(n, d, λ)-graphs: properties and applications

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What makes a graph random?

Questions:

- What are the essential properties of random graphs?
- How can one tell when a given graph behaves like a random graph?
- How to create deterministically graphs that look random-like?

A possible answer:

Probably the most important characteristic of truly random graph is its *edge distribution*. Thus may be a pseudo-random graph is a graph whose edge distribution resembles the one of a random graph with the same edge density.

Spectra of graphs

Notation:

The adjacency matrix A_G of a graph G has

$$a_{uv} = \text{number of edges from } u \text{ to } v.$$

It is a symmetric matrix with real eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.

Definition:

G is an (n, d, λ)-graph if it is d-regular, has n vertices, and

$$\max_{i \geq 2} |\lambda_i| \leq \lambda.$$

Remark:

- If G is d-regular, then $\lambda_1 = d$.
- If $d \leq n/2$ and G is (n, d, λ), then $\lambda \geq \sqrt{\frac{d(n-d)}{n-1}} = \Omega(\sqrt{d})$.

Edge distribution

Notation:

Let G be an (n, d, λ)-graph. For $B, C \subseteq V(G)$

$$e(B, C) = \left| \{(b, c) \in E(G) \mid b \in B, c \in C\} \right|$$

$$e(B) = \frac{1}{2} e(B, B) = \left| \{(b, b') \in E(G) \mid b, b' \in B\} \right|$$

Theorem: *(Alon, Alon-Chung 80's)*

- For any $B, C \subseteq V(G)$ (not necessarily disjoint)

$$\left| e(B, C) - \frac{d}{n} |B||C| \right| \leq \lambda \sqrt{|B||C|},$$

- For any $B \subseteq V(G)$

$$\left| e(B) - \frac{d}{n} \frac{|B|^2}{2} \right| \leq \frac{1}{2} \lambda |B| \left(1 - \frac{|B|}{n}\right).$$

Independence number and MaxCut

Corollary: (*Hoffman*)

The independence number of an (n, d, λ)-graph G is at most

$$\alpha(G) \leq \frac{\lambda}{d+\lambda} n$$

Corollary:

The maximum number of edges in a cut of G

$$\mathrm{MaxCut}(G) \leq \frac{d+\lambda}{4} n = \frac{e(G)}{2} + \frac{\lambda n}{4}.$$

Vertex expansion

Definition:

The vertex boundary of $X \subset V(G)$ in a graph G is

$$\partial X = \{ y \in V(G) \backslash X \mid \exists x \in X : \{x, y\} \in E(G) \}.$$

Corollary: (*Alon-Milman 84, Tanner 84*)

If G is an (n, d, λ)-graph G and $X \subset V(G)$ of size at most $n/2$, then

$$|\partial X| \geq \frac{2(d - \lambda)}{3d - 2\lambda} |X|.$$

Converse results

Theorem: (*Alon 1986*)

If G is d-regular graph with eigenvalues $\lambda_1 = d \geq \lambda_2 \geq \ldots \geq \lambda_n$ such that $|\partial X| \geq c|X|$ for every $X \subset V, |X| \leq n/2$, then

$$\lambda_2 \leq d - \frac{c^2}{4 + 2c^2}.$$

Theorem: (*Bilu and Linial 2004*)

If $G = (V, E)$ is d-regular graph with eigenvalues $\lambda_1 = d \geq \lambda_2 \geq \ldots \geq \lambda_n$ such that for every $B, C \subset V$

$$\left| e(B, C) - \frac{d}{n}|B||C| \right| \leq \alpha \sqrt{|B||C|},$$

then $\max\left\{|\lambda_2|, |\lambda_n|\right\} \leq O\big(\alpha \log(d/\alpha)\big)$.

Chromatic number

Definition:

Chromatic number $\chi(G)$ is the minimum number of colors needed to color $V(G)$ such that adjacent vertices get different colors.

Theorem: (*Hoffman*)

If G is and (n, d, λ)-graph then $\chi(G) \geq 1 + \frac{d}{\lambda}$.

Theorem: (*Alon, Krivelevich and S. 99*)

If G is (n, d, λ) and $d \leq 2n/3$ then $\chi(G) \leq O\left(\frac{d}{\log(1+d/\lambda)}\right)$.

