On Ramsey size-linear graphs and related questions

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Abstract

In this paper we prove several results on Ramsey numbers $R(H,F)$ for a fixed graph H and a large graph F, in particular for $F = K_n$. These results extend earlier work of Erdős, Faudree, Rousseau and Schelp and of Balister, Schelp and Simonovits on so-called Ramsey size-linear graphs.

1 Introduction

For two graphs H and F, the Ramsey number $R(H,F)$ is the smallest N such that for every graph G on N vertices, either G contains a copy of H or its complement $\overline{G}$ contains a copy of F. One of the central problems in graph Ramsey theory is the estimation of Ramsey numbers of complete graphs $R(K_s,K_n)$ for s fixed and large n. The classical Erdős-Szekeres [9] theorem implies that $R(K_s,K_n) = O(n^{s-1})$, and this was improved to $R(K_s,K_n) = O(n^{s-1}/\log^{s-2} n)$ by a celebrated result of Ajtai, Komlós and Szemerédi [1]. As for lower bounds, Spencer [15] showed that $R(H,K_n) = \tilde{\Omega}(n^{m_2(H)})$ [1] for every graph H, where $m_2(H)$ is the 2-density of H. This in particular implies that $R(K_s,K_n) = \tilde{\Omega}(n^{(s+1)/2})$. Kim [13] improved the implied logarithmic term in the case $s = 3$, obtaining the tight result $R(K_3,K_n) = \Theta(n^2/\log n)$. This was later generalized by Bohman and Keevash [4], who improved the logarithmic term for every s. On the other hand, no improvement to the exponent of n has been obtained for any $s \geq 4$. Very recently, Mubayi and Verstraëte [14] showed that the existence of optimally-dense pseudorandom K_s-free graphs would imply that $R(K_s,K_n) = \tilde{\Omega}(n^{s-1})$, matching the upper bound. This gives some evidence to the conjecture that $R(K_s,K_n) = \tilde{\Theta}(n^{s-1})$ for every s.

A more general problem is to estimate $R(H,K_n)$ for an arbitrary graph H. It is well-known that $R(H,K_n) = O(n)$ if and only if H is a forest. In fact, when H is a tree, a classical result of Chvátal [7] gives the exact value of $R(H,K_n)$. It is thus natural to ask which graphs satisfy $R(H,K_n) = O(n^k)$ for $k \geq 2$. Erdős, Faudree, Rousseau and Schelp [8] were the first to study this problem, proving several results for the case $k = 2$. They proved that $R(H,K_n) = O(n^2)$ for every connected graph H with $e(H) - v(H) \leq 1$. This result is tight, as $e(K_4) - v(K_4) = 2$ and $R(K_4,K_n) = \tilde{\Omega}(n^{5/2})$ (by the aforementioned result of Spencer [15]). We propose the following conjecture which generalizes the result of Erdős, Faudree, Rousseau and Schelp.

Conjecture 1. *Let $k \geq 1$. For every connected graph H with $e(H) - v(H) \leq \binom{k+1}{2} - 2$ it holds that $R(H,K_n) = O(n^k)$.*

Note that $e(K_{k+2}) - v(K_{k+2}) = \binom{k+1}{2} - 1$. Hence, if the aforementioned conjecture that $R(K_s,K_n) = \tilde{\Omega}(n^{s-1})$ is true, then the constant $\binom{k+1}{2} - 2$ in Conjecture 1 would be best possible. In this paper we prove the first open case of Conjecture 1, namely the case $k = 3$.

Theorem 2. *Let H be a connected graph with $e(H) - v(H) \leq 4$. Then $R(H,K_n) = O(n^3)$.*

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[1] As customary, for two functions f,g, we write $g(n) = \tilde{\Omega}(f(n))$ to mean that $g(n) \geq f(n)/\mathrm{polylog}(n)$, $g(n) = \tilde{O}(f(n))$ to mean that $g(n) \leq f(n) \cdot \mathrm{polylog}(n)$, and $g(n) = \tilde{\Theta}(f(n))$ to mean that $f(n)/\mathrm{polylog}(n) \leq g(n) \leq f(n) \cdot \mathrm{polylog}(n)$.

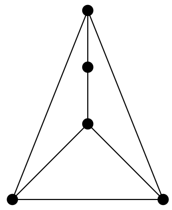

Figure 1: K_4^*

In the proof of Theorem 2, we make use of the following claim, which bounds the Ramsey number $R(H, K_n)$ in terms of the treewidth of H. This might be of independent interest. Recall that a graph is called a *k-tree* if it is K_{k+1} or if it is obtained from a smaller k-tree by adding a new vertex and connecting it to k vertices which form a clique. The treewidth of H is the minimal k for which H is a subgraph of a k-tree.

Proposition 1.1. *For every fixed graph H we have $R(H, K_n) = O(n^{tw(H)})$.*

In addition to proving Conjecture 1 in the case $k = 3$, we show that the full conjecture holds if K_n is replaced with $K_{n,n}$ (see Proposition 5.6). Along the way we also obtain bounds for $R(H, K_{n,n})$ for graphs H with bounded maximum degree or bounded degeneracy; see Corollaries 6 and 7.

Note that if $R(H, K_n) = O(n^2)$ then this can be written as $R(H, K_n) = O(e(K_n))$. This motivated Erdős, Faudree, Rousseau and Schelp [8] to define the so-called *Ramsey size-linear graphs*. A graph H is called Ramsey size-linear if

$$R(H, F) = O(e(F)) \tag{1}$$

holds for every graph F with no isolated vertices. This notion was introduced in [8], where the authors established some basic results and raised several intriguing questions. In particular, Erdős, Faudree, Rousseau and Schelp asked whether it is true that every graph H with $m_2(H) \leq 2$ is Ramsey size-linear. This would imply that every 2-degenerate graph is Ramsey size-linear. These questions seem to be still out of reach at the moment. Perhaps in light of this, Erdős et al. also asked about specific graphs H. In particular, they asked whether K_4^*, the graph obtained from K_4 by subdividing one edge, is Ramsey size-linear. This question was later reiterated by Balister, Schelp and Simonovits [3]. While we cannot supply an affirmative answer, we can show that (1) at the very least holds for every **bipartite** graph F.

Theorem 3. *For every bipartite graph F with no isolated vertices it holds that $R(K_4^*, F) = O(e(F))$.*

The above question of Erdős et al. for K_4^* motivates the study of Ramsey numbers for subdivisions of K_4. Balister, Schelp and Simonovits [3] showed (as part of a more general result) that the graph obtained from K_4 by subdividing an edge four times is Ramsey size-linear. Here we extend this further, showing that *every* subdivision of K_4 other than K_4^* is Ramsey size linear.

Theorem 4. *Every subdivision of K_4 on at least 6 vertices is Ramsey size-linear.*

The proofs of Theorems 3 and 4 both use dependent random choice (see e.g., [10] for a description of this method and a brief history).

The rest of this short paper is organized as follows. Section 2 contains some lemmas used in the proofs of Theorems 3 and 4. We then prove Theorem 3 in Section 3 and Theorem 4 in Section 4. Section 5 contains all results related to Conjecture 1. Finally, the last section includes some comments and related open questions. We use $\log n$ to denote the natural logarithm of n. We omit floor and ceiling signs whenever these are not crucial. We use $\delta(G), \Delta(G), d(G)$ to denote the minimum, maximum and average degree of G, respectively. We will frequently use the fact that a graph with n vertices and average degree d contains an independent set of size at least $n/(d+1)$. This is a well-known consequence of Turán's theorem.

2 Preliminary Lemmas

Lemma 2.1. *Let $r > 0$. Consider a bipartite graph with sides X, Y with $e(X,Y) \leq r|Y|$. Then there are $X' \subseteq X, Y' \subseteq Y$, $|X'| \geq \lfloor |X|/(r+1) \rfloor, |Y'| \geq \lfloor |Y|/(r+1) \rfloor$, with no edges between X' and Y'.*

Proof. By averaging, there is $X' \subseteq X$ with $|X'| = \lfloor |X|/(r+1) \rfloor$ such that $e(X',Y) \leq e(X,Y)/(r+1) \leq r|Y|/(r+1)$. Hence, there are at least $\lfloor |Y|/(r+1) \rfloor$ vertices $y \in Y$ which have no edge to X'. Take Y' to be the set of these vertices. □

In the proof of Theorem 3, it is convenient to assume that the host graph is (almost) regular. The following lemma allows us to assume that the maximum degree is larger than the average degree by no more than a logarithmic factor.

