Proof of Grinblat's conjecture on rainbow matchings in multigraphs

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Abstract

Many well-known problems in combinatorics can be reduced to finding a large rainbow structure in a certain edge-coloured multigraph. Two celebrated examples of this are Ringel's tree packing conjecture and Ryser's conjecture on transversals in Latin squares. In this paper, we answer such a question raised by Grinblat twenty years ago. Let an (n, v)-*multigraph* be an n-edge-coloured multigraph in which the edges of each colour span a disjoint union of non-trivial cliques that have in total at least v vertices. Grinblat conjectured that for all $n \geq 4$, every $(n, 3n-2)$-multigraph contains a rainbow matching of size n. Here, we prove this conjecture for all sufficiently large n.

1 Introduction

A *rainbow copy* of a graph H in an edge-coloured graph G is a subgraph of G isomorphic to H whose edges have different colours. There are many well-known problems in combinatorics that can be reduced to finding a large rainbow structure in a certain edge-coloured multigraph. One example of this is the famous conjecture of Ringel [21] from 1963 stating that the edges of the complete graph K_{2n+1} can be decomposed by copies of any tree on n vertices. Kötzig [22] noticed that this can be reduced to showing that a certain edge-colouring of K_{2n+1} contains a rainbow copy of any tree on n vertices. Recently, this problem and some other questions about finding rainbow trees were resolved in [14, 15].

Another celebrated problem in the area involves transversals in Latin squares, the study of which dates back to the work of Euler in the 1700s. A Latin square of order n is an $n \times n$ array filled with n symbols such that no symbol appears more than once in a row or column and a transversal is a set of entries such that no two of them have a symbol, row or column in common. The Ryser-Brualdi-Stein conjecture [4, 10, 23, 24] states that every Latin square contains a transversal using all but at most one symbol. It is not difficult to see that a Latin square of order n is actually equivalent to a proper n-edge-colouring of the complete bipartite graph $K_{n,n}$ and a transversal is now a rainbow matching in this graph. Thus, the conjecture states that there is always a rainbow matching of size $n-1$ in such a graph. Although this still remains open, the problem has atracted a lot of attention over the last 50 years (see, e.g., [12] and its references). Improving previous bounds from [11, 25], the best known result towards this conjecture was recently obtained by Keevash, Pokrovskiy, Sudakov and Yepremyan [12] who showed that there is always a rainbow matching of size $n - O\left(\frac{\log n}{\log\log n}\right)$.

There are now many variants and generalisations of the Ryser-Brualdi-Stein conjecture. One of these is the Aharoni-Berger conjecture [1], which states that every edge-coloured bipartite multigraph with n colours, each consisting of a matching of size $n+1$, contains a rainbow matching using all the colours. Note that indeed this implies the Ryser-Brualdi-Stein conjecture since any properly n-edge-coloured $K_{n,n}$ can be transformed into such a graph by adding to it a disjoint edge repeated in each one of the n colours. This problem has been extensively studied (see, e.g., [2, 3, 5, 13, 20]) and the conjecture was shown to hold asymptotically in [19] (see also [16] for a very short proof).

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In this paper, we will consider another open problem in this area. An (n, v)*-multigraph* is an n-edge-coloured multigraph in which each *colour class*, i.e., the graph formed by the edges of each colour, forms a disjoint union of non-trivial cliques (i.e., cliques of size at least 2) that in total have at least v vertices. These can be seen as a generalisation of the type of edge-coloured multigraphs mentioned earlier. Indeed, note that the Aharoni-Berger conjecture is equivalent to the statement that every bipartite $(n, 2n+2)$-multigraph contains a rainbow matching of size n. Twenty years ago, Grinblat [8] raised the question of how large v should be so that every (n, v)-multigraph contains a rainbow matching using all the colours. In fact, initially, Grinblat's question was formulated as a measure-theoretic problem in the context of his work on algebras of sets, in which he looked at sufficient conditions for a family of algebras over a set X to cover the whole power set $\mathcal{P}(X)$. His question was later reformulated as a graph theoretic problem and gained attention of combinatorialists.

Let us note first that if $v \geq 4n-3$ we can greedily find a rainbow matching of size n. Indeed, given a rainbow matching M of size at most $n-1$, if any colour not used in it has at least $4n-3$ vertices in its colour class, which is a disjoint union of non-trivial cliques, then note that it must have an edge outside M and thus, we can add it and get a larger rainbow matching. Grinblat [8] first showed that v should be larger than $3n-3$ and is at most $4n - \lfloor \frac{n+3}{2} \rfloor$. The lower bound follows by considering a disjoint union of $n-1$ triangles, whose edges are repeated in each one of the n colours. Note that a matching cannot have more than one edge from each triangle and so, there is no matching of size n. For the smaller values of $n = 2, 3$, one can do slightly better. E.g., for $n = 3$, consider two disjoint copies of a proper edge-colouring of K_4 with three colours. Nevertheless, Grinblat [8] conjectured the following.

Conjecture 1.1. *For all $n \geq 4$, every $(n, 3n-2)$-multigraph contains a rainbow matching of size n.*

In [9], he subsequently improved the upper bound to $10n/3 + \sqrt{2n/3}$. Later, Nivasch and Omri [18] lowered it to $16n/5 + O(1)$. Finally, Clemens, Ehrenmüller and Pokrovskiy [6] gave the first asymptotic proof of the conjecture showing that there is a constant c such that every $(n, 3n + c\sqrt{n})$-multigraph has a rainbow matching of size n. In this paper, we resolve Grinblat's conjecture for all sufficiently large n.

Theorem 1.2. *There is n_0 such that for all $n \geq n_0$, every $(n, 3n-2)$-multigraph contains a rainbow matching of size n.*

In the next section we will give some notation, preliminary lemmas and a proof outline. We will then prove two technical lemmas in Section 3 and use these to prove Theorem 1.2 in Section 4. The last section of the paper contains some concluding remarks and further questions.

2 Preliminaries

We begin by first giving some notation and definitions. Throughout the paper we use standard graph-theoretic notation. We will often consider an edge-coloured multigraph G, which is not necessarily properly coloured. For a colour c we will say that an edge e of colour c is *c-coloured* and define the function *col* so that $col(e) = c$. The edge e is naturally associated to a pair of vertices xy which are its endpoints and we will also then say that the pair xy is c-coloured. The edge e, or the pair xy, will be said to be *repeated in at least t colours* if there are at least t different colours c' such that xy is c'-coloured. More generally, for a set of colours C, we will say that the edge e (or the pair xy) is C-coloured if $col(e) \in C$. For two sets of vertices X, Y, we will let $E[X, Y]$ denote the set of edges which have one endpoint in X and the other in Y. For a set of edges F, we will let $V(F)$ denote the set of endpoints of these edges and $col(F)$ denote the set of colours c with a c-coloured edge in F. Given a matching M and a vertex $x \in V(M)$, we let $m(x)$ denote its 'opposite vertex' in M, that is, the one such that $xm(x)$ is an edge of M. Similarly, for a set $A \subseteq V(M)$, we let $m(A)$ denote the set of opposite vertices in M of the vertices in A, that is, $m(A) := \{m(x) : x \in A\}$ (note that we can have $m(A) \cap A \neq \emptyset$) - usually the small m notation will be unambiguous. Also, in some situations, it will be clear that although we are referring to an edge e, we are instead considering its endpoints xy - for example, we might refer to an edge-set of the form $E[e, X]$, where we are obviously considering the set $E[\{x, y\}, X]$. Furthermore, we might say that $xy \in E[X, Y]$ where we mean that any edge with those endpoints belongs to that edge-set. We also make the following definition.