Theorem: (*Alon, Krivelevich and S. 99 and Vu 00*)

The choice number of G satisfies a similar inequality.

Hamiltonicity

Definition:

Graph G is hamiltonian if it has Hamilton cycle, i.e., a cycle containing all vertices of G.

Theorem: (*Krivelevich and S. 02*)

If G is and (n, d, λ)-graph with

$$\lambda < \frac{d}{\log n},$$

then G is hamiltonian.

Conjecture:

There exist an $\epsilon > 0$ such that if $\lambda < \epsilon\, d$ then G is hamiltonian.

Small subgraphs

Setting:

- $H =$ fixed graph with s vertices, r edges and max. degree Δ.
- $G = (V, E)$ is an (n, d, λ)-graph and $U \subseteq V$ of size m.

Theorem: *(Alon)*

If $m \gg \lambda \left(\frac{n}{d}\right)^{\Delta}$ then U contains

$$(1 + o(1)) \frac{s!}{|Aut(H)|} \binom{m}{s} \left(\frac{d}{n}\right)^{r}$$

copies of H

Remark:

If $d^r \gg \lambda n^{r-1}$ then G contains a complete graph K_{r+1}.

Spectral Turán's theorem

Question:

How large can be K_{r+1}-free subgraph of (n, d, λ)-graph?

(*Every G has such subgraph with at least $\frac{r-1}{r}e(G)$ edges.*)

Theorem: (*S., Szabó, Vu 2005*)

Let $r \geq 2$, and let G be an (n, d, λ)-graph with $d^r \gg \lambda n^{r-1}$. Then the size of the largest K_{r+1}-free subgraph of G is

$$\frac{r-1}{r}e(G) + o\big(e(G)\big).$$

Remarks:

- The complete graph K_n has $d = n - 1$ and $\lambda = 1$. Thus we have an asymptotic extension of Turán's theorem.
- The theorem is tight for $r = 2$. By a result of Alon, there are (n, d, λ)-graphs with $d^2 = \Theta(\lambda n)$ which contain no triangles.

Examples of (n, d, λ)-graphs

Friedman 03:

For every fixed $\epsilon > 0$ and $d \geq 3$, a random d-regular graph on n vertices is, asymptotically almost surely, an (n, d, λ)-graph with
$$\lambda = 2\sqrt{d-1} + \epsilon.$$

Paley graph:

- $V(G) = \mathbb{Z}_p$, where p is a prime $p = 1 (\bmod 4)$.
- $(i, j) \in E(G)$ iff $i - j = r^2 (\bmod p)$ is a quadratic residue.

G is an (n, d, λ)-graph with

$$n = p, \quad d = \frac{p-1}{2}, \quad \lambda = \frac{1+\sqrt{p}}{2}.$$

Examples of (n, d, λ)-graphs

Erdős-Rényi graph:

G is polarity graph of lines-point incidence graph of finite projective plane of order q.

- $V(G) =$ lines through the origin in $\mathbb{F}_q^3$, q is a prime power.
- Two lines are adjacent if they orthogonal.

G has no 4-cycles and is an (n, d, λ)-graph with

$$n = q^2 + q + 1, \quad d = q + 1, \quad \lambda = \sqrt{q}.$$

Examples of (n, d, λ)-graphs

Lubotzky-Phillips-Sarnak 86, Margulis 88:

For every $d = p + 1$ where p is prime $p = 1 \pmod 4$, there are infinitely many $(n, d, 2\sqrt{d-1})$-graphs.

Alon 94:

For every $k, 3 \nmid k$ there is a triangle-free (n, d, λ)-graph with

$$n = 2^{3k}, \quad d = (1/4 + o(1))n^{2/3}, \quad \lambda = (9 + o(1))n^{1/3}.$$

Applications: MaxCut

Definition:

$f(G) =$ the number of edges in MaxCut, i.e., a maximum bipartite subgraph of G.

Claim: (*Folklore*)

Every graph G with m edges contains a cut of size at least $m/2$.