Lemma 2.2. *Let G be a graph on N vertices. Then there is an induced subgraph G' of G with average degree $d(G')$ such that $|V(G')| \geq N/6$ and $\Delta(G') \leq d(G') \cdot \log |V(G')|$.*

Proof. If $N \leq 12$ then take G' to be a 2-vertex graph. Suppose then that $N \geq 12$. We run the following process for $2N/3$ steps. If the current graph G satisfies $\Delta(G) \leq d(G) \cdot \log |V(G)|$, then stop. Otherwise, take $v \in V(G)$ with $d_G(v) > d(G) \cdot \log |V(G)|$, and replace G with $G - v$. Letting d denote the old average degree and d_{new} the new average degree, we have

$$\begin{aligned} d_{\text{new}} &= \frac{d|V(G)| - 2d_G(v)}{|V(G)|-1} < \frac{d|V(G)| - 2d\log|V(G)|}{|V(G)|-1} = d \cdot \left(1 - \frac{2\log|V(G)|-1}{|V(G)|-1}\right) \\ &\leq d \cdot \left(1 - \frac{2\log N - 1}{N-1}\right), \end{aligned}$$

where the last inequality holds because the function $x \mapsto \frac{2\log x - 1}{x-1}$ is decreasing for $x \geq 4$, say, and $|V(G)| \geq N/3 \geq 4$ (as we only run the process for $2N/3$ steps and in each step remove one vertex). So we see that if the process did not stop, then the average degree of the final graph G is at most $(N-1) \cdot \left(1 - \frac{2\log N - 1}{N-1}\right)^{2N/3} \leq (N-1) \cdot \exp\left(-\frac{2N/3 \cdot (2\log N - 1)}{N-1}\right) \leq (N-1) \cdot e^{-\log N} < 1$. Therefore, the final graph G contains an empty subgraph on at least $|V(G)|/2 = N/6$ vertices. This subgraph satisfies the assertion of the lemma. □

The following lemma shows that if some number of highest-degree vertices of a graph F has already been embedded into the complement $\overline{G}$ of a graph G, and if $N = |V(G)|$ is large enough compared to the average degree of G and the degrees of the vertices already used in the embedding, then one can complete the embedding of F into $\overline{G}$. The proof uses a basic greedy embedding argument.

Lemma 2.3. *Let F be a graph with m edges. Let $0 \leq k \leq |V(F)|$, and let A be the set of the k highest-degree vertices in F. Let G be a graph with N vertices and average degree d. Let $\sigma' : A \to V(G)$ be an embedding of $F[A]$ into $\overline{G}$. Suppose that*

$$N \geq \frac{4m}{k+1} \cdot \max\left(\max_{v \in \sigma'(A)} d_G(v),\ 2d\right) + 2|V(F)|. \tag{2}$$

Then there is an embedding σ of F into $\overline{G}$ which extends σ' (i.e. $\sigma(x) = \sigma'(x)$ for every $x \in A$).

Proof. Let W be the set of $v \in V(G)$ with $d_G(v) \leq 2d$. Then $|W| \geq N/2$. We embed the vertices of $V(F) \setminus A$ one-by-one into W. Let $x \in V(F) \setminus A$. We want to choose $\sigma(x) \in W$ which is different from all previously embedded vertices, such that if $y \in V(F)$ has already been embedded and is adjacent in F to x, then $\sigma(x)$ is not adjacent in G to $\sigma(y)$. If $y \in A$, then the degree of $\sigma(y)$ in G is of course not larger than $\max_{v \in \sigma'(A)} d_G(v)$. And if $y \notin A$, then $\sigma(y) \in W$ and hence its degree in G is at most $2d$. The total number of vertices y which we need to consider is $d_F(x) \leq 2m/(k+1)$, where the inequality holds by the choice of A, as $|A| = k$ and $e(F) = m$. So in total, the number of vertices which **cannot** play the role of $\sigma(x)$ is at most $\frac{2m}{k+1} \cdot \max\left(\max_{v \in \sigma'(A)} d_G(v),\ 2d\right) + |V(F)| - 1 < N/2 \leq |W|$. Hence, there is a suitable choice for $\sigma(x) \in W$. □

Corollary 2.4. *Let F be a graph with m edges, and let G be a graph with N vertices and average degree $d \leq \sqrt{\frac{N^2-2N\cdot|V(F)|}{48m}}$. Then $\overline{G}$ contains a copy of F.*

Proof. Let W be the set of $v \in V(G)$ with $d_G(v) \leq 2d$. Then $\Delta(G[W]) \leq 2d$ and $|W| \geq N/2$. Hence, $G[W]$ contains an independent set I of size $k \geq \frac{N/2}{2d+1} \geq \frac{N}{6d}$. Let A be the set of $\min\{k, |V(F)|\}$ highest-degree vertices of F. Mapping $F[A]$ arbitrarily into I gives an embedding of $F[A]$ into $\overline{G}$, since I is independent in G. If $A = V(F)$ then we are done. Else, we apply Lemma 2.3 to complete the embedding of F into $\overline{G}$. We only need to verify the condition (2). Since $I \subseteq W$, all vertices in I have degree at most $2d$ in G. So for (2) to hold, it suffices that $N \geq \frac{4m}{N/6d} \cdot 2d + 2|V(F)|$, which holds by the assumption of the lemma. $\square$

Next, we need a bipartite version of Lemma 2.3.

Lemma 2.5. *Let F be a bipartite graph with sides A, B and m edges. Let $k, \ell \geq 0$, let A' be the set of the k highest-degree vertices in A, and let B' be the set of the ℓ highest-degree vertices in B. Let G be a graph with N vertices and average degree d. Let $\sigma' : A' \cup B' \to V(G)$ be an embedding of $F[A' \cup B']$ into $\overline{G}$. Suppose that*

1. *$B' = B$ or $N \geq \frac{2m}{\ell+1} \cdot \max_{v \in \sigma'(A')} d_G(v) + 2|V(F)|$.*
2. *$A' = A$ or $N \geq \frac{m}{k+1} \cdot \max\left(\max_{v \in \sigma'(B')} d_G(v),\ 2d\right) + |V(F)|$.*

Then there is an embedding σ of F into $\overline{G}$ which extends σ'.

Proof. Let W be the set of $v \in V(G)$ with $d_G(v) \leq 2d$. Then $|W| \geq N/2$. We will embed the vertices of $V(F) \setminus (A' \cup B')$ one-by-one. We first embed the vertices of $B \setminus B'$ into W. Let $b \in B \setminus B'$. We want to choose $\sigma(b) \in W$ such that $\sigma(b)$ is not adjacent in G to $\sigma'(a)$ for any $a \in A'$ with $(a, b) \in E(F)$, and such that $\sigma(b)$ is different from all previously embedded vertices. We have $d_F(b) \leq m/(\ell+1)$, because $b \notin B'$ and B' is the set of the ℓ highest-degree vertices in B. So the number of vertices which **cannot** play the role of $\sigma(b)$ is at most $\frac{m}{\ell+1} \cdot \max_{v \in \sigma'(A')} d_G(v) + |V(F)| - 1 < N/2 \leq |W|$, where the first inequality uses Item 1. Therefore, there is a suitable choice for $\sigma(b) \in W$.

Suppose now that we have embedded $B \setminus B'$, and let us embed the vertices of $A \setminus A'$ (here we no longer insist that vertices are embedded into W). Let $a \in A \setminus A'$. We need to show that there is $\sigma(a) \in V(G)$ such that $\sigma(a)$ is not adjacent in G to $\sigma(b)$ for any $b \in B$ with $(a, b) \in E(F)$, and such that $\sigma(a)$ is different from all previously embedded vertices. As above, we have $d_F(a) \leq m/(k+1)$. For each $b \in B \setminus B'$, we have $d_G(\sigma(b)) \leq 2d$ because $\sigma(b) \in W$. Therefore, the number of vertices which **cannot** play the role of $\sigma(a)$ is at most $\frac{m}{k+1} \cdot \max\left(\max_{v \in \sigma'(B')} d_G(v),\ 2d\right) + |V(F)| - 1 < N$, using Item 2. So there is a valid choice for $\sigma(a) \in V(G)$. $\square$

Finally, we will need the following well-known result on the independence number of graphs with few triangles, see e.g. [5, Lemma 12.16].