Definition 2.1. For a matching M, a *horn in M* is an edge $e \in M$ such that there exist two disjoint edges $e_1, e_2 \in E[e, V \setminus V(M)]$. It is a *$c$-horn* if both edges e_1, e_2 are c-coloured and it is a *C-rainbow horn* if there exist two distinct $c_1, c_2 \in C$ such that e_1 is c_1-coloured and e_2 is c_2-coloured.

Finally, we make a simple remark which we will refer to in various places throughout the paper.

Remark. Every (n, v)-multigraph has a subgraph which is also an (n, v)-multigraph and such that every colour is a disjoint union of K_2's and K_3's.

Indeed, let G be an (n, v)-multigraph. The desired subgraph $G' \subseteq G$ can be obtained by removing edges from G in the following manner. Let c be a colour and recall that its edges span a disjoint union of cliques with in total at least v vertices. For each such clique, partition its vertices into sets of size two and possibly one set of size three. Then, replace the clique with the disjoint edges corresponding to these sets of size two and possibly a K_3 corresponding to the set of size three. Doing this for every colour c produces a subgraph G' which is still an (n, v)-multigraph and such that every monochromatic clique is either an edge or a K_3.

2.1 A few simple lemmas

In this section, we will give some very simple lemmas which will be used throughout the paper. First, let us note the following.

Observation 2.2. *Let M be matching, $e \in M$ and suppose there is a set C of three colours such that e is a c-horn for two colours from C and there is some edge in $E[e, V \setminus V(M)]$ coloured in the third colour from C. Then, e is a C-rainbow horn.*

It is quick to check the above observation and further, as a trivial corollary, note the following.

Lemma 2.3. *Let N be a matching and C a set of colours such that for each $c \in C$ there are at least k c-horns in N. If $k|C| > 2|N|$, there exists a C-rainbow horn in N.*

Proof. Suppose there is no C-rainbow horn. By the previous observation, then no edge in N is a c-horn for at least three colours $c \in C$. Therefore, since there are k c-horns for each colour c, we must have $k|C| \leq 2|N|$, which is a contradiction. $\square$

We also give a lemma which, in various situations, will essentially allow us to forget about the various monochromatic triangles which can appear in (n, v)-multigraphs and only look at each colour class as a matching.

Lemma 2.4. *Let M be a matching and $A \subseteq V(M)$ be such that $A \cap m(A) = \emptyset$. Let H be a disjoint union of non-trivial cliques with no edges contained in $A \cup (V \setminus V(M))$ and with in total at least $2|M| + s$ vertices. Then, there exists a matching in H of size s consisting of edges with one endpoint in $V(M) \setminus (A \cup m(A))$ and the other in $A \cup (V \setminus V(M))$.*

Proof. Define the set $S := V(M) \setminus A$ so that there is no edge of H contained outside S. Note that H has at least $|A|+s$ vertices outside S and so, there is an edge of H from each such vertex to S. Since the graph H is a disjoint union of non-trivial cliques, these edges must be pairwise disjoint, as otherwise, there would be an edge of H connecting their endpoints outside S. This then gives a matching in H contained in $E[S, V \setminus S]$ of size at least $|A|+s$. Since at most $|A|$ edges of this matching intersect $m(A)$, we are done. $\square$

We will finally need a standard probabilistic concentration inequality, which can be found in most probabilistic textbooks (e.g., [7]).

Lemma 2.5. *Let X be the sum of independent random variables $X_1, \ldots, X_n$ such that each $0 \le X_i \le k$. Then, for all $0 < \varepsilon < 1$,*

$$\mathbb{P}\left(|X - \mathbb{E}[X]| > \varepsilon \mathbb{E}[X]\right) \le 2e^{-\varepsilon^2 \mathbb{E}[X]/3k^2}.$$

2.2 Auxiliary matchings

For a rainbow matching M, a *t-auxiliary matching for M* is a matching $N \subseteq E[V \setminus V(M), V(M)]$ such that the following properties hold: no two of its edges intersect the same edge in M; each one of its edges is repeated in at least t colours not belonging to *col*(M). We will let $M_N \subseteq M$ denote the set of edges of M which intersect an edge of N. For $e \in M_N$, we will let x_e denote the endpoint of e which is contained in $V(N)$, $m(x_e)$ denote its opposite vertex in M (as usual) and v_e denote its opposite vertex in N. Finally, we let C_N denote the set of colours used in M_N (not those in N), that is, the set *col*(M_N). The reader might want to refer to Figure 1 for an illustration.

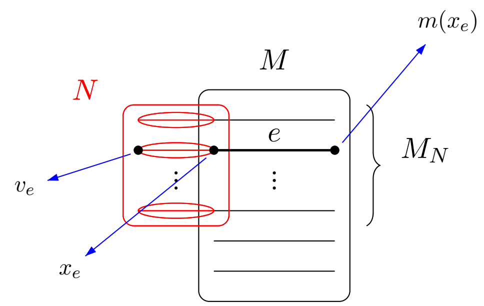

Figure 1: A 3-auxiliary matching for M.

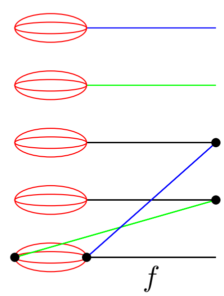

Figure 2: A C_N-rainbow horn in $N \cup (M \setminus M_N)$.

As a first simple observation, we will show the following lemma, which will be the basis of most of the local considerations done throughout the paper. Indeed, a particular case of the lemma is Observation 2.2.

Lemma 2.6. *Let M be a maximum rainbow matching, C_0 be the set of colours not in col(M), N be a t-auxiliary matching for M and $M' \subseteq M$ be a set of size less than $t/3-1$. Then there is no $(C_0 \cup C_N)$-coloured rainbow matching of size $|M'|+1$ vertex disjoint from the matching $\{v_e x_e : e \in M_N \setminus M'\} \cup (M \setminus (M_N \cup M'))$.*

Proof. Suppose otherwise and let L be such a $(C_0 \cup C_N)$-coloured rainbow matching. Let $e_1, \ldots, e_i$ denote the edges of $M_N \setminus M'$ which intersect edges of L and let $e'_1, \ldots, e'_j$ denote the edges of $M_N \setminus (M' \cup \{e_1, \ldots, e_i\})$ whose colours are represented in L. First, note that we have $|C_0 \cap col(L)| \le |L| - j$, since the colours $col(e'_l)$ belong to $col(L)$ but not to C_0. Further, we have that $i \le |V(L)| = 2|L|$ and so, $|C_0 \cap col(L)| + i + j \le (|L| - j) + i + j \le 3|L| = 3|M'| + 3 < t$. By the definition of a t-auxiliary matching this implies that we can pick distinct colours in $C_0 \setminus col(L)$ for the edges $v_{e_l}x_{e_l}, v_{e'_l}x_{e'_l}$ so that

$$\Big(M \setminus \big(M' \cup \{e_1, \ldots, e_i, e'_1, \ldots, e'_j\}\big)\Big) \cup L \cup \{v_{e_1}x_{e_1}, \ldots, v_{e_i}x_{e_i}, v_{e'_1}x_{e'_1}, \ldots, v_{e'_j}x_{e'_j}\}$$

forms a larger rainbow matching than M, which is a contradiction. $\square$

Next, we give two corollaries of the above lemma. In both of them, let N be a t-auxiliary matching for M.