Theorem: (*Edwards 73,75*)

Every graph G with m edges contains a cut (a bipartite subgraph) of size at least

$$f(G) \geq \frac{m}{2} + \frac{-1 + \sqrt{8m+1}}{8} = \frac{m}{2} + \Omega(\sqrt{m}).$$

MaxCut in triangle-free graphs

Conjecture: *(Erdős 70's)*

If G contains no short cycles than it has bigger cut.

Theorem: *(Alon 96, improving Erdős-Lovász, Poljak-Tuza, Shearer)*

If G is triangle-free and has m edges then

$$f(G) \geq \frac{m}{2} + \Omega\left(m^{4/5}\right).$$

The constant $4/5$ tight

Proof of tightness:

Use an (n, d, λ)-graph with $d \approx \frac{1}{4}n^{2/3}$, $\lambda \approx 9n^{1/3}$, no triangles.

Theorem: (*Alon, Bollobás, Krivelevich and S. 02*)

If G has girth (length of the shortest cycle) r and m edges, then

$$f(G) \geq \frac{m}{2} + \Omega\left(m^{\frac{r}{r+1}}\right).$$

This is tight for $r = 5$ (and $r = 4$).

Proof of tightness:

Uses a random modification of Erdős-Renyi graph, which is C_4-free $(n, d \approx n^{1/2}, \lambda \approx n^{1/4})$-graph. Hence $m = \Omega(n^{3/2})$ and

$$\text{MaxCut} \leq \frac{m}{2} + \frac{\lambda n}{4} = \frac{m}{2} + O(n^{5/4}) = \frac{m}{2} + O(m^{5/6}).$$

Conjecture:

Exponent $\frac{r}{r+1}$ is tight also for all $r > 5$.

MaxCut in H-free graphs

Conjecture:

For every fixed H there is $c_H > 3/4$ such that if G is an H-free graph with m edges, then

$$f(G) \geq \frac{m}{2} + \Omega\left(m^{c_H}\right).$$

Theorem: *(Alon, Krivelevich and S. 05)*

- $H =$ cycle of length $r = 4, 6, 10$ then $c_H = \frac{r+1}{r+2}$.
- $H = K_{2,s}$ complete bipartite graph with parts of size 2 and $s \geq 2$ then $c_H = 5/6$.
- $H = K_{3,s}$ complete bipartite graph with parts of size 3 and $s \geq 3$ then $c_H = 4/5$.

A GEOMETRIC PROBLEM

PROBLEM: (*Lovász 79*)

Estimate $f(n) = \max ||\sum_{i=1}^{n} v_i||$, where

- $v_i \in \mathbb{R}^n$ and $||v_i|| = 1$.
- Among any three v_i's some two are orthogonal.

RESULTS:

- Konyagin 81: $\Omega(n^{0.54}) \leq f(n) \leq n^{2/3}$.
- Kashin-Konyagin 83: $\Omega\left(\frac{n^{2/3}}{\log^{1/2} n}\right) \leq f(n)$.

THEOREM: (*Alon 94*)

$$f(n) \geq (1/6 - o(1))\, n^{2/3}.$$

PROOF OF LOWER BOUND:

G is a triangle-free (n, d, λ)-graph with $d = \Omega(n^{2/3})$, $\lambda = O(n^{1/3})$. A is its adjacency matrix.

$\frac{1}{\lambda}(A + \lambda I)$ is positive semidefinite, so there is matrix B such that

$$B^T B = \frac{1}{\lambda}(A + \lambda I).$$

Let $v_1, v_2, \ldots, v_n$ be the columns of B. Then

- Each $||v_i|| = 1$.
- Among any three v_i's some two are orthogonal.
- $$\begin{aligned} ||\sum_{i=1}^{n} v_i||^2 &= \sum_{i,j} \left[\frac{1}{\lambda}(A + \lambda I)\right]_{ij} \\ &= n + \frac{nd}{\lambda} = \Omega(n^{4/3}). \end{aligned}$$

Universal graphs

Definition:

Given $\mathcal{H}$ a family of graphs (e.g., all trees, planar graphs and etc.), G is called *$\mathcal{H}$-universal* if it contains copy of every $H \in \mathcal{H}$.

Goal: (*motivated by VLSI design*)

Find sparse universal graph G for $\mathcal{H}$.

