TWO EXTENSIONS OF RAMSEY'S THEOREM

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Abstract

Ramsey's theorem, in the version of Erdős and Szekeres, states that every 2-coloring of the edges of the complete graph on $\{1, 2, \ldots, n\}$ contains a monochromatic clique of order $(1/2) \log n$. In this article, we consider two well-studied extensions of Ramsey's theorem. Improving a result of Rödl, we show that there is a constant $c > 0$ such that every 2-coloring of the edges of the complete graph on $\{2, 3, \ldots, n\}$ contains a monochromatic clique S for which the sum of $1/\log i$ over all vertices $i \in S$ is at least $c \log \log \log n$. This is tight up to the constant factor c and answers a question of Erdős from 1981. Motivated by a problem in model theory, Väänänen asked whether for every k there is an n such that the following holds: for every permutation π of $1, \ldots, k-1$, every 2-coloring of the edges of the complete graph on $\{1, 2, \ldots, n\}$ contains a monochromatic clique $a_1 < \cdots < a_k$ with

$$a_{\pi(1)+1} - a_{\pi(1)} > a_{\pi(2)+1} - a_{\pi(2)} > \cdots > a_{\pi(k-1)+1} - a_{\pi(k-1)}.$$

That is, not only do we want a monochromatic clique, but the differences between consecutive vertices must satisfy a prescribed order. Alon and, independently, Erdős, Hajnal, and Pach answered this question affirmatively. Alon further conjectured that the true growth rate should be exponential in k. We make progress towards this conjecture, obtaining an upper bound on n which is exponential in a power of k. This improves a result of Shelah, who showed that n is at most double-exponential in k.

1. Introduction

Ramsey theory refers to a large body of deep results in mathematics whose underlying philosophy is captured succinctly by the statement that "Every large system contains a large well-organized subsystem." This subject is currently one of the most active areas of research within combinatorics, overlapping substantially with number theory,

geometry, analysis, logic, and computer science (see [13] for details). The cornerstone of this area is Ramsey's theorem, which guarantees the existence of Ramsey numbers.

The Ramsey number $r(k)$ is the minimum n such that in every 2-coloring of the edges of the complete graph K_n there is a monochromatic K_k. Ramsey's theorem [17] states that $r(k)$ exists for all k. Classical results of Erdős [5] and Erdős and Szekeres [10] give the quantitative bounds $2^{k/2} \leq r(k) \leq 2^{2k}$ for $k \geq 2$. Over the last sixty years, there have been several improvements on these bounds (see, e.g., [2]). However, despite efforts by various researchers, the constant factors in the above exponents remain the same.

Given these difficulties, it is natural that the field has stretched in different directions. One such direction is to try to strengthen Ramsey's theorem, asking that the monochromatic clique have some additional structure. This allows us to test the limits of current methods and may also lead to the development of new techniques which could be relevant to the original problem of estimating classical Ramsey numbers. Furthermore, for some applications such additional structure is needed. One famous example of a theorem that strengthens Ramsey's theorem is the Paris–Harrington theorem [14]. In the following, we consider two further strengthenings, both of which have already been studied in some detail.

1.1. Ramsey's theorem with skewed vertex distribution

In the early 1980s, Erdős, interested in the distribution of monochromatic cliques in edge-colorings, considered the following variant of Ramsey's theorem. For a finite set S of integers greater than 1, define its weight $w(S)$ by

$$w(S) = \sum_{s \in S} \frac{1}{\log s}.$$

For a red-blue edge-coloring c of the edges of the complete graph on $[2, n] = \{2, \ldots, n\}$, let $f(c)$ be the maximum weight $w(S)$ over all sets $S \subset [2, n]$ which form a monochromatic clique in coloring c. For each integer $n \geq 2$, let $f(n)$ be the minimum of $f(c)$ over all red-blue edge-colorings c of the edges of the complete graph on $\{2, \ldots, n\}$. Note that a simple application of $r(k) \leq 2^{2k}$ only gives $f(n) \geq (\log n/2) \cdot (1/\log n) = 1/2$.

In his article [6], Erdős conjectured that $f(n)$ tends to infinity and asked for an accurate estimate of $f(n)$. Soon after, Rödl [18] verified this conjecture, showing that $f(n) = \Omega(\frac{\log \log \log \log n}{\log \log \log \log \log n})$. In the other direction, by considering a uniform random coloring of the edges, one can easily obtain $f(n) = O(\log \log n)$. Rödl [18] improved this upper bound further to $f(n) = O(\log \log \log n)$. Nevertheless, there was still an exponential gap between the bounds for $f(n)$.

We will now discuss Rödl's coloring that gives the estimate $f(n) = O(\log \log \log n)$. Cover the interval $[2, n]$ by $t = \lceil \log \log n \rceil$ intervals, where the ith

interval is $[2^{2^{i-1}}, 2^{2^i})$. We first describe the coloring of the edges within each of these t intervals, and then the coloring of the edges between these intervals. Using the fact that the Ramsey number $r(k) \geq 2^{k/2}$, we can edge-color the complete graph in the ith interval so that the maximum monochromatic clique in this interval has order 2^{i+1}. Also note that the logarithm of any element in the ith interval is at least 2^{i-1}. Therefore, the maximum weight of any monochromatic clique in this interval is at most 4. It follows again from the lower bound on $r(k)$ that there is a red-blue edge-coloring of the complete graph on $t = \lceil \log \log n \rceil$ vertices whose largest monochromatic clique is of order $O(\log t)$. Color the edges of the complete bipartite graph between the ith and jth interval by the color of edge (i, j) in this coloring. We get a red-blue edge-coloring of the complete graph on $[2, n]$ such that any monochromatic clique in this coloring has a nonempty intersection with at most $O(\log t)$ intervals. Since, as we have already explained, every interval can contribute at most 4 to the weight of this clique, the total weight of any monochromatic clique is $O(\log t) = O(\log \log \log n)$.

The key idea behind Rödl's lower bound for $f(n)$ is to try and force the type of situation that arises in this upper bound construction. We follow the basic line of his argument but add two extra ideas, dependent random choice and a weighted variant of Ramsey's theorem, to achieve a tight result. That is, we prove that $f(n) = \Omega(\log \log \log n)$, which, by the above construction of Rödl, is tight up to a constant factor. This determines the growth rate of $f(n)$ and answers Erdős's question.