Lemma 2.6. *Let G be a graph with N vertices, average degree d, and at most T triangles. Then G contains an independent set of size at least $0.1\frac{N}{d} \cdot \left(\log d - \frac{1}{2}\log(T/N)\right)$.*

3 Proof of Theorem 3

Let us first sketch the proof of Theorem 3 in the case $F = K_{n,n}$. So let G be a graph on $N = Cn^2$ vertices with no copy of K_4^*. We need to show that $\overline{G}$ contains a copy of $K_{n,n}$. First, it is easy to see that by deleting some $N/2$ (say) vertices, we may assume that the minimum degree of G is $\Omega(Cn)$. (Else, G contains an independent set of size $2n$, so $\overline{G}$ contains a $K_{n,n}$, as required.) Let S be the set of pairs of vertices (x, y) such that x, y have at most two common neighbours. Suppose first that there is $x \in V(G)$ such that $S(x) := \{y : (x, y) \in S\}$ has size at least $3n$. In this case, take disjoint sets $A \subseteq N_G(x), B \subseteq S(x)$, each of size $3n$ (this is possible because $d(x) = \Omega(Cn) \geq 6n$). By the definition of S, each vertex in B has at most two common neighbours with x, hence it has at most two neighbours in A. This allows us to find greedily an $n \times n$ empty bipartite graph between A, B, as required. So from now on suppose that $|S(x)| \le 3n$ for each $x \in V(G)$, implying that $|S| \le 3nN/2$. Now, taking a vertex $v \in V(G)$ at random, we see that the number of pairs $(x,y) \in S$ contained in $N(v)$ is on average at most $3n$ (each pair from S is counted at most twice when averaging over v, by the definition of S). So fix $v \in V(G)$ with at most $3n$ pairs $(x,y) \in S$ inside $N(v)$. Recall that $|N(v)| = \Omega(Cn)$. We may assume that the average degree inside $N(v)$ is at least $\Omega(C)$, because otherwise $N(v)$ would contain an independent set of size $2n$, and we would be done. It follows, by convexity, that $G[N(v)]$ contains at least $\Omega(C^3 n)$ paths of length two. Also, each pair x, y can be the endpoints of at most one such path of length two, because otherwise we get a C_4 inside $N(v)$, and hence a K_4^* together with v. So in $N(v)$ there are at least $\Omega(C^3 n) > 3n$ pairs (x,y) which are the endpoints of a path of length two. Hence, one of these pairs is not in S. Fix such a pair x, y, and let x, z, y be a path of length two inside $N(v)$. Since $(x,y) \notin S$, there is an additional neighbour $u \notin \{v, z\}$ of x, y. Now v, x, y, z, u form a copy of K_4^*.

Unfortunately, we were not able to adapt the above proof in a clean way to work for every bipartite graph F. Finding such a concise proof of Theorem 3 would be interesting. Instead, to make the proof work for an arbitrary bipartite F, we apply regularization to G via Lemma 2.2, ensuring that the maximum degree is at most a logarithmic factor away from the average degree. This "almost-regularity" of G will be useful when applying Lemmas 2.3 and 2.5 in certain steps of the proof. To compensate for the (extra) logarithmic factor, we use Lemma 2.6. The details follow.

Proof of Theorem 3. Let F be a bipartite graph with sides A, B, having m edges and no isolated vertices. Note that $|V(F)| \le 2m$. Let G be a graph on $N = Cm$ vertices with no copy of K_4^*, where C is a large enough constant. Our goal is to show that $\overline{G}$ contains a copy of F. By Lemma 2.2, there is an induced subgraph G' of G with $|V(G')| \ge N/6$ and $\Delta(G') \le d(G') \cdot \log |V(G')|$. With a slight abuse of notation, we will use the notation G for G' and N for $|V(G')|$; so $|V(G)| = N$ and $\Delta(G) \le d(G) \cdot \log N$. Put $d := d(G)$.

Let S be the set of pairs $(x,y) \in \binom{V(G)}{2}$ with $d_G(x,y) \le 2$. We proceed with several cases.

Case 1: $|S| \ge 2Nd \log N$. For each $x \in V(G)$, let $S(x)$ be the set of $y \in V(G)$ with $d(y) \le d(x)$ and $(x,y) \in S$. Then $\sum_{x \in V(G)} |S(x)| \ge |S|$. Hence, there is x with $|S(x)| \ge 2d \log N$. Since $\Delta(G) \le d \log N$, we have $|S(x) \setminus N_G(x)| \ge d \log N$. Each $y \in S(x)$ has at most 2 neighbours in $N_G(x)$, by the definition of S. By Lemma 2.1 with $r = 2$, $X = N_G(x)$ and $Y = S(x) \setminus N_G(x)$, there exist $X' \subseteq N_G(x)$ and $Y' \subseteq S(x) \setminus N_G(x)$ such that $|X'| \ge \lfloor d_G(x)/3 \rfloor$, $|Y'| \ge \lfloor |S(x) \setminus N_G(x)|/3 \rfloor \ge \lfloor d \log(N)/3 \rfloor$, and there are no edges in G between X' and Y'. Let $A' \subseteq A$ be the set of the $k := \min\{|Y'|, |A|\}$ highest-degree vertices in A, and let $B' \subseteq B$ be the set of the $\ell := \min\{|X'|, |B|\}$ highest-degree vertices in B. Map A' into Y' and B' into X' arbitrarily. This mapping σ' is an embedding of $F[A' \cup B']$ into $\overline{G}$, because there are no edges in G between X' and Y'. We now verify that Items 1-2 in Lemma 2.5 hold. All vertices in $\sigma'(A') \subseteq Y' \subseteq S(x)$ have degree at most $d_G(x)$ by the definition of $S(x)$. Hence (assuming $B' \ne B$), we have $\frac{2m}{\ell+1} \cdot \max_{v \in \sigma'(A')} d_G(v) \le \frac{2m}{|X'|+1} \cdot d_G(x) \le \frac{2m}{d_G(x)/3} \cdot d_G(x) = 6m$. Also, $|V(F)| \le 2m$. Therefore, Item 1 in Lemma 2.5 holds provided that $N \ge 10m$. Next, (assuming $A' \ne A$), we have $\frac{m}{k+1} \cdot \max\left(\max_{v \in \sigma'(B')} d_G(v),\, 2d\right) \le \frac{m}{|Y'|+1} \cdot 2\Delta(G) \le \frac{m}{d \log(N)/3} \cdot 2d \log N \le 6m$. So Item 2 in Lemma 2.5 holds as well, provided that $N \ge 8m$. Hence, $\overline{G}$ contains a copy of F, as required.

Case 2: $|S| \le 2Nd \log N$ and $d \ge 576\sqrt{m \log N}$. For each $x \in V(G)$, let $N'(x)$ denote the set of neighbours y of x with $d(y) \le d(x)$, and let $d'(x) = |N'(x)|$. Then $\sum_{x \in V(G)} d'(x) \ge e(G) = dN/2$. Let $t(x)$ be the number of pairs $(y,z) \in S$ such that $y, z \in N'(x)$. We have $\sum_{x \in V(G)} t(x) \le 2|S|$, because each pair in S is counted at most twice in this sum, by the definition of S. Observe that

$$\sum_{x \in V(G)} \Big[16 \log N \cdot \Big(d'(x) - d(x)/8 - d/8\Big) - t(x)\Big] \ge 16 \log N \cdot e(G)/2 - \sum_{x \in V(G)} t(x)$$
$$= 4dN \log N - \sum_{x \in V(G)} t(x) \ge 4dN \log N - 2|S| \ge 0.$$

Hence, there is $x \in V(G)$ with $16 \log N \cdot \Big(d'(x) - d(x)/8 - d/8\Big) - t(x) \geq 0$. In particular, $d'(x) \geq d(x)/8, d/8$, and the number $t(x)$ of pairs $(y,z) \in S$ with $y,z \in N'(x)$ is at most $16d'(x)\log N$. Fix such x, let $G_1 := G[N'(x)]$ and let $d_1 := d(G_1)$ be the average degree of G_1. We claim that $d_1 \leq 6\sqrt{\log N}$. Suppose otherwise. Then, by convexity, the number of paths of length 2 in G_1 is at least $|N'(x)| \cdot \binom{6\sqrt{\log N}}{2} > 16d'(x)\log N \geq t(x)$. A pair of vertices from $N'(x)$ can be the endpoints of at most one path of length two in G_1, because otherwise $G_1 = G[N'(x)]$ would contain a copy of C_4, which together with x would give a copy of K_4^* in G, a contradiction. So we see that there are more than $t(x)$ pairs $(y,z) \in \binom{N'(x)}{2}$ which are the endpoints of a path of length 2 in G_1. Hence, there is such a pair (y,z) which does **not** belong to S. Let w be the middle vertex of the path of length two between y and z in G_1. Since $(y,z) \notin S$, there is a common neighbour u of y,z with $u \neq x, w$. Now x, y, z, w, u span a copy of K_4^*, a contradiction. Hence, our claim that $d_1 \leq 6\sqrt{\log N}$ holds.

Let $A \subseteq V(F)$ be the set of $k := \min\{d'(x)/3, |V(F)|\}$ highest-degree vertices in F. Our goal is to embed $F[A]$ into $\overline{G}[N'(x)]$, and then use Lemma 2.3 to extend this into an embedding of F into $\overline{G}$. To embed $F[A]$ into $\overline{G}[N'(x)]$, we use Corollary 2.4 with $F[A]$ in the role of F and $G_1 = G[N'(x)]$ in the role of G. To apply the corollary, we need to verify the condition

$$d_1 \leq \sqrt{\frac{|V(G_1)|^2 - 2|V(G_1)| \cdot |A|}{48e(F[A])}}. \tag{3}$$

By definition, $|A| \leq d'(x)/3$. Also, $|V(G_1)| = d'(x)$, $d_1 \leq 6\sqrt{\log N}$ and $e(F[A]) \leq m$. So the RHS of (3) is at least $\frac{d'(x)}{12\sqrt{m}}$, and hence (3) holds provided that $d'(x) \geq 72\sqrt{m \log N}$, as $d_1 \leq 6\sqrt{\log N}$. But $d'(x) \geq d/8 \geq 72\sqrt{m\log N}$ by the assumption of Case 2, so (3) indeed holds. By Corollary 2.4, there is an embedding σ' of $F[A]$ into $\overline{G}[N'(x)]$.