Lemma 2.7. *Let $t \ge 7$ and M be a maximum rainbow matching and C_0 be the set of colours not in $col(M)$. Then, there is no $(C_0 \cup C_N)$-coloured edge disjoint from the matching $N \cup (M \setminus M_N)$ and there is no $(C_0 \cup C_N)$-rainbow horn in the matching $N \cup (M \setminus M_N)$.*

Lemma 2.8. *Let $t \ge 7$ and M be a maximum rainbow matching and C_0 be the set of colours not in $col(M)$. Then, if $v_e z$ is a $(C_0 \cup C_N)$-coloured edge with $e \in M_N$ and $z \notin V(N) \cup V(M \setminus M_N)$, then $z = m(x_e)$ or the edge is of colour $col(e)$.*

The first lemma can be easily checked by Lemma 2.6. Indeed, note that the first part is a corollary of it with $M' = \emptyset$ and the second part follows when M' consists of a single edge - the reader might want to refer to Figure 2 where the case of a rainbow horn of the form $v_f x_f$ for some $f \in M_N$ is illustrated. The second lemma, Lemma 2.8, is actually a corollary of the first - indeed, if $v_e z$ is a $((C_0 \cup C_N) \setminus \{col(e)\})$-coloured edge with $z \ne m(x_e)$ then the edges e and $v_e z$ form a $(C_0 \cup C_N)$-coloured rainbow matching and so, the edge $v_e x_e$ is a $(C_0 \cup C_N)$-rainbow horn in the matching $N \cup (M \setminus M_N)$, thus contradicting Lemma 2.7.

2.3 An outline of the proof

In the next section we prove two key technical lemmas. Then, in Section 4, we use these lemmas to prove Theorem 1.2. Here we give a brief overview of the main ideas. The first crucial thing to observe is that the Grinblat problem behaves differently when one restricts the edge-multiplicity of the (n, v)-multigraph. For example, it was shown in [16] (improving upon results from [17]) that any $(n, 2n + 2m + o(n))$-multigraph with edge-multiplicity at most m contains a rainbow matching using all the colours. This indicates that in general, lower multiplicity will make it easier to find such a rainbow matching.

Our first step towards the proof of Theorem 1.2 will be in this direction. We will show that in any (n, v)-multigraph with multiplicity at most $(1 - \delta)n$ and with v close enough to $3n$, we can always find a rainbow matching of size n (see Theorem 4.2). In order to prove this, we will first show that one can find a rainbow matching of size $n - f(\delta)$ (see Proposition 4.1), which will be a quick corollary of Lemmas 3.1 and 3.2, and then use the sampling trick introduced in [16] to transform this into a result (Theorem 4.2) giving a rainbow matching of size n.

The next step of the proof is to note that given an $(n, 3n - 2)$-multigraph G, with possibly some edges of multiplicity larger than $(1 - \delta)n$, we can assume that there exists a matching L consisting of r such edges, for some large constant r and $\delta = 1/10r$. Informally, we can find the matching L by greedily adding one edge after another, since otherwise, we will at some point be left with a graph with no edges of multiplicity larger than $(1 - \delta)n$ and thus the previously mentioned bounded multiplicity result (Theorem 4.2) will immediately guarantee the existence of a full rainbow matching. Now, notice that the graph $G' := G - V(L)$ which remains after deleting the vertices used in L is now not necessarily an $(n, 3(n - r) - 2)$-multigraph. However, by the definition of an (n, v)-multigraph, note that the set

C' of at least $(1 - r\delta)n = 0.9n$ colours which were repeated in every edge of L is such that each one of those colours still have at least $3n - 2 - 3|L| = 3(n - r) - 2$ vertices in its colour class in G'.

The final step will be to find a rainbow matching M of size $n-r$ in G' such that the colours not used in it belong to C' - after we can choose colours from C' for the edges of L to make $M \cup L$ a rainbow matching of size n in G. In fact, as C' is very large and each colour class has size much larger than $2n$, we will be able to transform any rainbow matching of size $n - r$ into one that does not use colours from C'. Hence, finding a rainbow matching of size $n - r$ in G' is the main part of the proof.

In fact, we can already sketch how Lemma 3.1 alone can be used to relatively easily establish a bound which is only by one more than Grinblat's conjecture. Indeed, suppose we started with G being an $(n, 3n - 1)$-multigraph, so that all colours in C' have at least $3(n - r) - 1$ vertices in their colour class in G'. Take a maximum rainbow matching M in G' and for the sake of contradiction, assume that $|M| < n - r$. By Lemma 3.1, there is a 7-auxiliary matching N' for M of size, say, at least $0.9n$. Then $|C_{N'} \cap C'| \geq |N'| - (n - |C'|) \geq 0.8n$. So we can restrict to an auxiliary matching $N \subseteq N'$ of size $0.8n$ such that $C_N \subset C'$. Now, by Lemma 2.7 there is no C_N-coloured edge in G' disjoint from the matching $N \cup (M \setminus M_N)$. Hence, by Lemma 2.4, we have that for each colour $c \in C_N$ there is a c-coloured matching M_c in G' of size $3(n - r) - 1 - 2|M| \geq |M| + 2$ consisting of edges going from the matching $N \cup (M \setminus M_N)$ to the outside. Also, by Lemma 2.7 there cannot exist a C_N-rainbow horn in the matching $N \cup (M \setminus M_N)$ and thus, Observation 2.2 implies that every edge in $N \cup (M \setminus M_N)$ must intersect at most $|C_N| + 1$ edges of $\bigcup_{c \in C_N} M_c$. By double-counting, we must then have $|C_N| \cdot (|M| + 2) = \sum_{c \in C_N} |M_c| \leq |N \cup (M \setminus M_N)| \cdot (|C_N| + 1) = |M| \cdot (|C_N| + 1)$ and so, $0.8n \leq |N| = |C_N| \leq |M|/2 \leq n/2$, which gives a contradiction.

In order to prove Theorem 1.2, we will need an additional observation, which consists of considering a different type of auxiliary matching.

3 Key lemmas

We are now ready to give the two main facts about auxiliary matchings that we will need. The first will essentially show that maximum rainbow matchings have large auxiliary matchings.

Lemma 3.1. *For all $\delta > 0$ and t, there exist $\delta' > 0$ and r such that the following holds for all sufficiently large n. Let G be an $(n, (3 - \delta')\, n)$-multigraph, M a maximum rainbow matching in G and suppose $|M| < n - r$. Then, there is a t-auxiliary matching for M of size at least $(1 - \delta)n$.*

Proof. Let $\delta' > 0$ be arbitrarily small as a function of t and δ and define $r := \max\left(10/\delta', 2t/\sqrt{\delta'}\right)$. Let C denote the set of at least r colours not used in M and note that by Lemma 2.7 applied with an empty auxiliary matching $N := \emptyset$, we have that there is no C-coloured edge outside the matching M and no C-rainbow horn in M. We now apply Lemma 2.3 as follows. Suppose that more than $|C|/2 \geq r/2$ colours $c \in C$ are such that M has at least $4|M|/r$ c-horns. This would be a contradiction as it would imply a C-rainbow horn in M by applying Lemma 2.3 since $(4|M|/r) \cdot (r/2) = 2|M|$. Therefore, it follows that at most half of the colours $c \in C$ are such that M has at least $4|M|/r$ c-horns. Let C_0 be the set of colours from C such that M has less than $4|M|/r$ c-horns for every $c \in C_0$. By the above discussion we have $|C_0| \geq r/2$.