THEOREM 1.1
For n sufficiently large, every 2-coloring of the edges of the complete graph on the interval $\{2, \ldots, n\}$ contains a monochromatic clique with vertex set S such that

$$\sum_{s \in S} \frac{1}{\log s} \geq 2^{-8} \log \log \log n.$$

Hence, $f(n) = \Theta(\log \log \log n)$.

Ramsey's theorem continues to hold if we use more than 2 colors. We define the Ramsey number $r(k; q)$ to be the minimum n such that in every q-coloring of the edges of the complete graph K_n there is a monochromatic K_k. The upper bound proof of Erdős and Szekeres [10] implies that $r(k; q) \leq q^{qk}$. On the other hand, a simple product coloring shows that, for q even, $r(k; q) \geq r(k; 2)^{q/2} \geq 2^{kq/4}$. Phrased differently, we see that any q-coloring of K_n contains a monochromatic clique of size $c_q \log n$ and that this is, up to the constant, the best possible.

It therefore makes sense to consider the function $f_q(n)$, defined now as the minimum over all q-colorings of the edges of the complete graph on $\{2, 3, \ldots, n\}$ of the maximum weight of a monochromatic clique. However, as observed by Rödl, the analogue of Erdős's conjecture for three colors instead of two does not hold. Indeed,

again cover the interval $[2, n]$ by $t = \lceil \log \log n \rceil$ intervals, where the ith interval is $[2^{2^{i-1}}, 2^{2^i})$. The edges inside the intervals are colored red-blue as in the above construction and the edges between the intervals are colored green. Then the maximum weight of any red or blue clique is at most 4, since any such clique must lie completely within one of the intervals, and the maximum weight of the green clique is at most $\sum_{i \geq 1} 2^{-i+1} \leq 2$.

1.2. Ramsey's theorem with fixed-order type

We also consider another extension of Ramsey's theorem. For a positive integer n, let $[n] = \{1, \ldots, n\}$. Motivated by an application in model theory, Väänänen asked (see [16] and [19]) whether, for any positive integers k and q and any permutation π of $[k-1]$, there is a positive integer R such that, for any q-coloring of the edges of the complete graph on vertex set $[R]$, there is a monochromatic K_k with vertices $a_1 < \cdots < a_k$ satisfying

$$a_{\pi(1)+1} - a_{\pi(1)} > a_{\pi(2)+1} - a_{\pi(2)} > \cdots > a_{\pi(k-1)+1} - a_{\pi(k-1)}.$$

That is, we not only want a monochromatic K_k, but the differences between consecutive vertices must satisfy a prescribed order. The least such positive integer R is denoted by $R_\pi(k; q)$, and we let $R(k; q) = \max_\pi R_\pi(k; q)$; that is, $R(k; q)$ is the maximum of $R_\pi(k; q)$ over all permutations π of $[k-1]$.

Väänänen's question was popularized by Spencer, and it was positively answered by Alon and, independently, by Erdős, Hajnal, and Pach [7]. Alon's proof (see [16]) uses the Gallai–Witt theorem and gives a weak bound on $R(k; q)$. The proof by Erdős, Hajnal, and Pach uses a compactness argument and gives no bound on $R(k; q)$. Later, Alon, Shelah, and Stacey all independently found proofs giving tower-type bounds for $R(k; q)$, though these were never published (see [19]).

A natural conjecture, made by Alon (see [19]), is that $R(k; q)$ should grow exponentially in k. For monotone sequences, this was confirmed by Alon and Spencer. A breakthrough on this problem was obtained by Shelah in [19], who proved the double-exponential upper bound $R(k; q) \leq 2^{(q(k+1)^3)^{qk}}$. Here, we make further progress, showing that, for fixed q, $R(k; q)$ grows as a single exponential in a power of k.

THEOREM 1.2

For any positive integers k and q and any permutation π of $[k-1]$, every q-coloring of the edges of the complete graph on vertex set $[R]$ with $R = 2^{k^{20q}}$ contains a monochromatic K_k with vertices $a_1 < \cdots < a_k$ satisfying

$$a_{\pi(1)+1} - a_{\pi(1)} > a_{\pi(2)+1} - a_{\pi(2)} > \cdots > a_{\pi(k-1)+1} - a_{\pi(k-1)}.$$

That is, $R(k; q) \leq 2^{k^{20q}}$.

Common to the proofs of both Theorems 1.1 and 1.2 is a simple yet power-ful lemma whose proof, which we present in the next section, uses a probabilistic argument known as *dependent random choice*. Early versions of this technique were developed in [12], [15], and [20]. Several variants have since been discovered and applied to various problems in Ramsey theory and extremal graph theory (see the survey [11] for more details).

Organization of the paper. We prove Theorem 1.1 in Section 3 and Theorem 1.2 in Section 4. In Section 3, we make use of a weighted variant of Ramsey's theorem, Lemma 3.2, which may be of independent interest. In Section 5, we make several additional related remarks. These include discussing the asymptotic behavior of $f(n)$, considering what happens for other weight functions, showing that some natural vari-ants of both problems have simple counterexamples, and presenting a simple coloring that gives a lower bound on Ramsey numbers for cliques with increasing consecutive differences. All logarithms are base 2 unless otherwise indicated. For the sake of clar-ity of presentation, we systematically omit floor and ceiling signs whenever they are not crucial. We also do not make any serious attempt to optimize absolute constants in our statements and proofs.

2. Dependent random choice

The following lemma shows that every dense graph contains a large vertex subset U such that every small subset $S \subset U$ has many common neighbors. For a vertex v in a graph, let $N(v)$ denote the set of neighbors of v. For a set T of vertices, let $N(T)$ denote the set of common neighbors of T.

LEMMA 2.1
Suppose that $p > 0$ and that s, t, N_1, N_2 are positive integers satisfying $\binom{N_1}{s}(m/N_2)^t \leq p^t N_1/2$. If $G = (V_1, V_2, E)$ is a bipartite graph with $|V_i| = N_i$ for $i = 1, 2$ and at least pN_1N_2 edges, then G has a vertex subset $U \subset V_1$ such that $|U| \geq p^t N_1/2$ and all s vertices in U have at least m common neighbors.