We now use Lemma 2.3 to embed F into $\overline{G}$. Recall that all vertices in $\sigma'(A) \subseteq N'(x)$ have degree at most $d(x) \leq 8d'(x)$ in G (by the definition of $N'(x)$), and that the average degree of G is $d \leq 8d'(x)$. Also, $k = d'(x)/3$ assuming $A \neq V(F)$, by our choice of k. Therefore, we can bound the first term on the RHS of (2) as follows: $\frac{4m}{k+1} \cdot \big(\max_{v \in \sigma'(A)} d_G(v),\ 2d\big) \leq \frac{12m}{d'(x)} \cdot 16d'(x) = 192m$. Also, $|V(F)| \leq 2m$. So (2) holds for $N \geq 196m$. We conclude that $\overline{G}$ contains a copy of F, as required.

Case 3: $d \leq 576\sqrt{m \log N} = O(\sqrt{N \log N})$ (recall that $N = Cm$ so $m \leq N$). Let W be the set of vertices of G of degree at most $2d$. Then $|W| \geq N/2$. Let $d_0 := d(G[W])$ be the average degree of $G[W]$. Then $d_0 \leq \Delta(G[W]) \leq 2d = O(\sqrt{N\log N})$.

Let us bound the number of triangles in $G[W]$. We have $\Delta(G[W]) \leq 2d = O(\sqrt{N \log N}) = N^{1/2+o(1)}$. For each vertex x, the neighbourhood of x in G contains no C_4, because otherwise G would contain a copy of K_4^*. Hence, x participates in at most $\Delta(G[W])^{3/2} \leq N^{3/4+o(1)} \leq d_0^{3/2+o(1)}$ triangles of $G[W]$. So the overall number of triangles in $G[W]$ is at most $T := |W| \cdot d_0^{3/2+o(1)}$. By Lemma 2.6, applied to $G[W]$, there is an independent set I in $G[W]$ of size at least

$$\begin{aligned} |I| &\geq 0.1\frac{|W|}{d_0} \cdot \left(\log d_0 - \frac{1}{2}\log\left(\frac{T}{|W|}\right)\right) = 0.1\frac{|W|}{d_0} \cdot \left(\frac{1}{4} - o(1)\right)\log d_0 \\ &\geq 0.01\frac{N}{d_0}\log d_0 \geq \Omega\left(\sqrt{N\log N}\right) \geq \Omega(d). \end{aligned}$$

Here, the penultimate inequality uses that $d_0 \leq O(\sqrt{N \log N})$ and that the function $x \mapsto \frac{\log x}{x}$ is decreasing (for $x \geq e$), and the last inequality uses that $d \leq O(\sqrt{N\log N})$. Let A be the set of the $k := \min\{|I|, |V(F)|\}$ highest-degree vertices in F. We use Lemma 2.3 to embed F into $\overline{G}$, starting with an arbitrary embedding of $F[A]$ into I. Recall that all vertices in $I \subseteq W$ have degree at most $2d$ in G. Hence, assuming $A \neq V(F)$, the RHS of condition (2) is at most $\frac{4m}{k+1} \cdot 2d + 2|V(F)| \leq \frac{4m}{|I|} \cdot 2d + 4m \leq O(m)$, using that $|I| = \Omega(d)$. Therefore, condition (2) holds for $N = Cm$, provided that C is large enough compared to the implied constant in the O-notation in the previous sentence. So $\overline{G}$ contains a copy of F. This completes the proof. $\square$

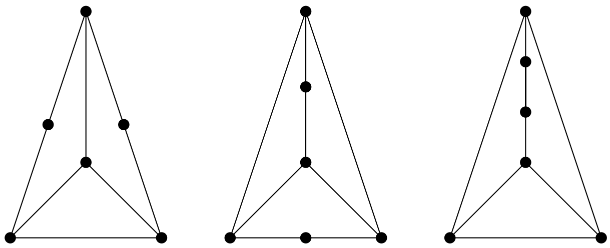

Figure 2: H_1, H_2, H_3 (from left to right)

4 Proof of Theorem 4

Proof. There are three subdivisions of K_4 on 6 vertices, and we denote these by H_1, H_2, H_3; see the figure below. Every subdivision of K_4 on more than six vertices is a subdivision of H_i for some $i = 1, 2, 3$. So let H be a subdivision of H_1, H_2 or H_3. Let $h := |V(H)|$, and let ℓ be the largest length of a subdivision path in H. Fix constants $1 \ll C_0 \ll C_1 \ll C$, to be chosen implicitly later. Let F be a graph with m edges and no isolated vertices. Let G be a graph on $N = Cm$ vertices. We assume that $\overline{G}$ has no copy of F and our goal is to show that G has a copy of H. We begin with some general preparation that will be used in all three cases of H_1, H_2, H_3. Let $d = d(G)$ be the average degree of G. Under the assumption that $\overline{G}$ has no copy of F (and that C is large enough in terms of C_1), we now prove the following:

Claim 4.1. *The following holds:*

1. $d \geq 0.1N/\sqrt{m} \geq C_1\sqrt{N}$.
2. *There is no independent set I such that $|I| \geq d/C_1$ and all vertices in I have degree at most $C_1 \cdot |I|$.*

Proof. We show that if Item 1 or 2 does not hold, then $\overline{G}$ contains a copy of F. For Item 1, if $d \leq 0.1N/\sqrt{m}$ then $\overline{G}$ contains a copy of F by Corollary 2.4, using that $|V(F)| \leq 2m$. The inequality $0.1N/\sqrt{m} \geq C_1\sqrt{N}$ in Item 1 holds because $C \gg C_1$, as $N = Cm$. Suppose now that Item 2 fails and let I be as in that item. We apply Lemma 2.3. Take A to be the set of $k = \max\{|I|, |V(F)|\}$ highest-degree vertices in F. Map A arbitrarily into I; this is an embedding of $F[A]$ into $\overline{G}$ because I is independent in G. Let us verify (2) in Lemma 2.3. All vertices in I have degree at most $C_1 \cdot |I|$. Also, the average degree d of G satisfies $d \leq C_1 \cdot |I|$. Hence, assuming $A \neq V(F)$, the RHS in (2) is at most $\frac{4m}{|I|+1} \cdot 2C_1 \cdot |I| + 2|V(F)| \leq 8C_1 m + 4m$. So (2) holds for $N = Cm$ if $C \gg C_1$. $\square$

Let $\mathcal{T}$ be the set of triangles in G. We run the following process. As long as there is an edge e which is contained in at least one and at most C_0 triangles from $\mathcal{T}$, we delete from $\mathcal{T}$ all triangles containing e (we do not make any changes to the graph, only to the collection $\mathcal{T}$). We say that e is *eliminated* at this step. Note that at a given step of the process, a triangle (still) belongs to $\mathcal{T}$ if and only if none of its edges have been eliminated. By the end of the process we are left with a collection of triangles $\mathcal{T}_0$ with the property that if an edge is contained in a triangle from $\mathcal{T}_0$ then it is contained in at least C_0 such triangles. An edge is called *good* if it is contained in a triangle from $\mathcal{T}_0$, and *bad* otherwise. We denote the set of good edges by E_{good} and the set of bad edges by E_{bad}. Observe that:

(a) Every good edge is on at least C_0 triangles, all of whose edges are good.

(b) For every set $B \subseteq E_{\text{bad}}$ of bad edges, there is $e \in B$ such that there are at most C_0 triangles in G which contain e and only use edges from $B \cup E_{\text{good}}$. In particular, the total number of triangles containing bad edges is at most $C_0 \cdot e(G)$.

Property (a) holds by the definition of good edges and the above-mentioned property of $\mathcal{T}_0$. For property (b), take e to be the earliest eliminated edge among the edges in B. Before e is eliminated, all triangles in G which only use edges from $B \cup E_{\text{good}}$ are still in $\mathcal{T}$, because none of the edges in

$B \cup E_{\text{good}}$ has been eliminated yet (the edges in E_{good} are never eliminated). At the moment that e is eliminated, the number of triangles in $\mathcal{T}$ which contain e must be at most C_0.

We will use the following terminology. We call y a *good neighbour* of x if (x,y) is a good edge. The *good neighbourhood* of x is the set of all good neighbours of x. Similarly, a *common good neighbour* of x,y is a vertex z such that $(x,z),(y,z)$ are good edges.

Claim 4.2. *For every good edge (x,y), there are length-ℓ paths $P_1^{x,y},\dots,P_h^{x,y}$, all starting at x and intersecting only at x, such that every vertex on these paths is adjacent to y.*

Proof. Let $A(y)$ be the good neighbourhood of y. Then $x \in A(y)$. Observe that by Property (a), $G[A(y)]$ has minimum degree at least C_0. Hence, one can greedily find the above paths $P_1^{x,y},\dots,P_h^{x,y}$ inside $A(y) \subseteq N(y)$, provided that C_0 is large enough in terms of ℓ and h. □

For $v \in V(G)$, let $b(v)$ denote the number of bad edges contained in $N(v)$. By Property (b),

$$\sum_{v \in V(G)} b(v) \le 3C_0 \cdot e(G). \tag{4}$$

Indeed, the sum $\sum_{v\in V(G)} b(v)$ counts triangles which contain a bad edge, and each such triangle is counted at most 3 times.