Now, each $c \in C_0$ has at least $(3 - \delta')n$ vertices in its colour class and, as previously observed, there is no c-coloured edge contained in $V \setminus V(M)$. Therefore, by Lemma 2.4 (applied with $A = \emptyset$), there is a c-coloured matching $M_c \subseteq E[V \setminus V(M), V(M)]$ of size $(1 - \delta')n$. Furthermore, since M has less than $4|M|/r$ c-horns, which are the only edges in M that can intersect two edges of M_c, we have that M_c intersects at least $|M_c| - 4|M|/r \geq (1 - \delta')n - 4|M|/r \geq (1 - 2\delta')n$ edges of M. Therefore the following holds.

Claim. *There is a subset $C_0' \subseteq C_0$ of $\lfloor t/\sqrt{\delta'} \rfloor$ colours and a subset $M' \subseteq M$ of size at least $(1-2t\sqrt{\delta'})n$ such that every edge in M' intersects all matchings M_c with $c \in C_0'$.*

Proof. Let $C_0' \subseteq C_0$ be an arbitrary subset consisting of precisely $\lfloor t/\sqrt{\delta'} \rfloor$ colours. Note that this exists since $t/\sqrt{\delta'} \leq r/2 \leq |C_0|$, where the first inequality follows by the definition of r and the second follows by the arguments before. Now, recall that each matching M_c consists of edges in $E[V \setminus V(M), V(M)]$, in particular, edges which intersect edges of M. Moreover, as observed above, each M_c intersects at least $(1-2\delta')n$ edges of M and so, there are at most $|M| - (1-2\delta')n$ edges of M which do not intersect any edge of M_c. By a union bound, we can now see that there are at most $(|M| - (1-2\delta')n) \cdot |C_0'| \leq (|M| - (1-2\delta')n) \cdot (t/\sqrt{\delta'})$ edges $e \in M$ with the property that there is a colour $c \in C_0'$ such that no edge of M_c intersects e. To conclude we now define $M' \subseteq M$ to be the set of all edges of M without this property, that is, those which intersect all matchings M_c with $c \in C_0'$ as desired. This will have size at least

$$|M| - (|M| - (1-2\delta')n) \cdot (t/\sqrt{\delta'}) \geq n - (n - (1-2\delta')n) \cdot (t/\sqrt{\delta'}) = (1 - 2t\sqrt{\delta'})n,$$

where the first inequality follows since δ' is arbitrarily small and thus $t/\sqrt{\delta'} > 1$, and also $|M| \leq n$. □

Let us now define an edge $e \in M'$ to be a *bad edge* if it has an endpoint u such that there are at least ten distinct vertices $z \notin V(M)$ with $uz \in \bigcup_{c \in C_0'} M_c$. Let M'_{bad} denote the set of bad edges and note the following.

Claim. $|M'_{bad}| \leq 2\delta'|M|$.

Proof. Suppose otherwise and let $V_{\text{bad}} \subseteq V(M'_{\text{bad}})$ denote the set of endpoints witnessing the badness of the edges in M'_{bad} (that is, for each edge in M'_{bad} we include in V_{bad} an endpoint which witnesses the badness of that edge). Letting $A := m(V_{\text{bad}})$, we can check that there cannot exist a C_0-coloured edge contained in $A \cup (V \setminus V(M))$. Indeed, for the sake of contradiction, let e be such an edge and let $\{e_1, e_2\} \subseteq M'_{\text{bad}}$ be a set containing all edges of M'_{bad} which intersect e. Since e_1, e_2 are bad edges, note that there exist edges $e_1' \in E[e_1 \cap V_{\text{bad}}, V \setminus V(M)]$ and $e_2' \in E[e_2 \cap V_{\text{bad}}, V \setminus V(M)]$ such that $\{e, e_1', e_2'\}$ forms a C_0-coloured rainbow matching. In turn, then $(M \setminus \{e_1, e_2\}) \cup \{e, e_1', e_2'\}$ is a larger rainbow matching than M, which is a contradiction.

We can now use Lemma 2.4 and have that for each $c \in C_0$ (all of which have at least $(3-\delta')n$ vertices in their colour class) there is a c-coloured matching $L_c \subseteq E[V(M \setminus M'_{\text{bad}}), A \cup (V \setminus V(M))]$ of size at least $(1-\delta')n \geq (1-\delta')|M|$. Since by assumption, $|M \setminus M'_{\text{bad}}| = |M| - |M'_{\text{bad}}| < (1-2\delta')|M|$, we have that each $c \in C_0$ is such that there are at least $(1-\delta')|M| - (1-2\delta')|M| = \delta'|M| > 2|M|/|C_0|$ edges $e \in M \setminus M'_{\text{bad}}$ which intersect two edges of L_c (the previous inequality followed since by definition of r we have $\delta' \geq 10/r$ and further, recall that $|C_0| \geq r/2$). Since clearly $(2|M|/|C_0|) \cdot |C_0| = 2|M| \geq 2|M \setminus M'_{\text{bad}}|$, it must be then that there exists an edge $e \in M \setminus M'_{\text{bad}}$ for which there are at least three colours $c \in C_0$ such that L_c intersects e in two edges. We claim that this also contradicts the maximality of M. Indeed, by using a similar argument as for Observation 2.2, one can note that there must exist a C_0-coloured rainbow matching of two edges going from the edge e to $A \cup (V \setminus V(M))$. In turn, just like in the last paragraph, one can check that this contradicts the maximality of M.

□

Given the above claim, let us now delete the bad edges from M' so that we still have $|M'| \geq \left(1 - 2t\sqrt{\delta'}\right) n - 2\delta'|M| \geq \left(1 - 4t\sqrt{\delta'}\right) n$. To finish, we make a final observation implied by the definition of a bad edge. Recall first that the set $C_0' \subseteq C_0$ is by definition such that each edge of M' intersects all matchings M_c with $c \in C_0'$. Therefore, since we have deleted all bad edges from M' the following holds.