(*Use limited resources to achieve max. flexibility*)

Theorem: (*Bhatt, Chung, Leighton, Rosenberg 89*)

If $\mathcal{H}$ is all trees on n vertices of maximum degree at most D, then there is universal G of order n with maximum degree $\leq f(D)$.

Nearly spanning trees in (n, d, λ)-graphs

Theorem: *(Alon-Krivelevich-S. 06, extending Friedman-Pippenger 87)*

Let $D \geq 2$, $0 < \epsilon < 1/2$ and let G be an (n, d, λ)-graph such that

$$\frac{d}{\lambda} \geq \Omega\left(\frac{D^{5/2}\log(2/\epsilon)}{\epsilon}\right)$$

Then G contains a copy of every tree with $(1-\epsilon)n$ vertices and with maximum degree at most D.

Remark:

Random regular graphs, Lubotzky-Phillips-Sarnak graphs etc. are universal for almost spanning trees of bounded degree.

Embedding strategy

Very brief sketch:

- Cut tree T into pieces $T_1, \ldots, T_s, s = f(D, \epsilon)$ of decreasing size. Embed T piece by piece respecting previous embedding.
- Use result of Friedman-Pippenger that if every subset X of graph G of size at least $2k$ satisfies that $|\partial X| \geq D|X|$, then G contains every tree on k vertices with maximum degree D.
- Use the fact that if induced subgraph of (n, d, λ)-graph has minimal degree at least $\Omega(\lambda\sqrt{D})$, then it is a very good expander.

Conjecture:

There is a constant C_D such that (n, d, λ)-graph with $d/\lambda > C_D$ contains every spanning tree of maximum degree at most D.

Edge-deletion problems

Definition:

A graph property $\mathcal{P}$ is *monotone* if it is closed under deleting edges and vertices. It is *dense* if there are n-vertex graphs with $\Omega(n^2)$ edges satisfying it.

Examples:

- $\mathcal{P} = \{G \text{ is 5-colorable}\}$.
- $\mathcal{P} = \{G \text{ is triangle-free}\}$.
- $\mathcal{P} = \{G \text{ has a 2-edge coloring with no monochromatic } K_6\}$

Definition:

Given a graph G and a monotone property $\mathcal{P}$, denote by

$$E_{\mathcal{P}}(G) = \text{smallest number of edge deletions needed to turn } G \text{ into a graph satisfying } \mathcal{P}.$$

Approximation and hardness

Theorem: (*Alon, Shapira, S. 2005*)

- For every monotone $\mathcal{P}$ and $\epsilon > 0$, there exists a linear time, deterministic algorithm that given graph G on n vertices computes number X such that $\left|X - E_{\mathcal{P}}(G)\right| \leq \epsilon n^2$.
- For every monotone dense $\mathcal{P}$ and $\delta > 0$ it is NP-hard to approximate $E_{\mathcal{P}}(G)$ for graph of order n up to an additive error of $n^{2-\delta}$.

Remark:

Prior to this result, it was not even known that computing $E_{\mathcal{P}}(G)$ *precisely* for dense $\mathcal{P}$ is NP-hard. We thus answer (in a stronger form) a question of Yannakakis from 1981.

Hardness proof: example

Setting:

$\mathcal{P}$ = property of being H-free, $\chi(H) = r + 1$.
$E_{r\text{-}col}(F)$ = number of edge-deletions needed to make graph F r-colorable. Computing $E_{r\text{-}col}(F)$ is NP-hard.

Reduction:

- Given F, let $F' =$ blow-up of F : vertex $\leftarrow$ large independent set, edge $\leftarrow$ complete bipartite graph. Take union of F' with an appropriate (n, d, λ)-graph to get a graph G with large minimum degree.
- $E_{r\text{-}col}(F)$ changes in a controlled way, i.e., knowledge of an accurate estimate for $E_{r\text{-}col}(G)$ tells us the value of $E_{r\text{-}col}(F)$. Moreover $\left|E_{r\text{-}col}(G) - E_{\mathcal{P}}(G)\right| \leq n^{2-\gamma}$.
- Thus, approximating $E_{\mathcal{P}}(G)$ up to an additive error of $n^{2-\delta}$ is as hard as computing $E_{r\text{-}col}(F)$.