Proof
Consider a set T of t vertices in V_2 picked uniformly at random with repetition. Let $W = N(T)$ and let X denote the cardinality of W. We have

$$\mathbb{E}[X] = \sum_{v \in V_1}\left(\frac{|N(v)|}{N_2}\right)^t = N_2^{-t}\sum_{v \in V_1}|N(v)|^t \geq N_1 N_2^{-t}\left(\frac{\sum_{v \in V_1}|N(v)|}{N_1}\right)^t \geq p^t N_1,$$

where the second-to-last inequality is by Jensen's inequality applied to the convex function $f(z) = z^t$.

Let Y be the random variable which counts the number of subsets $S \subset W$ of size s with fewer than m common neighbors. For a given $S \subset V_1$, the probability that it is a subset of W equals $(|N(S)|/N_2)^t$. Since there are at most $\binom{N_1}{s}$ such sets, it follows that

$$\mathbb{E}[Y] \leq \binom{N_1}{s} \left(\frac{m}{N_2} \right)^t.$$

By linearity of expectation,

$$\mathbb{E}[X - Y] = \mathbb{E}[X] - \mathbb{E}[Y] \geq p^t N_1 - \binom{N_1}{s} \left(\frac{m}{N_2} \right)^t \geq p^t N_1/2,$$

where the last inequality uses the assumption of the lemma. Hence, there is a choice of T such that the corresponding set W satisfies $X - Y \geq p^t N_1/2$. Delete one vertex from each subset S of W of size s with fewer than m common neighbors. We let U be the remaining subset of W. We have $|U| \geq X - Y \geq p^t N_1/2$ and all subsets of size s have at least m common neighbors. $\square$

3. Monochromatic cliques of large weight

The off-diagonal Ramsey number is the smallest natural number n such that any red-blue edge-coloring of K_n contains either a red copy of K_s or a blue copy of K_t. The Erdős–Szekeres bound for Ramsey numbers says that, for any $s, t \geq 2$,

$$r(s,t) \leq \binom{s+t-2}{s-1}.$$

Note that this implies $r(s,t) \leq 2^{s+t}$ and hence that every 2-coloring of K_n contains a monochromatic clique of order $(1/2) \log n$. The following lemma is a further simple consequence of this formula. Note that here and throughout the rest of this section we will use the natural logarithm $\ln$ as well as the log base 2.

LEMMA 3.1
Suppose that $0 < a \leq 1/4$. Then, every 2-coloring of the edges of K_n contains either a red clique of order $a \ln n$ or a blue clique of order $e^{1/4a} \ln n$.

Proof
From the Erdős–Szekeres bound, we have

$$r(s,t) \leq \binom{s+t}{s} \leq \left(\frac{e(s+t)}{s} \right)^s.$$

Applying this with $s = a \ln n$ and $t = a(e^{(1/a)-1} - 1) \ln n$ tells us that, since

$$\left(\frac{e(a \ln n + a(e^{(1/a)-1} - 1) \ln n)}{a \ln n} \right)^{a \ln n} = e^{\ln n} = n,$$

there is either a red clique of order s or a blue clique of order t. For $0 < a \le 1/4$, we have

$$a(e^{(1/a)-1} - 1) \ge e^{1/4a}.$$

The result follows.   □

We would now like to prove a weighted version of Ramsey's theorem. The set-up is that each vertex v is given two weights r_v and b_v which are balanced in a certain sense. We would then like to show that it is possible to find a red clique K or a blue clique L for which either the sum of r_v over the vertices of K, or the sum of b_v over the vertices of L, is large.

LEMMA 3.2
Suppose that the edges of K_n have been two-colored in red and blue and that each vertex v has been given positive weights r_v and b_v satisfying $b_v \ge \ln(4/r_v)$ if $r_v \le b_v$ and satisfying $r_v \ge \ln(4/b_v)$ if $b_v \le r_v$. Then there exists either a red clique K for which $\sum_{v \in K} r_v \ge (1/2) \ln n$ or a blue clique L for which $\sum_{v \in L} b_v \ge (1/2) \ln n$.

Proof
Let $w(n)$ be the infimum, over all red-blue edge-colorings of K_n, for the sum of the maximum of $\sum_{v \in K} r_v$ over all red cliques K and the maximum of $\sum_{v \in L} b_v$ over all blue cliques L. We will show by induction on n that $w(n) \ge \ln n$. This clearly implies the desired bound.

The base cases $n = 1, 2$ clearly hold. Suppose, therefore, that $n \ge 3$ and that $w(n') \ge \ln n'$ for all positive integers $n' < n$.

Consider a red-blue edge-coloring of K_n, and let w be the sum of the maximum of $\sum_{v \in K} r_v$ over all red cliques K and the maximum of $\sum_{v \in L} b_v$ over all blue cliques L. It suffices to show that $w \ge \ln n$.

Let v be a vertex in K_n. By symmetry, we may suppose without loss of generality that $r_v \ge b_v$. Since $r_v \ge \ln(4/b_v)$ and $r_v \ge b_v$, we have $r_v \ge 1$. We may assume $r_v \le \ln n$ as otherwise we could pick the red clique K to consist of just the vertex v. Hence, $b_v \ge 4/n$.

Let R be the set of red neighbors of v, and let B be the set of blue neighbors of v, so $|R| + |B| = n - 1$. Let $\alpha = |R|/n$. We can add v to the largest red clique in R in terms of weight, and thus $w \ge r_v + w(\alpha n) \ge r_v + \ln(\alpha n) \ge r_v + \ln \alpha + \ln n$.

We may assume that $r_v + \ln \alpha < 0$, as otherwise we are done. So $\alpha < e^{-r_v} \leq b_v/4$. From $r_v \geq 1$, we have $\alpha < 1/e$. From the above lower bounds on b_v, we have $b_v \geq 4 \max(\alpha, \frac{1}{n}) \geq 2\beta$, where $\beta = \alpha + \frac{1}{n} < 1/e + 1/3 < 3/4$. We can add v to the largest blue clique in B in terms of weight, and thus

$$w \geq b_v + w(|B|) \geq b_v + \ln\left(1 - \alpha - \frac{1}{n}\right) + \ln n \geq 2\beta + \ln(1 - \beta) + \ln n \geq \ln n,$$

where we used $0 < \beta < 3/4$, which completes the proof. $\square$

Scaling all weights by a factor $c > 0$, we have the following equivalent version.