Claim. *For each edge $e \in M'$, there are at most $20t$ colours $c \in C_0'$ such that there is an edge $uz \in M_c$ with $u \in e$, $z \notin V(M)$ and such that uz is repeated in at most t colours in C_0'.*

Proof. Let $e \in M'$ have endpoints u, v and note that since it is not a bad edge, there must be at most 10 vertices $z \notin V(M)$ such that $uz \in \bigcup_{c \in C_0'} M_c$, and similarly for v. Let now $Z_u \subseteq V \setminus V(M)$ denote the set of vertices $z \notin V(M)$ such that $uz \in \bigcup_{c \in C_0'} M_c$ and it is repeated in at most t colours of C_0' and similarly define Z_v. By before, we must have that $|Z_u|, |Z_v| \leq 10$ and moreover, note that then in total, at most $(|Z_u| + |Z_v|) \cdot t \leq 20t$ C_0'-coloured edges are of the form uz for some $z \in Z_u$ or vz for some $z \in Z_v$. Concluding, there are at most $20t$ colours in C_0' with the property described in the statement of the claim. $\square$

Now, for each such edge $e \in M'$, let $C_e \subseteq C_0'$ denote the set of at most $20t$ colours given by the claim. There must then exist a colour $c \in C_0'$ which belongs to at most

$$\frac{1}{|C_0'|} \cdot \sum_{e \in M'} |C_e| \leq \frac{1}{|C_0'|} \cdot 20t|M'| = \frac{1}{\lfloor t/\sqrt{\delta'} \rfloor} \cdot 20t|M'| \leq 40\sqrt{\delta'} \cdot |M'|$$

sets C_e. Finally, note that the matching N formed by the edges of M_c which intersect those edges $e \in M'$ such that $c \notin C_e$ forms a t-auxiliary matching since each one of its edges is repeated in more than t colours of C_0'. In turn, N is of size at least

$$\left(1 - 40\sqrt{\delta'}\right)|M'| \geq \left(1 - 40\sqrt{\delta'}\right)\left(1 - 4t\sqrt{\delta'}\right)n \geq (1-\delta)n,$$

as desired, since δ' was chosen to be arbitrarily small in terms of δ and t. $\square$

Next, given a t-auxiliary matching N for a rainbow matching M, define the subset $N_\alpha \subseteq N$ to be the set of edges $v_e x_e \in N$ such that the edge $v_e m(x_e)$ is repeated in at most $\alpha|C_N|$ colours of C_N, and define M_{N_α} to be the set of edges $e \in M_N$ such that $v_e x_e \in N_\alpha$. The lemma below will tell us that these sets cannot be very large.

Lemma 3.2. *For every $\gamma > 0$ and $1 > \alpha > 0$, the following holds for all sufficiently large n. Let G be an $(n, (3-\gamma)n)$-multigraph and M a maximum rainbow matching in G with a 10-auxiliary matching N of size at least $(1-\gamma)|M|$. Then, $|N_\alpha| \leq \left(\frac{20\gamma}{1-\alpha}\right) n$.*

Proof. For the sake of contradiction, suppose otherwise. First, for each colour $c \in C_N$, delete the c-coloured edges which intersect the vertex v_e where $e \in M_N$ is such that $col(e) = c$. Secondly, delete those c-coloured edges which intersect $V(M \setminus M_N)$. Note that after these deletions, we can delete some final c-edges so that c is now a disjoint union of non-trivial cliques with in total at least $(3-\gamma)n - 3 - 3 \cdot |V(M \setminus M_N)| \geq (3-\gamma)n - 3 - 6\gamma|M| \geq (3-8\gamma)n$ vertices, since n is sufficiently large. In turn, because of the first deletion process and the maximality of M, we now have the following.

Claim. *There is no C_N-coloured edge contained in $V \setminus V(M)$.*

Proof. Suppose otherwise and let e be such an edge. Let also $M' \subseteq M_N$ be the subset of at most two edges $f \in M_N$ such that v_f intersects e. Then, since the first deletion process implies that none of these edges is the edge in M_N of the same colour as e, we have that $M' \cup e$ forms a rainbow matching of size $|M'| + 1$. This in turn contradicts Lemma 2.6 applied with $t = 10$. $\square$

Observe also that the deletion processes above can only increase the size of N_α. Now, let us define $A := \{m(x_e) : e \in M_N\}$ and note that we can claim some further assertions about the appearance of C_N-coloured edges using Lemmas 2.7 and 2.8.

Claim. *There is no C_N-coloured edge contained in A and for any C_N-coloured edge of the form $m(x_f)y$ for $y \notin V(M), f \in M_N$ we have that $y = v_f$.*

Proof. The first assertion follows by directly applying Lemma 2.7. For the second, suppose for the sake of contradiction that $m(x_f)y$ is a C_N-coloured edge with $y \notin V(M)$ and $y \neq v_f$. Let us first look at the case that $y = v_e$ for some $e \in M_N \setminus \{f\}$. Here, Lemma 2.8 directly implies that $m(x_f)v_e$ must have colour $col(e)$, which cannot occur since if such an edge existed, it was deleted in the first deletion process at the beginning of the proof. Suppose now the other case that $y \notin V(N)$. Then, $m(x_f)y$ is disjoint from the matching $N \cup (M \setminus M_N)$, which directly contradicts Lemma 2.7. $\square$

Now, a consequence of the second assertion of the claim above is that, because of the definition of N_α, for each $e \in M_{N_\alpha}$, there are at most $\alpha|C_N|$ C_N-coloured edges of the form $m(x_e)y$ with $y \notin V(M)$. A simple counting argument then gives the following.

Claim. *There exist at least $(1-\alpha)|C_N|/2$ colours $c \in C_N$ for which there exists a subset $A_c \subseteq A$ of size at least $(1-\alpha)|N_\alpha|/2$ such that there are no c-coloured edges contained in $A_c \cup (V \setminus V(M))$.*

Proof. For each edge $e \in M_{N_\alpha}$, let $C_e \subseteq C_N$ denote the set of at least $(1-\alpha)|C_N|$ colours which do not appear in the edge $v_e m(x_e)$. Let $C'_N \subseteq C_N$ denote the set of colours which belong to at least $(1-\alpha)|N_\alpha|/2$ sets C_e. Then, we must have

$$(1-\alpha)|C_N||N_\alpha| \leq \sum_{e \in M_{N_\alpha}} |C_e| \leq |C_N \setminus C'_N| \cdot (1-\alpha)|N_\alpha|/2 + |C'_N| \cdot |N_\alpha|,$$

since every colour in $C_N \setminus C'_N$ appears, by definition, in at most $(1-\alpha)|N_\alpha|/2$ sets C_e and every colour in C'_N can appear in at most all sets C_e, of which we have $|N_\alpha|$ many. Note thus that since $|C_N \setminus C'_N| \leq |C_N|$, the above inequality implies that $|C'_N| \geq (1-\alpha)|C_N|/2$.

Now, note that from the two previous claims, we have that for each $c \in C_N$, the only possible c-coloured edges which are contained in $A \cup (V \setminus V(M))$ must be of the form $v_e m(x_e)$ for some $e \in M_N$. Therefore, letting $A_c \subseteq A$ denote the set of vertices $m(x_e)$ with $e \in M_{N_\alpha}$ and $c \in C_e$, we have that there is no c-coloured edge contained in $A_c \cup (V \setminus V(M))$. We are then done since $|A_c| \geq (1-\alpha)|N_\alpha|/2$ for each $c \in C'_N$, by definition, and $|C'_N| \geq (1-\alpha)|C_N|/2$. $\square$

Let $C'_N \subseteq C_N$ denote the set of colours given by the above claim. For each $c \in C'_N$, by Lemma 2.4 applied to A_c and recalling that in the second deletion process at the start of the proof we deleted all c-edges intersecting $V(M \setminus M_N)$, we have that there exists a c-coloured matching $L_c \subseteq E[V(M_N) \setminus (A_c \cup m(A_c)), A_c \cup (V \setminus V(M))]$ of size $(1-8\gamma)n$. Since $|A_c| \geq (1-\alpha)|N_\alpha|/2 \geq 10\gamma n$ (recall that we are assuming that $|N_\alpha| > \left(\frac{20\gamma}{1-\alpha}\right) n$), there are at least $(1-8\gamma)n-(|M_N|-|A_c|) \geq (1-8\gamma)n-(n-10\gamma n) \geq 2\gamma n$ edges of M_N which intersect two edges of L_c. Moreover, since $|C'_N| \geq (1-\alpha)|C_N|/2 \geq (1-\alpha)|N_\alpha|/2 \geq 10\gamma n$ and so, $|C'_N| \cdot \binom{2\gamma n}{45} \geq 10\gamma n \binom{2\gamma n}{45} > 3\binom{n}{45}$ (because n is sufficiently large), there exist three colours $c_!, c_2, c_3 \in C'_N$ such that there are at least 45 edges in M_N each intersecting two edges of each L_{c_i}. This in turn implies the following.

