \vartheta(\overline{G[U]}) + \vartheta(\overline{G[W]}) \leq c_{t,s} n^{1-2/t+1/(t2^{t-1})}. \qquad \square$$

4 Concluding Remarks

We have seen that for $H \in \{C_{2t+1}, C_4, C_6, C_{10}, K_{2,t}\}$ fixed and n large, Theorem 1.4 and Corollary 1.6 provide bounds on $\lambda(n, H)$ that are asymptotically tight. However, the lower bound in Theorem 1.4 for $\lambda(n, K_{t,s})$ with $s \geq t \geq 3$ does not match the upper bound obtained in Theorem 1.7, so determining the correct asymptotic dependence on n is an interesting problem. Indeed, for $n \gg t \to \infty$, we have

$$1/2 - o(1) \leq \log_n \lambda(n, K_{t,s}) \leq 1 - o(1),$$

where the lower bound is coming from graphs with optimal spectral gaps which are almost extremal for the Turán number, so that we cannot hope to do better with such constructions. On the other hand, we know

$$1 - o(1) \leq \log_n \lambda(n, K_t) \leq 1 - o(1)$$

for $n \gg t \to \infty$, and it would therefore be interesting to determine if the asymptotic behavior of $\lambda(n, H)$ is different for $H = K_t$ versus $H = K_{t,s}$.

For $H = K_{2,t}$, even though we know the asymptotic behavior of $\lambda(n, H)$, we are only able to show that

$$\Omega(t^{1/4} n^{1/4}) \leq \lambda(n, K_{2,t}) \leq O(t^{3/4} n^{1/4}),$$

so it would be interesting to determine the correct dependence of $\lambda(n, K_{2,t})$ on t.

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