LEMMA 3.3
Let $c > 0$. Suppose that the edges of K_n have been two-colored in red and blue and that each vertex v has been given positive weights r_v and b_v satisfying $b_v \geq c \ln(4c/r_v)$ if $r_v \leq b_v$ and satisfying $r_v \geq c \ln(4c/b_v)$ if $b_v \leq r_v$. Then there exists either a red clique K for which $\sum_{v \in K} r_v \geq (c/2) \ln n$ or a blue clique L for which $\sum_{v \in L} b_v \geq (c/2) \ln n$.

Before proving Theorem 1.1, we sketch the proof. We begin with a collection of $d = O(\sqrt{\log \log n})$ disjoint subsets $S_{1,0}, S_{2,0}, \ldots, S_{d,0}$ of $[2, n]$. We then find a sequence $T_{1,2} \supseteq T_{1,3} \supseteq \cdots \supseteq T_{1,d}$ of subsets of $S_{1,0}$, where $T_{1,k}$ has the property that there exists a color $\chi(1,k)$, either red or blue, such that all small subsets of $T_{1,k}$ have many common neighbors in $S_{k,0}$ in that color.

We now find maximum-sized red and blue cliques in $T_{1,d}$. If either of these cliques is sufficiently large, then that clique will be the desired clique for the whole theorem. We may therefore assume that they are both small. We let $S_{1,1} \subseteq T_{1,d} \subseteq S_{1,0}$ be their union. The fundamental property of $T_{1,d}$ now implies that, for $2 \leq k \leq d$, the set $S_{1,1}$ has many common neighbors in color $\chi(1,k)$ in $S_{k,0}$. We call this set of neighbors $S_{k,1}$.

We now repeat the entire process, first within $S_{2,1}$ to find sets $S_{k,2}$ for $2 \leq k \leq d$, then in $S_{3,2}$ to find sets $S_{k,3}$ for $3 \leq k \leq d$, and so on. In the end, we find subsets $S_{i,i}$ such that each $S_{i,i}$ is the union of a red and blue clique and the color of the edges between $S_{i,i}$ and $S_{j,j}$ only depends on i and j. If we consider the reduced graph on d vertices whose edges have color $\chi(i, j)$ and whose vertices have blue and red weights representing the sizes of the red and blue cliques in $S_{i,i}$, an application of Lemma 3.3 allows us to complete the proof.

We are now ready to prove Theorem 1.1, which we restate for convenience.

THEOREM 3.1
For sufficiently large n, in every red-blue edge-coloring of the complete graph on the

interval $\{2, \ldots, n\}$ *there is a monochromatic clique with vertex set* I *such that*

$$\sum_{i \in I} \frac{1}{\log i} \geq 2^{-8} \log \log \log n.$$

Proof

Let $d = (1/2)\sqrt{\log \log n} - 1$ and $c = 1/4$. For $i = 1, \ldots, d$, let $S_i = \{n_i, n_i + 1, \ldots, 2n_i - 1\}$ be the interval of size n_i beginning at the integer n_i, where $\log \log n_i = i\sqrt{\log \log n} + (1/2) \log \log n$. For each $j = 0, 1, \ldots, d$, we will find, by induction, a collection of sets $S_{i,j}$ for $i \geq j$ such that $S_{i,j} \subseteq S_{i,j-1} \subseteq \cdots \subseteq S_{i,0} = S_i$ and such that

- the set $S_{j,j}$ is the union of two monochromatic cliques, one in red of order $(1/4)r_j \log n_j$ and the other in blue of order $(1/4)b_j \log n_j$, where $r_j \geq c \ln(4c/b_j)$ if $b_j \leq r_j$ and $b_j \geq c \ln(4c/r_j)$ if $r_j \leq b_j$;
- for each $i > j$, the set $S_{i,j}$ satisfies $|S_{i,j}| \geq n_i^{1-(j/2i)}$;
- for each $i > j$, there exists a color $\chi(j,i)$ such that every vertex in $S_{j,j}$ is connected to every vertex in $S_{i,j}$ by an edge with color $\chi(j,i)$.

To begin the induction, we let $S_{i,0} = S_i$ for each i. The required conclusion then holds trivially for $j = 0$. Suppose therefore that the result holds for j. We will prove it also holds for $j + 1$.

For each $i \geq j + 1$, we will find a subset $S_{i,j+1}$ of $S_{i,j}$ satisfying the conditions. To do this we apply another induction, finding for each $j + 1 \leq k \leq d$, a subset $T_{j+1,k}$ of $S_{j+1,j}$ such that $T_{j+1,d} \subseteq T_{j+1,d-1} \subseteq \cdots \subseteq T_{j+1,j+1} = S_{j+1,j}$ and such that

- $|T_{j+1,k}| \geq n_{j+1}^{\frac{1}{2} - \frac{k}{4d}}$;
- for every $j + 1 < i \leq k$, there is a color $\chi(j + 1, i)$ such that every collection of $\log n_{j+1} \log \log \log n$ vertices in $T_{j+1,k}$ have at least $n_i^{1-\frac{j+1}{2i}}$ common neighbors in color $\chi(j + 1, i)$ in the set $S_{i,j}$.

Once this induction is complete, we consider $T_{j+1,d}$. Let $\frac{1}{4}r_{j+1} \ln n_{j+1}$ and $\frac{1}{4}b_{j+1} \ln n_{j+1}$ denote the orders of the largest red clique and the largest blue clique, respectively, in $T_{j+1,d}$. Since $|T_{j+1,d}| \geq n_{j+1}^{1/4}$, the remark before Lemma 3.1 implies that if $r_{j+1} \geq b_{j+1}$ then $T_{j+1,d}$ contains a red clique of order $\frac{1}{2} \log |T_{j+1,d}| \geq \frac{1}{8} \log n_{j+1}$. That is, $r_{j+1} \geq \frac{1}{2}$. Using Lemma 3.1, we see that if $r_{j+1} \geq b_{j+1}$, then either $b_{j+1} \geq 1/4$ and $r_{j+1} \geq 1/2 \geq c \ln(4c/b_{j+1})$ or $b_{j+1} < 1/4$ and $r_{j+1} \geq e^{1/(4b_{j+1})} \geq c \ln(4c/b_{j+1})$. Similarly, if $b_{j+1} \geq r_{j+1}$, then $b_{j+1} \geq c \ln(4c/r_{j+1})$.