Claim. *There exist two colours $c, c' \in C'_N$ for which there are at least 15 edges $e \in M_N$ which intersect an edge of L_{c_1} and an edge of L_{c_2} which are vertex-disjoint.*

Proof. Take the three colours c_1, c_2, c_3 given just before the statement of the claim, with the property that there are at least 45 edges $e \in M_N$ each intersecting two edges of each L_{c_i}. By using the simple fact that a family of three matchings of size two contains a rainbow matching of size two, each such edge e is such that for some two colours $c, c' \in \{c_1, c_2, c_3\}$ there are two disjoint edges $f \in L_c$ and $f' \in L_{c'}$ which intersect e. To achieve the desired statement, take now c, c' to be the pair of colours for which the most number of edges e have this property, which will be at least 15. $\square$

To finish the proof of the lemma, we will show that the claim above contradicts the maximality of M. Indeed, let $e_c, e_{c'} \in M_N$ be the edges in M of colours c, c' and let $S := e_c \cup e_{c'} \cup \{v_{e_c}, v_{e_{c'}}\}$. By the claim above, there exists an edge $e \in M_N$ matched to $A \cup (V \setminus V(M))$ by edges $f \in L_c$ and $f' \in L_{c'}$ such that f, f' are disjoint from S. This is the case since each L_c, L'_c are matchings, and thus, there exist at most $2|S| \leq 12$ edges of $L_c \cup L'_c$ intersecting S; therefore, since the claim above gives us more than 12 edges in M_N with the given property (along with edges $f \in L_c$ and $f' \in L_{c'}$), such an edge e must exist. Now, suppose without loss of generality that f is the edge with $m(x_e)$ as one of its endpoints. Note that its other endpoint must be a vertex $v_{e'}$ for some $e' \in M_N \setminus \{e_c, e_{c'}\}$, as otherwise this edge, which is C_N-coloured, would be contained outside $V(N) \cup V(M \setminus M_N)$ and thus contradict Lemma 2.7. Furthermore, if $e' \neq e$, then the edges e' and f form a rainbow matching and thus, $x_{e'}v_{e'}$ is a C_N-rainbow horn in the matching $N \cup (M \setminus M_N)$, which also contradicts the Lemma 2.7. Therefore, $f = v_e m(x_e)$. Now, let $f' = x_e z$. By the definition of the matching $L_{c'}$, we have $z \in A \cup (V \setminus V(M))$. Also note that if $z \in A$ or $z \in V \setminus (V(M) \cup V(N))$, we have that $v_e x_e$ is another C_N-rainbow horn in the matching $N \cup (M \setminus M_N)$ which contradicts Lemma 2.7. Hence, much like in the argument before, we have that $z = v_{e'}$ for some $e' \in M_N \setminus \{e_c, e_{c'}\}$. Finally, consider $M' = \{e, e'\} \subset M_N$ and note that the rainbow matching $\{f, f', e'\}$, whose edges are fully contained in the set of vertices $e \cup e' \cup \{v_e, v_{e'}\}$, contradicts Lemma 2.6. $\square$

To finish the section, we conclude with the following quick corollary of Lemmas 3.1 and 3.2.

Corollary 3.3. *For all $\delta > 0$ and $t \geq 10$, there exist $\delta' > 0$ and r such that the following holds for all sufficiently large n. Let G be an $(n, (3 - \delta')\, n)$-multigraph, M a maximum rainbow matching in G and suppose that $|M| < n - r$. Then, there exists a t-auxiliary matching N for M of size at least $(1 - \delta)n$ with $N_{1/4} = \emptyset$.*

Proof. Let us assume without loss of generality that $\delta < 1/5$ and let $\delta_0 = \delta/41$ and note that from Lemmas 3.1 and 3.2, there are δ' and r which imply the existence of a t-auxiliary matching N' for M of size at least $(1 - \delta_0)n$ with $|N'_{1/2}| \leq 40\delta_0 n$. Let $N := N' \setminus N'_{1/2}$ which is also a t-auxiliary matching, and of size at least $(1 - 41\delta_0)n = (1 - \delta)n$. Then, for each edge $e \in M_N$, which is then not in $M_{N'_{1/2}}$, we have, since $|C_{N'} \setminus C_N| = |N'_{1/2}|$, that the pair $m(x_e)v_e$ is repeated in at least $|N'|/2 - |N'_{1/2}| > (1/2 - \delta_0/2 - 40\delta_0)n > n/4$ C_N-colours. Therefore, $N_{1/4} = \emptyset$. $\square$

4 Proof of Theorem 1.2

As discussed in the proof outline, we will first need to prove a result concerning (n, v)-multigraphs with bounded edge-multiplicity.

Proposition 4.1. *For all $\delta > 0$ there exists $\delta' > 0$ and r such that for all sufficiently large n, every $(n, (3 - \delta')n)$-multigraph with multiplicity at most $(1 - \delta)n$ contains a rainbow matching of size $n - r$.*

Proof. Without loss of generality, let us assume that $\delta < 1/100$. Let G be an $(n, (3 - \delta')n)$-multigraph with multiplicity at most $(1 - \delta)n$ and, for the sake of contradiction, let M be a maximum rainbow matching of size at most $n - r$. By Lemma 3.1, there exists a sufficiently small δ' and a sufficiently large r which implies the existence of a 10-auxiliary matching N for M of size at least $(1 - \delta^2)n$. Additionally, by Lemma 3.2 (with $\gamma = \delta^2$), we have that $|N_{1-\delta/2}| \leq 40\delta n$. In turn, the multiplicity condition on G implies that for each $e \in M_N$, there exist at most $(1-\delta)n \leq (1-\delta/2)(1-\delta^2)n \leq (1-\delta/2)|C_N|$ C_N-colours repeated in the edge $v_e m(x_e)$. Thus, $N = N_{1-\delta/2}$, which leads to a contradiction since then $(1-\delta^2)n \leq |N| \leq 40\delta n$. □

We can now use the sampling trick introduced in [16] to transform the proposition above into the following theorem.

Theorem 4.2. *For all $\delta > 0$ there exists $\delta' > 0$ such that for all sufficiently large n, every $(n,(3-\delta')n)$-multigraph with multiplicity at most $(1-\delta)n$ contains a rainbow matching of size n.*

Proof. Let G be an $(n,(3-\delta')n)$-multigraph with multiplicity at most $(1-\delta)n$. Recall that by the remark given just before Section 2.1, we can assume that G is such that every colour is a disjoint union of K_3's and K_2's. Then, for each colour c, let t_c denote the number of triangles in its colour class and l_c the number of edges, so that $3t_c + 2l_c \geq (3-\delta')n$. Let $S \subseteq V(G)$ be a random set obtained by choosing each vertex independently with probability $p = 2n^{-1/4}$.