Claim 4.3. *There are at least $e(G)/2$ good edges.*

Proof. Suppose by contradiction that $|E_{\text{good}}| \le e(G)/2$, and hence $|E_{\text{bad}}| \ge e(G)/2$. For $v \in V(G)$, let $N'(v)$ be the set of vertices u such that (u,v) is a bad edge and $d(u) \le d(v)$, and let $d'(v) = |N'(v)|$. Then $\sum_{v\in V(G)} d'(v) \ge |E_{\text{bad}}| \ge e(G)/2$. Observe that

$$\sum_{v\in V(G)} \left(d'(v) - d(v)/16 - d/16 - \frac{1}{12C_0}\cdot b(v) \right) \ge e(G)/2 - e(G)/8 - e(G)/8 - \frac{1}{12C_0}\cdot \sum_{v\in V(G)} b(v) \ge 0,$$

where the last inequality uses (4). So there is $v \in V(G)$ such that $d'(v) \ge d(v)/16, d/16$ and $b(v) \le 12C_0 \cdot d'(v)$. In particular, this means that there are at most $12C_0 \cdot d'(v)$ bad edges inside $N'(v)$. So the average degree of the graph of bad edges inside $N'(v)$ is at most $24C_0$. Hence, there is $A \subseteq N'(v)$, $|A| \ge d'(v)/(24C_0+1)$, such that $G[A]$ has no bad edges. Recall that (v,u) is a bad edge for every $u \in A$, by the definition of $N'(v)$. We claim that the graph $G[A]$ is C_0-degenerate. Indeed, given any $S \subseteq A$, apply Property (b) to the set of bad edges $B := \{(v,u) : u \in S\}$, and let $e = (v,u) \in B$ be the edge given by Property (b). Since all edges in $G[S]$ are good, every edge of the form $(u,w) \in E(G[S])$ forms a triangle u,v,w in which $(v,u),(v,w) \in B$ and (u,w) is good. By our choice of e, there are at most C_0 such triangles, so $d_{G[S]}(u) \le C_0$, as required. It follows that $G[A]$ contains an independent set I of size at least $|I| \ge \frac{|A|}{C_0+1} \ge \frac{d'(v)}{(C_0+1)(24C_0+1)} \ge d/C_1$, as $d'(v) \ge d/16$ (provided $C_1 \gg C_0$). Also, all vertices in $I \subseteq N'(v)$ have degree at most $d(v)$ in G, by the definition of $N'(v)$, and $d(v) \le 16d'(v) \le 16(C_0+1)(24C_0+1)\cdot|I| \le C_1 \cdot |I|$. But this contradicts Item 2 of Claim 4.1. □

We now proceed by case analysis over the cases of H_1, H_2, H_3.

Case 1: H is a subdivision of H_1. For $v \in V(G)$, denote by $N^*(v)$ the set of all $u \in V(G)$ such that (u,v) is a good edge and $d(u) \le d(v)$, and let $d^*(v) = |N^*(v)|$. Then $\sum_{v\in V(G)} d^*(v) \ge |E_{\text{good}}| \ge e(G)/2$, by Claim 4.3. Let $P(v)$ denote the set of pairs $(u,w) \in \binom{N^*(v)}{2}$ such that u,w have at most C_0 common good neighbours. Let $p(v) = |P(v)|$. Observe that

$$\sum_{v\in V(G)} p(v) \le C_0 \binom{N}{2}, \tag{5}$$

because each pair (u,w) is counted at most C_0 times in the sum on the LHS of (5), by the definition of the sets $P(v)$. Now, note that

$$\sum_{v\in V(G)}\left(d^*(v)-d(v)/16-d/16-\frac{1}{24C_0}\cdot b(v)-\frac{e(G)}{4C_0N^2}\cdot p(v)\right)=$$
$$\sum_{v\in V(G)}d^*(v)-e(G)/4-\frac{1}{24C_0}\cdot\sum_{v\in V(G)}b(v)-\frac{e(G)}{4C_0N^2}\cdot\sum_{v\in V(G)}p(v)\geq$$
$$e(G)/2-e(G)/4-e(G)/8-e(G)/8=0.$$

Here we used (4) and (5). So there is $v\in V(G)$ such that $d^*(v)\geq d(v)/16, d/16$, $b(v)\leq 24C_0\cdot d^*(v)$ and

$$p(v)\leq\frac{4C_0N^2}{e(G)}\cdot d^*(v)=\frac{8C_0N}{d}\cdot d^*(v)\leq\frac{d^*(v)\cdot(d^*(v)-3)}{8},\tag{6}$$

where the last inequality holds if $C_1\gg C_0$ because $d^*(v)\geq d/16\geq C_1\sqrt{N}/16$, using Item 1 of Claim 4.1. Let $A\subseteq N^*(v)$ be the set of all $u\in N^*(v)$ which participate in at most $\frac{d^*(v)-3}{2}$ of the pairs in $P(v)$. Then $p(v)\geq\frac{1}{2}(d^*(v)-|A|)\cdot\frac{d^*(v)-3}{2}$, so $|A|\geq d^*(v)/2$ by (6). We claim that $e(G[A])>48C_0\cdot|A|$. Indeed, otherwise $G[A]$ would contain an independent set I of size $|I|\geq\frac{|A|}{96C_0+1}\geq\frac{d^*(v)}{2(96C_0+1)}\geq\frac{d}{C_1}$, as $d^*(v)\geq d/16$ and $C_1\gg C_0$. Also, $I\subseteq N^*(v)$ and all vertices in $N^*(v)$ have degree at most $d(v)\leq 16d^*(v)\leq C_1\cdot|I|$ in G. This would contradict Item 2 of Claim 4.1. So indeed $e(G[A])>48C_0\cdot|A|$. On the other hand, $G[A]$ contains at most $b(v)$ bad edges. Hence, $G[A]$ contains at least $e(G[A])-b(v)>48C_0\cdot|A|-b(v)\geq 48C_0\cdot d^*(v)/2-b(v)\geq 0$ good edges. Fix such a good edge $(u_1,u_2), u_1,u_2\in A$. By the definition of A, each u_i participates in at most $\frac{d^*(v)-3}{2}$ of the pairs in $P(v)$. Hence, there is $w\in N^*(v)$ different from u_1,u_2 such that $(u_1,w),(u_2,w)\notin P(v)$. By the definition of $P(v)$, this means that u_i,w have at least C_0 common good neighbours for $i=1,2$. Let z_i be a common good neighbour of u_i,w such that v,u_1,u_2,w,z_1,z_2 are all distinct. Observe that these six vertices form a copy of H_1 in which all edges are good. To obtain a copy of H, we need to replace the edges of this copy of H_1 with internally-disjoint paths of appropriate lengths. We find these paths one-by-one. Suppose that the path between vertices $x,y\in\{v,u_1,u_2,w,z_1,z_2\}$ needs to have length $k\leq\ell$. The edge (x,y) is good, hence we can apply Claim 4.2 to it to obtain the paths $P_i^{x,y}$, $i=1,\dots,h$. One of the paths $P_i^{x,y}$, $1\leq i\leq h$, must be internally disjoint from all the vertices we embedded so far. Also, since all vertices of this path are adjacent to y, we can shorten it to a path that ends in y and has length exactly k. This completes the proof for the case of H_1.

Case 2: H is a subdivision of H_2. For a vertex $v\in V(G)$, denote by $d_{\text{good}}(v)$ the number of good neighbours of v. By Claim 4.3, we have $\sum_v d_{\text{good}}(v)\geq e(G)$. Sample two distinct vertices $v_1,v_2\in V(G)$ uniformly at random and let A be the common good neighbourhood of v_1,v_2. For each $u\in V(G)$, the probability that $u\in A$ is $\binom{d_{\text{good}}(u)}{2}/\binom{N}{2}$. By Jensen's inequality,

$$\mathbb{E}[|A|]=\frac{1}{\binom{N}{2}}\cdot\sum_u\binom{d_{\text{good}}(u)}{2}\geq\frac{N}{\binom{N}{2}}\cdot\binom{\frac{1}{N}\cdot\sum_u d_{\text{good}}(u)}{2}\geq\frac{N}{\binom{N}{2}}\cdot\binom{e(G)/N}{2}$$
$$\geq\frac{2}{N}\cdot\frac{(e(G)/N)^2}{4}=\frac{e(G)^2}{2N^3}\geq C_0^2,$$

where the last inequality holds because $e(G)=dN/2$, $d\geq C_1\sqrt{N}$ by Item 1 of Claim 4.1, and $C_1\gg C_0$. Let P be the set of pairs of vertices $u_1,u_2\in A$ such that u_1,u_2 have at most C_0 common good neighbours. For a given pair $u_1,u_2\in V(G)$ with at most C_0 common good neighbours, the probability that $u_1,u_2\in A$ is at most $\binom{C_0}{2}/\binom{N}{2}$. Hence, $\mathbb{E}[|P|]\leq\binom{C_0}{2}$. By linearity of expectation, $\mathbb{E}[|A|-|P|]\geq C_0^2/2\geq 3$. Hence, there is a choice of v_1,v_2 for which $|A|-|P|\geq 3$. By removing one vertex from each pair in P, we obtain a subset $A'\subseteq A$, $|A'|\geq 3$, such that no pair of vertices in A' belongs to P. Fix distinct $u_1,u_2,u_3\in A'$. Since $(u_1,u_2)\notin P$, there are more than $C_0\geq 4$ common good neighbours of u_1, u_2. Hence, there is $w \notin \{v_1, v_2, u_3\}$ which is a common good neighbour of u_1, u_2. Observe that $v_1, v_2, u_1, u_2, u_3, w$ form a copy of H_2 in which all edges are good. We can obtain a copy of H by using the paths $P_i^{x,y}$ given by Claim 4.2, as is done in Case 1.