Note that we may assume that r_{j+1} and b_{j+1} are each less than $(1/2) \log \log \log n$. Suppose otherwise and that $r_{j+1} \geq (1/2) \log \log \log n$. Let R_{j+1} be the red clique of order $(1/4)r_{j+1} \log n_{j+1}$. Then

$$\sum_{i \in R_{j+1}} \frac{1}{\log i} \geq \frac{1}{4} r_{j+1} \ln n_{j+1} \frac{1}{\log 2n_{j+1}} \geq \frac{1}{16} \log \log \log n,$$

so we would be done.

Let $S_{j+1,j+1}$ be the union of the largest red and blue cliques in $T_{j+1,d}$. Note that $r_{j+1} + b_{j+1} \leq \log \log \log n$. Hence, $|S_{j+1,j+1}| \leq \log n_{j+1} \log \log \log n$ and therefore, for every $j + 1 < i \leq d$, the collection of vertices in $S_{j+1,j+1}$ has at least $n_i^{1-\frac{j+1}{2i}}$ common neighbors, in color $\chi(j + 1, i)$, in $S_{i,j}$. We let this set of common neighbors be $S_{i,j+1}$. It is now elementary to verify that the $S_{i,j+1}$ satisfy the conditions of the first induction. Hence, it only remains to show that the second induction holds good.

To begin the induction, we let $T_{j+1,j+1}$ be $S_{j+1,j}$. This clearly satisfies the required conditions. Suppose, therefore, that $T_{j+1,k}$ has been defined and we now wish to find a subset $T_{j+1,k+1}$ of $T_{j+1,k}$ satisfying the conditions of the induction. Consider the graph between $T_{j+1,k}$ and $S_{k+1,j}$. Either red or blue will have density at least $1/2$ in this graph. We let $\chi(j + 1, k + 1)$ be such a color, breaking a tie arbitrarily.

Now apply Lemma 2.1 to the bipartite graph of color $\chi(j + 1, k + 1)$ between $T_{j+1,k}$ and $S_{k+1,j}$. We take $N_1 = |T_{j+1,k}|$, $N_2 = |S_{k+1,j}|$, $m = N_2^{1-1/2\sqrt{\log \log n}}$, $s = \log n_{j+1} \log \log \log n$, and $t = \frac{\log n_{j+1}}{4\sqrt{\log \log n}}$. We need to verify that $\binom{N_1}{s}(\frac{m}{N_2})^t \leq \frac{p^t N_1}{2}$ with $p = 1/2$. It will be enough to show that $N_1^s (\frac{m}{pN_2})^t \leq 1$. But this is easy to check, since

$$N_1^s \left(\frac{m}{pN_2}\right)^t \leq (2N_1)^s N_2^{-t/2\sqrt{\log \log n}}$$

$$= \left(\frac{(2N_1)^{\log \log \log n}}{N_2^{1/8 \log \log n}}\right)^{\log n_{j+1}} \leq \left(\frac{(2n_{j+1})^{\log \log \log n}}{n_{k+1}^{1/16 \log \log n}}\right)^{\log n_{j+1}} < 1.$$

Here we used the fact that $N_1 = |T_{j+1,k}| \leq n_{j+1}$, $N_2 = |S_{k+1,j}| \geq n_{k+1}^{1/2}$ and, whenever $k > j$ and n is sufficiently large, the fact that

$$n_{k+1} \geq n_{j+1}^{2\sqrt{\log \log n}} \geq (2n_{j+1})^{16 \log \log n \log \log \log n}.$$

Therefore, there exists a subset M_{k+1} of $T_{j+1,k}$ of order $\frac{p^t N_1}{2}$ such that every vertex subset of order s has at least m common neighbors in $S_{k+1,j}$. We let $T_{j+1,k+1} = M_{k+1}$. Note that

$$|T_{j+1,k+1}| \geq \frac{p^t |T_{j+1,k}|}{2} \geq \frac{1}{2} 2^{-\frac{\log n_{j+1}}{4\sqrt{\log \log n}}} n_{j+1}^{\frac{1}{2}-\frac{k}{4d}} = \frac{1}{2} n_{j+1}^{-\frac{1}{8(d+1)}} n_{j+1}^{\frac{1}{2}-\frac{k}{4d}} \geq n_{j+1}^{\frac{1}{2}-\frac{k+1}{4d}},$$

as required. Moreover, since $k \leq d \leq (1/2)\sqrt{\log \log n}$, every subset of $T_{j+1,k+1}$ of order $\log n_{j+1} \log \log \log n$ has at least

$$m \geq |S_{k+1,j}|^{1-1/2\sqrt{\log\log n}} \geq (n_{k+1}^{1-\frac{j}{2(k+1)}})^{1-1/2\sqrt{\log\log n}} \geq n_{k+1}^{1-\frac{j+1}{2(k+1)}}$$

common neighbors in $S_{k+1,j}$, so the second requirement of the induction scheme also holds.

To complete the proof, note that for each $i = 1, \ldots, d$, we have found a red clique R_i and a blue clique B_i of orders $(1/4)r_i \ln n_i$ and $(1/4)b_i \ln n_i$, respectively, such that every vertex in $R_i \cup B_i$ is connected to every vertex in $R_j \cup B_j$ by color $\chi(i, j)$. Consider the 2-colored complete graph on the vertex set $\{1, 2, \ldots, d\}$, where i and j are joined in color $\chi(i, j)$. We give each vertex the two weights r_i and b_i. Since $b_i \geq c \ln(4c/r_i)$ if $r_i \leq b_i$ and $r_i \geq c \ln(4c/b_i)$ if $b_i \leq r_i$, we may apply Lemma 3.3 to find a red clique R such that

$$\sum_{i \in R} r_i \geq \frac{c}{2} \ln d \geq \frac{1}{32} \log\log\log n$$

or a blue clique B such that $\sum_{i \in B} b_i \geq (1/32) \log\log\log n$. Suppose, without loss of generality, that there is a red clique R such that $\sum_{i \in R} r_i \geq (1/32) \log\log\log n$.