For each colour c, let $e_c(S)$ be the random variable counting the number of c-coloured edges contained in S. This is a sum of independent $[0,3]$-valued random variables, since each colour is a disjoint union of cliques of size at most 3 and each clique can contribute $0, 1$ or 3 edges. Moreover the contribution from disjoint cliques are independent random variables. Since every component of colour c contributes an edge with probability at least p^2, we have $\mathbb{E}[e_c(S)] \geq p^2(t_c + l_c) \geq p^2 n/2 = 2\sqrt{n}$. Thus, by Lemma 2.5, we have $\mathbb{P}(X_c \leq \sqrt{n}) \leq e^{-\Omega(\sqrt{n})} = o(n^{-1})$, so by a union bound, we have that with probability $1 - o(1)$, all $e_c(S) > \sqrt{n}$. Also for each colour c, let $v_c(S)$ denote the number of vertices in the initial colour class of c (that is, those incident to edges of colour c in G) which belong to S. Again using Lemma 2.5 it is easy to note that $\mathbb{P}\left(v_c(S) \geq 2(3-\delta')np\right) \leq e^{-\Omega_{\delta'}(n^{3/4})} = o(n^{-1})$. Moreover, since every vertex counted in $v_c(S)$ belongs to one clique of the original colour class, by deleting S we might destroy at most $v_c(S)$ such cliques which cover at most $3v_c(S)$ vertices. Therefore, by union bound, with probability $1 - o(1)$, every colour class in $G - S$ is a disjoint union of non-trivial cliques covering in total at least $(1-6p)(3-\delta')n \geq (3-2\delta')n$ vertices.

Now, fix a subset S satisfying all the conditions discussed above. Then, provided that δ' is sufficiently small, Proposition 4.1 implies that for some constant r there is a rainbow matching M in $G - S$ of size at least $n - r$ and every colour has at least $\sqrt{n}$ edges in $G[S]$. Let C_0 denote the set of colours not used in M. Since each colour class in C_0 has maximum degree two and at least $\sqrt{n} \gg 4r = 4|C_0| = 2 \cdot 2 \cdot |C_0|$ edges in $G[S]$, we can greedily find a rainbow matching $N \subseteq G[S]$ which uses all colours in C_0. As a result, $M \cup N$ is a full rainbow matching in G. □

Now that we have the desired bounded multiplicity result, we will next show that we can indeed find a rainbow matching of size $n - r$ in the situation discussed in the end of the proof outline.

Lemma 4.3. *Let r be sufficiently large, n sufficiently large in terms of r and G an $(n, 3n-10r)$-multigraph such that there are at least $0.9n$ colours with at least $3(n-r)-2$ vertices in its colour class. Then, G contains a rainbow matching of size $n - r$.*

Proof. Let C' be the set of at least $0.9n$ colours mentioned in the statement, let M be a maximum rainbow matching in G and for the sake of contradiction, suppose that $|M| \leq n - r - 1$. By Corollary 3.3, we know that provided that r is sufficiently large and n is sufficiently large in terms of r, there is a 13-auxiliary matching N for M of size at least $0.9n$ such that $N_{1/4} = \emptyset$. By deleting at most $0.1n$ edges from M_N whose colours are not in C', we can further redefine N so that $C_N \subseteq C'$ and still $|N| \geq 0.8n > 2n/3$. Then, recalling the definition of $N_{1/4}$ and taking into account the previous modification made to N, we now have that for every $e \in M_N$, the edge $v_e m(x_e)$ is repeated in at least $(1/4) \cdot 0.9n - 0.1n \geq 0.1n$ colours in C_N. Now, define the matching $L := \{v_e m(x_e) : e \in M_N\} \cup (M \setminus M_N)$ and note the following (the reader might want to refer to Figures 3 and 4 for an illustration).

Claim. *There is no C_N-coloured edge contained in $V \setminus V(L)$ and L contains no C_N-rainbow horn.*

Proof. For the first part, let e' be a C_N-coloured edge contained in $V \setminus V(L)$ and let $e_1, \dots, e_i$ denote the edges of M_N which intersect e'. Since we observed before that each of the edges $v_e m(x_e)$ (for $e \in M_N$) is repeated in at least $0.1n \geq 3$ colours of C_N, and since $i \leq 2$, we can pick distinct colours in $C_N \setminus \{col(e')\}$ for the edges $v_{e_1} m(x_{e_1}), \dots, v_{e_i} m(x_{e_i})$ so that $\{e', v_{e_1} m(x_{e_1}), \dots, v_{e_i} m(x_{e_i})\}$ is a rainbow matching of size $i+1$ contradicting Lemma 2.6 with $M' = \{e_1, \dots, e_i\} \subseteq M$.

For the second part, suppose first that some edge $e \in M_N$ is such that $v_e m(x_e)$ is a C_N-rainbow horn in L. That is, there exist distinct vertices $z_1, z_2 \notin V(L)$ such that the edges $v_e z_1$ and $m(x_e) z_2$ are distinctly coloured in C_N. Let c_1, c_2 denote these colours (painted as green and blue in Figure 4). Let also $e_1, \dots, e_i$ denote the edges of M_N which intersect $\{z_1, z_2\}$. Much like before, since $0.1n \geq 4$ and $i \leq 2$, we can pick distinct colours in $C_N \setminus \{c_1, c_2\}$ for the edges $v_{e_1} m(x_{e_1}), \dots, v_{e_i} m(x_{e_i})$ so that $\{v_e z_1, m(x_e) z_2, v_{e_1} m(x_{e_1}), \dots, v_{e_i} m(x_{e_i})\}$ is a rainbow matching of size $i+2$ disjoint from the vertices in $V(M \setminus \{e, e_1, \dots, e_i\}) \cup \{v_f : f \in M_N \setminus \{e, e_1, \dots, e_i\}\} \cup \{x_f : f \in M_N \setminus \{e, e_1, \dots, e_i\}\}$. Since N is a 13-auxiliary matching, this contradicts Lemma 2.6 for $M' = \{e, e_1, \dots, e_i\} \subset M$ and $t = 13$.

A similar analysis can be done when the C_N-rainbow horn in L is an edge in $M \setminus M_N$. Indeed, suppose for some $e = x m(x) \in M \setminus M_N$ there exist distinct vertices $z_1, z_2 \notin V(L)$ such that the edges $x z_1$ and $m(x) z_2$ are distinctly coloured in C_N. Let c_1, c_2 denote these colours. Let also $e_1, \dots, e_i$ denote the edges of M_N which intersect $\{z_1, z_2\}$ and just like before, since $0.1n \geq 4$ and $i \leq 2$, note that we can pick distinct colours in $C_N \setminus \{c_1, c_2\}$ for the edges $v_{e_1} m(x_{e_1}), \dots, v_{e_i} m(x_{e_i})$ so that $\{x z_1, m(x) z_2, v_{e_1} m(x_{e_1}), \dots, v_{e_i} m(x_{e_i})\}$ is a rainbow matching of size $i+2$. Since N is a 13-auxiliary matching, this contradicts Lemma 2.6 for $M' = \{e, e_1, \dots, e_i\} \subset M$ and $t = 13$. $\square$

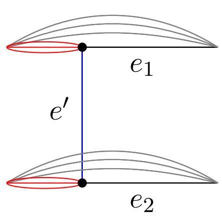

Figure 3: A C_N-coloured edge outside of L.