Case 3: H is a subdivision of H_3. We start by observing that finding certain subgraphs in G allows us to embed a copy of H. We need three such observations. Recall that K_4^* is the subdivision of K_4 where exactly one edge is subdivided once.

(i) H_3 is a subdivision of K_4^*, and hence H is a subdivision of K_4^* by transitivity. Therefore, if we find a copy of K_4^* consisting only of good edges, then we can use this copy to find a copy of H, as is done in the previous cases (by using the paths $P_i^{x,y}$ from Claim 4.2).

(ii) H_3 is obtained from K_4 by subdividing one edge twice. We distinguish between the original edges of K_4 and the subdivision edges; H_3 has five original edges and three subdivision edges. Observe that if we find a copy of H_3 in which all original edges are good and at least one subdivision edge is good, then we can find a copy of H. (Indeed, if to obtain a copy of H it is necessary to subdivide one of the bad subdivision edges in the copy of H_3, then we can instead subdivide a good subdivision edge of the copy of H_3.)

(iii) If H is obtained from K_4 by subdividing at least two edges (some number of times), then H is a subdivision of H_1 or H_2; so such H are already covered by Cases 1-2. Hence, we may assume that H is obtained from K_4 by subdividing exactly one edge (some number $\ell \geq 2$ times). It follows that if we find a copy of K_4 which contains at least one good edge, then we can find a copy of H.

Recall that a diamond is the graph consisting of two triangles sharing an edge. A diamond has two vertices of degree 2 and two vertices of degree 3; the vertices of degree 2 will be called the *tips* of the diamond, and the edge connecting the two vertices of degree 3 will be called the *middle edge* of the diamond. A diamond is called good if all of its edges are good. By Claim 4.3, G contains at least $e(G)/2$ good edges. By Property (a) above, every good edge is on at least C_0 triangles, all of whose edges are good. It follows that G contains at least $e(G)/2 \cdot \binom{C_0}{2} \geq 4e(G)$ good diamonds. For $v \in V(G)$, let $t(v)$ denote the number of good diamonds D such that v is a tip of D, and the other tip u of D satisfies $d(u) \leq d(v)$. Then $\sum_v t(v) \geq 4e(G)$. It follows that

$$\sum_{v \in V(G)} (t(v) - d(v) - d) = \sum_{v \in V(G)} t(v) - 4e(G) \geq 0.$$

Hence, there is a vertex v satisfying $t(v) \geq d(v), d$. Fix such a vertex v. By definition, there are good diamonds $D_1, \ldots, D_r$, $r = t(v)$, such that v is a tip of D_i, and the other tip u_i of D_i satisfies $d(u_i) \leq d(v)$ (for $i = 1, \ldots, r$). Let e_i be the middle edge of D_i. Suppose first that there is $1 \leq i \leq r$ such that $(v, u_i) \in E(G)$. Then the vertices of D_i form a K_4 in which all edges except possibly (v, u_i) are good. By (iii), this allows us to find a copy of H, as required. So from now on we may assume that v is not connected to any u_i. It follows that $\{u_1, \ldots, u_r\} \cap \bigcup_{i=1}^r e_i = \emptyset$, since v is connected with (good) edges to all vertices of $\bigcup_{i=1}^r e_i$. Next, suppose that there are $1 \leq i < j \leq r$ such that $u_i = u_j$. Since $D_i \neq D_j$, there is $x_j \in e_j$ such that $x_j \notin V(D_i)$. Observe that $V(D_i) \cup \{x_j\}$ spans a copy of K_4^* in which all edges are good. By (i), this is sufficient to find a copy of H. Hence, from now on we may assume that $u_1, \ldots, u_r$ are pairwise distinct. We claim that $U := \{u_1, \ldots, u_r\}$ is not an independent set of G. Indeed, observe that $|U| = r = t(v) \geq d$ and all vertices in U have degree at most $d(v) \leq t(v) = |U|$. So if U were independent then we would get a contradiction to Item 2 of Claim 4.1. Let us then fix $1 \leq i < j \leq r$ such that $(u_i, u_j) \in E(G)$. If $e_i = e_j$ then $e_i \cup \{u_i, u_j\}$ spans a copy of K_4 in which all edges except possibly (u_i, u_j) are good. This allows us to find a copy of H by (iii). Suppose finally that $e_i \neq e_j$. Let $x_j \in e_j \setminus V(D_i)$. It is easy to check that $V(D_i) \cup \{x_j, u_j\}$ contains a copy of H_3 in which all edges except possibly (u_i, u_j) are good, and (u_i, u_j) plays the role of a subdivision edge (the subdivision edges are $(v, x_j), (x_j, u_j), (u_j, u_i)$). By (ii), this allows us to find a copy of H. So we see that in all cases, G contains a copy of H. This completes the proof of Case 3 and hence the theorem. $\square$

5 On Conjecture 1

This section is broken into several parts. First, we prove Proposition 1.1, which then allows us to prove Theorem 2. Next, we show that Conjecture 1 holds if K_n is replaced by $K_{n,n}$ (see Proposition 5.6). To that end, we also study the Ramsey number $R(H, K_{n,n})$ for graphs H of bounded maximum degree and for degenerate graphs (see Section 5.3).

5.1 Proof of Proposition 1.1

Here we prove Proposition 1.1, which bounds the Ramsey number $R(H, K_n)$ in terms of the treewidth of H. We refer to [11] for the basic definitions related to treewidth. We will need the following lemma. For a graph G, let $\#K_r(G)$ denote the number of r-cliques in G.

Lemma 5.1. *For any $r \geq 1$ there is $C_r > 0$ such that the following holds. If G is a graph on $N \geq C_r n^r$ vertices with no independent set of size n, then $\#K_{r+1}(G) \geq \frac{N}{C_r n^r} \cdot \#K_r(G)$.*

Proof. We will show that one can take $C_1 = 4$ and $C_r = 8(r+1) \cdot C_{r-1}$ for $r \geq 2$. (We make no effort of optimising the value of C_r.) The proof is by induction on r. Suppose first $r = 1$. Let d be the average degree of G. We have $n > \alpha(G) \geq \frac{N}{d+1}$, and hence $\frac{2e(G)}{N} = d > \frac{N}{n} - 1 \geq \frac{N}{2n}$. It follows that $e(G) \geq \frac{N}{4n} \cdot N$, as required.

Let now $r \geq 2$, and let G be as in the statement of the lemma. By the induction hypothesis, we know that $\#K_r(G) \geq \frac{N}{C_{r-1}n^{r-1}} \cdot \#K_{r-1}(G)$. Let $\mathcal{C}$ be the set of all $(r-1)$-cliques X in G such that the number of r-cliques containing X is at least $\frac{r}{2} \cdot \frac{N}{C_{r-1}n^{r-1}}$. Observe that the number of r-cliques which do not contain any $(r-1)$-clique from $\mathcal{C}$ is at most $\frac{1}{r} \cdot \#K_{r-1}(G) \cdot \frac{r}{2} \cdot \frac{N}{C_{r-1}n^{r-1}} \leq \frac{1}{2} \cdot \#K_r(G)$; hence there are at least $\frac{1}{2} \cdot \#K_r(G)$ r-cliques which contain some $(r-1)$-clique from $\mathcal{C}$. For each $X \in \mathcal{C}$, let $N(X)$ be the set of vertices y such that $X \cup \{y\}$ is an r-clique. By definition, $|N(X)| \geq \frac{r}{2} \cdot \frac{N}{C_{r-1}n^{r-1}} \geq 4n$. By the case $r = 1$ of the lemma, applied to the graph $G[N(x)]$, we have $e(N(X)) \geq \frac{|N(X)|^2}{4n}$. Summing over all $X \in \mathcal{C}$, we see that

$$\begin{aligned}\binom{r+1}{2} \cdot \#K_{r+1}(G) &\geq \sum_{X \in \mathcal{C}} e(N(X)) \geq \sum_{X \in \mathcal{C}} \frac{|N(X)|^2}{4n} \geq \frac{1}{4n} \cdot \frac{r}{2} \cdot \frac{N}{C_{r-1}n^{r-1}} \cdot \sum_{X \in \mathcal{C}} |N(X)| \\ &\geq \frac{r}{8} \cdot \frac{N}{C_{r-1}n^r} \cdot \frac{\#K_r(G)}{2} = \binom{r+1}{2} \cdot \frac{N}{C_r n^r} \cdot \#K_r(G).\end{aligned}$$

$\square$

In the proof of Proposition 1.1, it is convenient to work with a tree-decomposition of H in which all bags have size $\text{tw}(H)+1$, and every two adjacent bags intersect in $\text{tw}(H)$ vertices. It is well-known that such a tree decomposition always exists, see e.g. [11, Lemma 2].