Consider now the set $\mathcal{R} = \bigcup_{i \in R} R_i$. Since R is a red clique by coloring χ, the edges between different R_i are red. Therefore, since also each R_i is a red clique, we see that $\mathcal{R}$ is a red clique in the original graph. Moreover,

$$\sum_{j \in \mathcal{R}} \frac{1}{\log j} \geq \sum_{i \in R} \sum_{j \in R_i} \frac{1}{\log j} \geq \sum_{i \in R} \frac{r_i}{4} \log n_i \frac{1}{\log 2n_i} \geq \sum_{i \in R} \frac{r_i}{8} \geq 2^{-8} \log\log\log n,$$

as required.   □

4. Monochromatic sets with differences satisfying a prescribed order

In this section, we prove Theorem 1.2, which gives an improved bound for Ramsey numbers with fixed-order type. We begin with several simple definitions and lemmas.

An *interval* I of integers is a set of consecutive integers. Let S be a nonempty set of integers, and let $\min(S)$ and $\max(S)$ denote the minimum and maximum integers in S. The density $d_I(S)$ of S with respect to an interval I of integers with $S \subset I$ is $|S|/|I|$.

The following definition is useful for finding cliques of a certain order type.

Definition
An ordered pair (T_1, T_2) of sets of integers are *separated* if, for $j = 1, 2$, we have

$$\min(T_2) - \max(T_1) > \max(T_j) - \min(T_j).$$

The next lemma shows that any dense subset S contains a pair of large dense subsets which are separated.

LEMMA 4.1

Let S be a finite set of integers with $|S| \geq 6$, and let $I = [a, b]$ be an interval with $S \subset I$. Then, for $j = 1, 2$, there are $T_j \subset S$ and an interval I_j with $T_j \subset I_j$, (T_1, T_2) separated, $d_{I_j}(T_j) \geq d_I(S)/2$, and $|I_j| \geq |S|/12$.

Proof

Let $i_1 \in I$ be the maximum integer (if it exists) such that the restriction of S to the interval $[a, i_1]$ has density at most $d_I(S)/2$. If no such i_1 exists, let $i_1 = a - 1$. Similarly, let $i_2 \in I$ be the minimum integer greater than i_1 (if it exists) such that the restriction of S to the interval $[i_2, b]$ has density at most $d_I(S)/2$. If no such i_2 exists, let $i_2 = b + 1$. Let S' be the restriction of S to the interval (i_1, i_2), that is, the set of $s \in S$ with $i_1 < s < i_2$. Since at most $1/2$ of the elements of S are deleted to obtain S', we have $|S'| \geq |S|/2$.

Let I' denote the interval $[\min(S'), \max(S')]$ of integers. Partition the interval I' into three intervals each of size as equal as possible, and let I_1 be the first interval and I_2 be the last interval. This guarantees that if $T_1 \subset I_1$ and $T_2 \subset I_2$, then (T_1, T_2) is separated. It follows from the definition of i_1 and i_2 that the restrictions of S to each of the two end intervals have density at least $d_I(S)/2$. Let $T_j = |S \cap I_j|$ for $j = 1, 2$. Since $S' \subset I'$, we have $|I'| \geq |S'| \geq |S|/2$. The end intervals have size at least $\lfloor |I'|/3 \rfloor$. Hence, for $j = 1, 2$,

$$|I_j| \geq \lfloor |I'|/3 \rfloor \geq \lfloor |S|/6 \rfloor \geq |S|/12.$$

The result follows.   □

We also need the following simple lemma which allows us to pass to a subinterval of a given size without the density decreasing significantly.

LEMMA 4.2

Suppose that S is a set of positive integers, that J is an interval containing S, and that $r \leq |J|$ is a positive integer. Then there is a subset $S' \subset S$ and an interval I of size r containing S' such that $d_I(S') \geq d_J(S)/2$.

Proof

We can cover the interval J with $\lceil |J|/r \rceil$ intervals of size r, some of which may be overlapping. If S restricted to any of these intervals has density at least $d_J(S)/2$, then we can pick S' to be this subset of S. Otherwise, since $\lceil |J|/r \rceil \leq 2|J|/r$, the total number of elements of S is less than

$$\lceil |J|/r \rceil r d_J(S)/2 \leq |S|,$$

a contradiction, which completes the proof.   □

For a permutation π of $[k-1]$, an increasing sequence $a_1, \ldots, a_k$ of k integers has *type* π if

$$a_{\pi(1)+1} - a_{\pi(1)} > a_{\pi(2)+1} - a_{\pi(2)} > \cdots > a_{\pi(k-1)+1} - a_{\pi(k-1)}.$$

Let G be a graph on a subset of the integers, let J be an interval, and let $S \subset J \cap V(G)$. For $0 < \alpha, \beta, \gamma, \delta, p < 1$, we say that G is $(\alpha, \beta, \gamma, \delta, p)$-*heavy* with respect to S if, for all subsets $S' \subset S$ with $|S'| \geq \gamma|S|$ for which there is an interval J' with $S' \subset J'$ and $d_{J'}(S') \geq \delta d_J(S)$, there are subsets $T_1, T_2 \subset S'$ and, for $j = 1, 2$, intervals I_j with $T_j \subset I_j$ such that (T_1, T_2) is a separated pair, $d_{I_j}(T_j) \geq \alpha d_{J'}(S')$, $|I_j| \geq \beta|S'|$, and the edge density of G across T_1, T_2 is at least p. That is, G is heavy with respect to S if every large subset S' of S which is dense in an interval J' contains a separated pair of subsets T_1 and T_2, dense in large intervals I_1 and I_2, such that there is a certain positive density of edges of G between them.