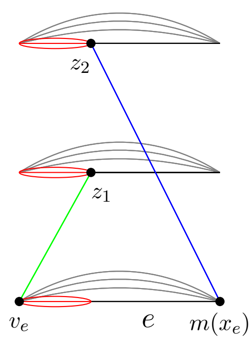

Figure 4: A C_N-rainbow horn in L.

Let now $c \in C_N \subseteq C'$. Since it has at least $3(n-r) - 2 \geq 3|L| + 1$ vertices in its colour class, by the first part of the claim above and Lemma 2.4 with $A = \emptyset$, there must exist a c-coloured matching $L_c \subseteq E[V(L), V \setminus V(L)]$ of size $|L|+1$. Also, using the claim above and Observation 2.2, any edge $e' \in L$ is a c-horn (in L) for at most two colours $c \in C_N$, and thus, in general, every edge in L intersects at most $|C_N| + 2$ edges belonging to $\bigcup_{c \in C_N} L_c$. Furthermore, suppose that $e' \in L$ is of the form $e' = v_e m(x_e)$, for some $e \in M_N$. If it is a c-horn (in L) for some $c \in C_N$, then recall that e' is repeated in at least $n/10 - 1 > n/20$ colours $c' \in C_N \setminus \{c\}$. Since each such colour c' is a disjoint union of non-trivial cliques, no edge of $L_{c'}$ can intersect e', otherwise this would imply that e' is a C_N-rainbow horn (with the two implicit colours being c, c'), which contradicts the above claim. Hence, the edge e' intersects at most $2 + |C_N| - n/20 \leq |C_N|$ edges from $\bigcup_{c \in C_N} L_c$. Trivially, if e' is not a c-horn for any $c \in C_N$, this also holds. Therefore, by double-counting we must have

$$|C_N|(|M|+1) = |C_N|(|L|+1) = \sum_{c \in C_N} |L_c| \leq |M_N| \cdot |C_N| + |M \setminus M_N| \cdot (|C_N|+2)$$

implying that $|C_N| \leq 2|M \setminus M_N| \leq 2n - 2|C_N|$, which is a contradiction since $|C_N| = |N| > 2n/3$. □

Following the ideas briefly discussed in the outline of the proof, we can now establish our main result.

Proof of Theorem 1.2. Let G be an $(n, 3n-2)$-multigraph. Let r be sufficiently large and n sufficiently large in terms of r, so that in particular, Lemma 4.3 holds. Note first that we can apply Theorem 4.2 with $\delta = 1/10r$ to find a rainbow matching of size n in G or find a matching L of size r such that each edge in it is repeated in at least $(1-\delta)n$ colours. Indeed, suppose that L is a maximum such matching and assume that $|L| < r$. Then, $G \setminus V(L)$ is an $(n, 3n-2-6|L|)$-multigraph with edge-multiplicity at most $(1-\delta)n$. Since $3n-2-6|L| = 3n - O_r(1)$ and n is sufficiently large in terms of r, Theorem 4.2 implies that there is indeed a rainbow matching of size n in $G \setminus V(L) \subseteq G$. We will now consider the case that such a matching L exists. Let C' be the set of colours which are repeated in all edges of L and note that by a union bound, we have $|C'| \geq n - r \cdot \delta n = 9n/10$. Since each colour class in G is a disjoint union of non-trivial cliques with in total $3n-2$ vertices, $G' := G - V(L)$ is now an $(n, 3n-10r)$-multigraph such that every colour in C' has at least $3(n-r)-2$ vertices in its colour class. Therefore, Lemma 4.3 implies that G' has a rainbow matching of size at least $n-r$.

In order to finish the proof, we only need to show that there exists a maximum rainbow matching M in G' such that the colours not used in it are contained in C'. Indeed, if this is the case, since $|M| \geq n-r$, we can then pick the colours of the edges of L, so that $M \cup L$ contains a rainbow matching of size n, and we are done. Take then a maximum rainbow matching M' in G' and suppose there is some colour $c \notin C'$ which is not used in M'. Since M' is maximum, there is no c-coloured edge in G' which is contained outside M' and thus, by Lemma 2.4, there is a c-coloured matching in $E[V(M'), V(G') \setminus V(M')]$ of size $3n - 10r - 2|M'| \geq n - 10r > 2n/3$. In particular, this matching then intersects at least $n/3$ edges of M' - and for each such edge, note we can switch it with the edge in the c-coloured matching which touches it. Thus, there exist at least $n/3$ colours c' which are used in M' for which there is a maximum rainbow matching M'' in G' such that $col(M'') = (col(M') \setminus \{c'\}) \cup \{c\}$. As $|C'| \geq 9n/10$, there is always such a colour c' which also belongs to C'. We can now take M'' and repeat the same operation until we have a maximum rainbow matching M in G' such that all the colours not used in it belong to C'. □

5 Concluding remarks

Although we have resolved the Grinblat problem for all sufficiently large n, there are still some related open questions which could be of interest. Firstly, one might want to understand what happens for small $n \geq 4$. Our proof yields the validity of Grinblat's conjecture starting with moderately large n, but it seems plausible that some of our ideas could help in this direction. Secondly, one can consider stability-type questions: Is the example consisting of a disjoint union of $n-1$ triangles repeated in each of the n colours the only $(n, 3n-3)$-multigraph without a rainbow matching of size n? How close to this example are all (n, v)-multigraphs with no rainbow matching of size n and v close to $3n$?

Finally, another interesting direction is to gain a better understanding of how Grinblat's problem varies with edge-multiplicity restrictions. Precisely, one would look to answer the following question.

Given n and $m \leq n$, what is the minimal v_m such that every (n, v_m)-multigraph with edge-multiplicity at most m contains a rainbow matching of size n? The difficulty of determining or estimating the value v_m will vary with m. For example, note that Theorem 1.2 states that $v_n = 3n-2$, whereas determining v_1 seems more difficult. The first author together with Yepremyan [17] showed that $v_1 = 2n+o(n)$. This might suggest that $v_1 = 2n+c$ for some constant c. However, recall that the famous Ryser-Brualdi-Stein conjecture can be formulated as stating that every properly n-edge-coloured $K_{n,n}$ contains a rainbow matching of size $n-1$. Let G be the graph formed by taking a disjoint union of $c/2$ stars of size n, each having edges of all the n colours. Note that the graph formed by the disjoint union of G and a properly n-edge-coloured $K_{n,n}$ is an $(n, 2n+c)$-multigraph with multiplicity at most 1. Therefore, if one proves that $v_1 = 2n+c$, then there exists a rainbow matching of size n in this graph. This will imply that $K_{n,n}$ contains a rainbow matching of size $n-c/2$, which would greatly improve on the best known bound for the Ryser-Brualdi-Stein conjecture from [12].

Recently, the two authors together with Pokrovskiy [16] showed that $v_m \leq 2n+2m+O(n/(\log n)^{1/4})$. This substantially improves the result in [17] and is asymptotically tight for $m = \varepsilon n$ and $\log^{-1/4} n \ll \varepsilon \ll 1$. Indeed, one can construct examples (see [16]) of $(n, 2n+2\varepsilon n - O(\varepsilon^2 n))$-multigraphs with edge-multiplicity at most εn and no rainbow matching of size n. However, this result does not give much information about what occurs when m is very small, nor when m is close to n.

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