Lemma 5.2 ([11])**.** *Let H be a graph with $r := tw(H)$. Then there is a tree-decomposition of H in which every bag has size $r+1$ and every two adjacent bags intersect in r vertices.*

We are now ready to prove Proposition 1.1, which we restate here for convenience.

Proposition 1.1. *For every fixed graph H we have $R(H, K_n) = O(n^{tw(H)})$.*

Proof. Put $r := \text{tw}(H)$. Take a tree-decomposition of H with the properties guaranteed in Lemma 5.2; let T be the corresponding tree, and let $B(t)$ be the bag corresponding to $t \in V(T)$. Let H' be the graph obtained by making each bag $B(t)$ a clique; so $V(H') = V(H)$ and H' contains H as a subgraph. We will show that $R(H', K_n) = O(n^r)$. Let G be a graph on $N = Cn^r$ vertices with no independent set of size n. By Lemma 5.1, if C is large enough then $\#K_{r+1}(G) > (v(H) - r - 1) \cdot \#K_r(G)$. We now run the following process with sets $\mathcal{C}_{r+1}, \mathcal{C}_r$. Initialize $\mathcal{C}_{r+1}$ to be the set of all $(r+1)$-cliques in G, and $\mathcal{C}_r$ to be the set of all r-cliques in G. As long as there is $X \in \mathcal{C}_r$ such that the number of $Y \in \mathcal{C}_{r+1}$ containing X is at most $v(H) - r - 1$, delete X from $\mathcal{C}_r$ and delete all such Y from $\mathcal{C}_{r+1}$. The number of elements of $\mathcal{C}_{r+1}$ deleted throughout the process is at most $\#K_r(G) \cdot (v(H) - r - 1) < \#K_{r+1}(G)$. Hence, the terminal set $\mathcal{C}_{r+1}$ is non-empty. By construction, this set has the property that for every $Y \in \mathcal{C}_{r+1}$ and every $X \subseteq Y$, $|X| = r$, there are at least $v(H) - r$ sets $Y' \in \mathcal{C}_{r+1}$ which contain X.

Fix an order $t_1, \dots, t_m$ of $V(T)$ such that t_i has exactly one neighbour in $\{t_1, \dots, t_{i-1}\}$. We now embed $B(t_1), \dots, B(t_m)$ one-by-one, such that the image of each $B(t_i)$ equals some $Y_i \in \mathcal{C}_{r+1}$. Fix an arbitrary $Y_1 \in \mathcal{C}_{r+1}$ and embed $B(t_1)$ onto Y_1. For $i \geq 2$, suppose that we already embedded $B(t_1), \dots, B(t_{i-1})$. There is a unique $1 \leq j \leq i-1$ such that t_j is a neighbour of t_i. By the definition of tree-decomposition, we have $B(t_i) \cap (B(t_1) \cup \dots \cup B(t_{i-1})) = B(t_i) \cap B(t_j)$. By our choice of the tree-decomposition and of H', the intersection $B(t_i) \cap B(t_j)$ is an r-clique. Hence, there is a unique vertex $v \in B(t_i) \setminus B(t_j)$. Let $X \subseteq Y_j$ be the r-clique playing the role of $B(t_i) \cap B(t_j)$. There are at least $v(H) - r$ different $(r+1)$-cliques $Y \in \mathcal{C}_{r+1}$ containing X; hence for one of these Y, the (unique) vertex in $Y \setminus X$ is "new", i.e. not contained in $Y_1 \cup \dots \cup Y_{i-1}$. We can now embed $B(t_i)$ onto $Y_i := Y$, mapping v to this new vertex. This completes the proof. $\square$

5.2 Proof of Theorem 2

The following lemma, appearing in [8], allows us to assume that H is 2-connected. For completeness, we include a proof.

Lemma 5.3 ([8]). *Let H be a graph obtained from graphs H_1, H_2 by gluing them together along a vertex. Then for every graph F, $R(H, F) = O(R(H_1, F) + R(H_2, F))$.*

Proof. Put $M = \max\{R(H_1, F), R(H_2, F)\}$, and let G be a graph on $N = (|V(H_1)| + 1) \cdot M$ vertices such that $\overline{G}$ contains no copy of F. We can then find in G vertex-disjoint copies $H_1^{(1)}, \dots, H_1^{(M)}$ of H_1. Let v be the unique common vertex of H_1 and H_2. For $i = 1, \dots, M$, let $v^{(i)}$ be the vertex of $H_1^{(i)}$ playing the role of v. The subgraph of G induced on $\{v^{(1)}, \dots, v^{(M)}\}$ contains a copy H_2' of H_2. Let $1 \leq i \leq M$ such that $v^{(i)}$ plays the role of v in H_2'. Then $H_1^{(i)} \cup H_2'$ form a copy of H. $\square$

Corollary 5 ([8]). *Let H be a graph with biconnected components $H_1, \dots, H_m$. Then for every graph F, $R(H, F) = O(R(H_1, F) + \dots + R(H_m, F))$.*

Proof of Theorem 2. By Corollary 5, we may assume that H is 2-connected. Indeed, let $H_1, \dots, H_m$ be the biconnected components of H. By Corollary 5, it is enough to prove that $R(H_i, K_n) = O(n^3)$ for every $i = 1, \dots, m$. Also, $e(H_i) - v(H_i) \leq e(H) - v(H) \leq 4$, because H is connected. Hence, from now on we assume that H is 2-connected.

Suppose first that $v(H) \leq 5$. In this case we show that $\text{tw}(H) \leq 3$, which would imply that $R(H, K_n) = O(n^3)$ by Proposition 1.1. If $v(H) = 4$ then $\text{tw}(H) \leq \text{tw}(K_4) = 3$. If $v(H) = 5$ then $e(H) \leq v(H) + 4 = 9$, so H is contained in $K_5 - e$. Note that $K_5 - e$ is obtained by gluing two copies of K_4 along a triangle. It is now easy to see that $\text{tw}(K_5 - e) \leq 3$, as required.

For the rest of the proof, suppose that $v(H) \geq 6$. If $\Delta(H) \leq 2$ then H is a cycle or a path, and it is well-known that in this case $R(H, K_n) = O(n^2)$ (for example, this follows from the case $k = 2$ of Conjecture 1, which was proved in [8]). Let $v \in V(H)$ be a vertex of maximum degree, $d_H(v) \geq 3$. Let $H' = H - v$. Note that H' is connected because H is 2-connected. Also, $e(H') - v(H') = (e(H) - d_H(v)) - (v(H) - 1) = e(H) - v(H) - d_H(v) + 1 \leq 5 - d_H(v)$. We claim that $R(H', K_n) = O(n^2)$. If $d_H(v) \geq 4$ then $e(H') - v(H') \leq 1$, so $R(H', K_n) = O(n^2)$ follows from the case $k = 2$ of Conjecture 1, which was proved in [8]. Suppose now that $d_H(v) = 3$, so $e(H') - v(H') \leq 2$. If $\text{tw}(H') \leq 2$ then $R(H', K_n) = O(n^2)$ by Proposition 1.1, so suppose that $\text{tw}(H') > 2$. It is known (see e.g. [6]) that a graph has treewidth larger than 2 if and only if it contains a subdivision of K_4. So H' contains a subdivision S of K_4. Observe that $e(S) - v(S) = 2$ (this holds for every subdivision of K_4). This implies that $e(H') - v(H') = 2$, and that every 2-connected component of H' other than S is a singleton (this can also be stated as saying that the 2-core of H' is S). It now follows from Corollary 5 that $R(H', K_n) = O(R(S, K_n))$. Now, if $v(S) \geq 6$ then by Theorem 4 we have $R(S, K_n) = O(n^2)$ and hence $R(H', K_n) = O(n^2)$. So suppose that $v(S) \leq 5$. If $v(S) = 4$, namely $S \cong K_4$, then, since

H is connected and has maximum degree 3, it holds that $H \cong K_4$, in contradiction to $v(H) \geq 6$. If $v(S) = 5$ then $S \cong K_4^*$. Note that K_4^* has four vertices of degree 3 and one vertex of degree 2. Let u be this vertex of degree 2 in S. We have $V(H) \setminus V(S) \neq \emptyset$ because $v(H) \geq 6$. Also, there are no edges in H between $V(S) \setminus \{u\}$ and $V(H) \setminus V(S)$, because the vertices in $V(S) \setminus \{u\}$ have degree 3 in S and $\Delta(H) = 3$. So u is a cut vertex of H, in contradiction to the fact that H is 2-connected. This concludes the proof of $R(H', K_n) = O(n^2)$. Now let G be a graph on $N = Cn^3$ vertices with no independent set of size n. There exists $x \in V(G)$ with $d(x) \geq Cn^2 - 1$. By choosing C large enough, we can make sure that $d(x) \geq R(H', K_n)$. Then, $G[N(x)]$ contains a copy of H'. Together with x, we get a copy of H, as required. □

5.3 On the Ramsey number $R(H, K_{n,n})$

We begin by proving upper bounds on $R(H, K_{n,n})$ for graphs H with $\Delta(H) = r$ and for r-degenerate H. Both of our results follow from Lemma 5.5 below. First, we need the following definition.