Let $\phi : [h-1] \to [k-1]$ be an injective function, $0 < \eta < 1$, and $r \in \mathbb{N}$. A clique in G of type (ϕ, η, r) consists of h pairwise adjacent vertices $a_1, \ldots, a_h$ such that $a_{i+1} - a_i \in [\eta^{\phi(i)}r, \eta^{\phi(i)-1}r)$ for $i \in [h-1]$. Note that if $h = k$ and ϕ is the inverse permutation of π, then a clique of type (ϕ, η, r) is also a clique of type π.

The following lemma shows that if a large subset S of a graph G is $(\alpha, \beta, \gamma, \delta, p)$-heavy with appropriate choices of parameters α, β, γ, δ, and p, then it must contain a clique of type (ϕ, η, r). We next describe the proof, which is by induction on the order h of the desired clique. The main intuition is that we find the largest gap first. Let τ be the minimum element of the image of ϕ, and let j be such that $\phi(j) = \tau$. We first pass to an interval I of size just smaller than $\eta^{\tau-1}r$ using Lemma 4.2. Using the heavy hypothesis, we find a separated pair (T_1, T_2) of large subsets of $S \cap I$ such that the edge density of G between T_1 and T_2 is at least p, and $\min(T_2) - \max(T_1) \geq \eta^{\tau}r$. This implies that, for any choice of $a_j \in T_1$ and $a_{j+1} \in T_2$, we have $a_{j+1} - a_j \in [\eta^{\tau}r, \eta^{\tau-1}r)$. Applying the dependent random choice lemma, Lemma 2.1, we find that there is a large subset $U \subset T_1$ such that all small subsets of U have many common neighbors in T_2. We find from the heavy hypothesis and induction that there is a clique with vertices $a_1, \ldots, a_j \in U$ such that, for $1 \leq i \leq j-1$, $a_{i+1} - a_i \in [\eta^{\phi(i)}r, \eta^{\phi(i)-1}r)$. Since every small subset of U has many common neighbors in T_2, the set W of common neighbors of $a_1, \ldots, a_j$ in T_2 is large. We again find from the heavy hypothesis and induction that there is a clique with vertices $a_{j+1}, \ldots, a_h \in W$ such that, for $j+1 \leq i \leq h-1$, $a_{i+1} - a_i \in [\eta^{\phi(i)}r, \eta^{\phi(i)-1}r)$. We conclude that $a_1, \ldots, a_h$ forms the desired clique in G of type (ϕ, η, r).

LEMMA 4.3

Suppose the following: G is a graph on a subset of the integers, J is an interval, $S \subset J \cap V(G)$, $\phi : [h-1] \to [k-1]$ is an injective function, $0 < \alpha, \beta, \gamma, \delta, \eta, p < 1$,

and $r \in \mathbb{N}$. Let $t = 2\sqrt{k \log_{1/p} |S|}$, let $\epsilon = p^t/2$, let $\lambda = (\epsilon\alpha/4)^{2h}$, and let $\kappa = \lambda\beta d_J(S)^2\eta^k r$. Provided that $\kappa \geq h$, $|J| \geq r$, $\eta \leq \beta\lambda d_J(S)^2$, $\delta \leq \lambda$, and $\gamma|S| \leq \kappa$, the following holds. If G is $(\alpha, \beta, \gamma, \delta, p)$-heavy with respect to S, then there is a clique in G of type (ϕ, η, r).

Proof

The proof is by induction on h. In the base case $h = 1$, it suffices to show that S is nonempty, which it clearly is. The induction hypothesis is that the lemma holds for all positive integers $h' < h$, where $h \geq 2$.

Let $\tau = \min_{i \in [h-1]} \phi(i)$ and $j \leq h-1$ be such that $\phi(j) = \tau$. Let $\phi_1 : [j-1] \to [k-\tau-1]$ and $\phi_2 : [h-j-1] \to [k-\tau-1]$ be the injective functions given by $\phi_1(x) = \phi(x) - \tau$ and $\phi_2(x) = \phi(x+j) - \tau$.

Let s be the largest integer less than $\eta^{\tau-1}r$. Since $\eta^{\tau-1}r \geq \eta^k r \geq \kappa \geq h \geq 2$, then $s \geq \eta^{\tau-1}r/2$. As $|J| \geq r \geq s$, we can apply Lemma 4.2 to obtain a subset $S' \subset S$ and an interval I with $|I| = s$ and $S' \subset I$ such that $d_I(S') \geq d_J(S)/2$.

We have $d_I(S') \geq d_J(S)/2 \geq \lambda d_J(S) \geq \delta d_J(S)$ and $|S'| = d_I(S')|I| \geq \frac{d_J(S)}{2}|I| \geq \frac{d_J(S)}{4}\eta^{\tau-1}r \geq \kappa \geq \gamma|S|$. Hence, by the heaviness hypothesis, for $i = 1, 2$, there is an interval I_i and a subset $T_i \subset I_i \cap S'$ such that (T_1, T_2) is a separated pair, $d_{I_i}(T_i) \geq \alpha d_I(S') \geq (\alpha/2)d_J(S)$, $|I_i| \geq \beta|S'|$, and the edge density of G between T_1 and T_2 is at least p. Note that $|T_i| = |I_i|d_{I_i}(T_i) \geq d_{I_i}(T_i)\beta|S'| \geq (1/2)\alpha\beta d_J(S)|S'|$.