Definition 5.4. *We say that a graph H is* r-strongly-degenerate *if there exists an ordering $v_1, \ldots, v_h$ of its vertices such that for all $i \in [h]$ one of the following holds:*

a) $|N(v_i) \cap \{v_1, \ldots, v_{i-1}\}| \leq r - 1$, *or*

b) $d(v_i) \leq r$.

Equivalently, a graph H is r-strongly-degenerate if its subgraph induced by the set of vertices with degree larger than r is $(r-1)$-degenerate.

Lemma 5.5. *For any r-strongly-degenerate graph H on h vertices, $R(H, K_{n,n}) \leq h^2 n^r$.*

Proof. Let $u_1, \ldots, u_h$ be an ordering of the vertices of H certifying that H is r-strongly-degenerate. We denote $d_i(j) = |N_H(u_j) \cap \{u_1, \ldots, u_{i-1}\}|$. Consider an arbitrary graph G on $N = h^2 n^r$ vertices. We show how to find either a copy of H or a copy of $\overline{K_{n,n}}$. Split the vertex-set into h parts $V_1, \ldots, V_h$ each of size hn^r. We will try to find an embedding $\phi \colon H \to V$ such that $\phi(u_j) \in V_j$, for all $j \in [h]$. For this purpose, we will maintain sets $A_{i,j}$ into which we can embed the vertices, starting with $A_{1,j} = V_j$, $j \in [h]$. We will maintain the following. For any $1 \leq i < j \leq h$,

$$|A_{i,j}| \geq h \cdot n^{r - d_i(j)}, \tag{7}$$

which is trivially satisfied for $i = 1$.

Next we describe how to embed H. Suppose we have embedded $u_1, \ldots, u_{i-1}$ and we wish to embed u_i. First suppose there exists a vertex $v \in A_{i,i}$ satisfying $|N(v) \cap A_{i,j}| \geq hn^{r - d_{i+1}(j)}$, for all $j > i$ such that $(u_i, u_j) \in E(H)$. Then, we set $\phi(u_i) = v$ and update the sets as follows:

$$A_{i+1,j} = \begin{cases} A_{i,j} \cap N(v) & \text{if } (u_i, u_j) \in E(H), \\ A_{i,j} & \text{otherwise.} \end{cases}$$

It directly follows that (7) is still satisfied. If we can embed all h vertices in this manner, we obtain a copy of H. Hence, for some i, there is no vertex $v \in A_{i,i}$ satisfying $|N(v) \cap A_{i,j}| \geq n^{r - d_{i+1}(j)}$, for all $j > i$ such that $(u_i, u_j) \in E(H)$. Since $A_{i,i} \neq \emptyset$ (by (7)), u_i has a neighbour u_j in H with $j > i$. By definition of an r-strongly-degenerate graph, it follows that $d_i(i) \leq r - 1$ so $|A_{i,i}| \geq hn$ by (7). By the pigeonhole principle, there is an index $k > i$ such that for at least n vertices $v \in A_{i,i}$ we have $|N(v) \cap A_{i,k}| < hn^{r - d_{i+1}(k)}$. Let $S \subseteq A_{i,i}$ be a set of n such vertices and let $R = A_{i,k} \setminus \bigcup_{v \in S} N(v)$. By assumption $|N(v) \cap A_{i,k}| \leq hn^{r - d_{i+1}(k)} - 1$ for every $v \in S$. Note that $d_{i+1}(k) = 1 + d_i(k)$ since $(u_i, u_k) \in E(H)$. Therefore,

$$|R| \geq |A_{i,k}| - n \cdot (hn^{r - d_{i+1}(k)} - 1) \geq hn^{r - d_i(k)} - n \cdot (hn^{r - d_{i+1}(k)} - 1) = n.$$

By construction, $G[R, S]$ is empty which completes the proof. □

Note that every graph with maximum degree r is r-strongly-degenerate, and every r-degenerate graph is $(r+1)$-strongly-degenerate. Hence, we have the following corollaries.

Corollary 6. *For any graph H, $R(H, K_{n,n}) = O(n^{\Delta(H)})$.*

Corollary 7. *For any r-degenerate graph H, $R(H, K_{n,n}) = O(n^{r+1})$.*

Finally, we show that Conjecture 1 holds if K_n is replaced with $K_{n,n}$.

Proposition 5.6. *Let $k \geq 1$. For every connected graph H with $e(H) - v(H) \leq \binom{k+1}{2} - 2$ it holds that $R(H, K_{n,n}) = O(n^k)$.*

Proof. The proof is by induction on k. As in the proof of Theorem 2, we may assume that H is 2-connected due to Corollary 5. If $\Delta(H) \leq k$ then we are done by Corollary 6. Else, let $v \in V(H)$ with $d(v) \geq k+1$, and let $H' = H - v$. Then $e(H') - v(H') \leq e(H) - (k+1) - v(H) + 1 \leq \binom{k}{2} - 2$. Also, H' is connected because H is 2-connected. So by the induction hypothesis, we have $R(H', K_{n,n}) = O(n^{k-1})$. Now let G be a graph on $N = Cn^k$ vertices with no $\overline{K_{n,n}}$. Then G has no independent set of size $2n$. Hence, there exists $x \in V(G)$ with $d(x) \geq \frac{1}{2}Cn^{k-1} - 1$. By choosing C large enough, we can make sure that $d(x) \geq R(H', K_{n,n})$. Then $G[N(x)]$ contains a copy of H', which gives a copy of H together with x. $\square$

6 Concluding remarks and open problems

- It is worth mentioning an intriguing conjecture of Alon, Krivelevich and Sudakov [2], that $R(H, K_n) \leq n^{O(r)}$ for every graph H with $\Delta(H) \leq r$. Using the dependent random choice method, [2] showed that $R(H, K_n) = O(n^{(2r-k+2)\cdot(k-1)/2})$, where $k = \chi(H)$. So in the worst case $k = r$, the exponent is quadratic in r. The problem for $K_{n,n}$ (in place of K_n) turned out to be much easier and is resolved in Corollary 6.

- In Corollary 7 we showed that $R(H, K_{n,n}) = O(n^{r+1})$ for an r-degenerate graph H. Can this be improved to $O(n^r)$? In particular, it would be very interesting to show that $R(H, K_{n,n}) = O(n^2)$ for every 2-degenerate graph H.

- Balister et al. [3] asked whether it is true that if H is 2-connected and has minimum degree 3, then H is **not** Ramsey size-linear. A recent result of Janzer [12] gives a negative answer to this question. Indeed, [12] constructed 2-connected 3-regular bipartite graphs H which have Turán number at most $O(n^{3/2})$ (in fact, at most $O(n^{4/3+\varepsilon})$). Erdős et al. [8] observed that a bipartite graph H with Turán number at most $O(n^{3/2})$ is Ramsey size-linear. Hence, the graphs of [12] are Ramsey size-linear.

- Theorem 3 implies that $R(K_4^*, K_{n,n}) = O(n^2)$. For K_n, it is not difficult to prove that $R(K_4^*, K_n) = O(n^{5/2})$. Indeed, suppose that G has $N = Cn^{5/2}$ vertices and no independent set of size n. We may assume that $\delta(G) \geq \Omega(N/n)$, and then the average degree inside each neighbourhood is $\Omega(N/n^2)$. Also, each neighbourhood is C_4-free or else G contains K_4^* and we are done. It follows that there are at least $N \cdot \Omega(N/n) \cdot \Omega(N/n^2)^2 \geq cN^4/n^5$ 4-tuples x, y, z, w with $x \sim y, z, w$ and $y \sim z, w$. Here c is some small absolute constant. On the other hand, the number of such 4-tuples x, y, z, w with $d(z, w) \leq 2$ is at most $4\binom{N}{2} \leq 2N^2$. For $N = Cn^{5/2}$ with large enough C (compared to c), we have $cN^4/n^5 > 2N^2$, so there is a 4-tuple x, y, z, w with $d(z, w) \geq 3$. This gives a K_4^*. It would be interesting to reduce the exponent $5/2$, hopefully all the way to 2.

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