We apply Lemma 2.1 to the bipartite subgraph of G with parts T_1 and T_2 and $s = j$, with t as defined in the statement of the lemma, $N_1 = |T_1|$, $N_2 = |T_2|$, and $m = \epsilon|T_2|$. Since $|T_1|^k \leq |S|^k = p^{-t^2/4}$, we can verify that

$$\binom{|T_1|}{j}\left(\frac{\epsilon|T_2|}{|T_2|}\right)^t \leq |T_1|^k\epsilon^t = |T_1|^k p^{t^2}/2^t \leq p^{3t^2/4}/2^t \leq p^t|T_1|/2.$$

Using $|T_1| \geq (1/2)\alpha\beta d_J(S)|S'|$, $|S'| \geq \frac{d_J(S)}{4}\eta^{\tau-1}r$, and $\eta \leq \beta\lambda d_J(S)^2 \leq \beta(\frac{\epsilon\alpha}{16})d_J(S)^2$, we conclude that there is a subset $U \subset T_1$ with

$$|U| \geq p^t|T_1|/2 = \epsilon|T_1| \geq \epsilon\alpha\beta\frac{d_J(S)}{2}|S'| \geq \epsilon\alpha\beta\frac{d_J(S)^2}{8}\eta^{\tau-1}r \geq \eta^\tau r$$

such that every j vertices in U have at least $\epsilon|T_2|$ common neighbors in T_2. Since (T_1, T_2) is separated and $|T_1| \geq |U| \geq \eta^\tau r$, we have, for any $a \in T_1$ and $b \in T_2$,

$$\eta^\tau r \leq |T_1| \leq b - a \leq |I| < \eta^{\tau-1}r.$$

We also have

$$d_{I_1}(U) = \frac{|U|}{|I_1|} \geq \frac{\epsilon|T_1|}{|I_1|} = \epsilon d_{I_1}(T_1) \geq \epsilon\alpha d_I(S') \geq \epsilon\frac{\alpha}{2}d_J(S).$$

Let $\delta' = \frac{d_J(S)}{d_{I_1}(U)}\delta$ and $\gamma' = \frac{|S|}{|U|}\gamma$. Since G is $(\alpha, \beta, \gamma, \delta, p)$-heavy with respect to S and $U \subset S$, then G is also $(\alpha, \beta, \gamma', \delta', p)$-heavy with respect to U.

Let $t' = 2\sqrt{(k-\tau)\log_{1/p}|U|}$, let $k' = k - \tau$, let $r' = \eta^\tau r$, and let $\epsilon' = p^{t'}/2$, so $\epsilon' \geq \epsilon$. Let $\lambda' = (\frac{\epsilon'\alpha}{4})^{2j}$ and let $\kappa' = \lambda'\beta d_{I_1}(U)^2\eta^{k'}r'$. Then $\lambda \leq (\frac{\epsilon\alpha}{4})^2\lambda'$, and therefore

$$\kappa' = \lambda'\beta d_{I_1}(U)^2\eta^{k'}r' \geq \lambda'\beta\left(\epsilon\frac{\alpha}{2}d_J(S)\right)^2\eta^k r \geq \lambda\beta d_J(S)^2\eta^k r = \kappa \geq h \geq j.$$

Since $|I_1| \geq |U| \geq \eta^\tau r = r'$, $\delta \leq \lambda = (\frac{\epsilon\alpha}{4})^{2h} \leq (\frac{\epsilon\alpha}{4})^2\lambda'$, $\epsilon' \geq \epsilon$, and $d_{I_1}(U) \geq (\epsilon\alpha/2)d_J(S)$, we have

$$\eta \leq \beta d_J(S)^2\lambda \leq \beta d_{I_1}(U)^2\lambda',$$

$$\delta' = \frac{d_J(S)}{d_{I_1}(U)}\delta \leq 2\epsilon^{-1}\alpha^{-1}\delta \leq \left(\frac{\epsilon\alpha}{4}\right)^{2h-1} \leq \lambda',$$

and

$$\gamma'|U| = \gamma|S| \leq \kappa \leq \kappa'.$$

Thus, we can apply the induction hypothesis and obtain a clique in G with vertices $a_1, \ldots, a_j$ in U which is of type $(\phi_1, \eta, \eta^\tau r)$.

Let W be the set of common neighbors of $a_1, \ldots, a_j$ in T_2, so $|W| \geq \epsilon|T_2|$. Let $\delta'' = \frac{d_J(S)}{d_{I_2}(W)}\delta$ and $\gamma'' = \frac{|S|}{|W|}\gamma$. As above, since $W \subset S$ and G is $(\alpha, \beta, \gamma, \delta, p)$-heavy with respect to S, then G is also $(\alpha, \beta, \gamma'', \delta'', p)$-heavy with respect to W. Again, by the induction hypothesis (exactly as done above, replacing U by W and j by $h - j$), there is a clique $b_1, \ldots, b_{h-j}$ in G with vertices from W of type $(\phi_2, \eta, \eta^\tau r)$. Then, letting $a_{j+i} = b_i$ for $1 \leq i \leq h - j$, we conclude that $a_1, \ldots, a_h$ form a clique of type (ϕ, η, r) in G, completing the proof. $\square$

The following theorem is a restatement of Theorem 1.2. Recall that if $h = k$ and if ϕ is the inverse permutation of π, then a clique of type (ϕ, η, r) is also a clique of type π. In the proof of Theorem 1.2, we show that a q-colored complete graph on sufficiently many vertices must contain a subset which is appropriately heavy in the graph of one of the colors. Lemma 4.3 then implies that the graph of this color contains the desired monochromatic clique with order type π. To find such a heavy subset, we suppose for contradiction that none exists. We then find a large interval I_q and a dense subset S_q of I_q such that for each color i, every separated pair (T_1, T_2) of subsets of S_q such that there are large intervals J_1, J_2 with T_j a dense subset of J_j has edge density less than $p = 1/q$ in color i between T_1 and T_2. But, by Lemma 4.1, S_q contains a separated pair (T_1, T_2) of large dense subsets. By the pigeonhole principle,

the edge density between T_1 and T_2 in one of the q colors is at least $1/q$, contradicting the existence of S_q.

THEOREM 4.1
Let $k, q \geq 2$ be integers and let π be a permutation of $[k-1]$. Every q-coloring of the complete graph on $[n]$ with $n = 2^{k^{20q}}$ contains a monochromatic clique of type π.

Proof
Suppose for contradiction that there is a q-coloring of the edges of the complete graph on $[n]$ without a monochromatic copy of K_k of type π. We label the q colors $1, \ldots, q$. Let $S_0 = I_0 = [n]$, so $d_{I_0}(S_0) = 1$ and $|S_0| = |I_0| = n$. Let $\phi = \pi^{-1}$, let $p = 1/q$, let $t = 2\sqrt{k \log_{1/p} n}$, and let $\epsilon = p^t/2$.

For $q \geq i \geq 1$, we define $\alpha_i, \beta_i, \gamma_i, \delta_i, \eta_i$ recursively as follows, starting with $i = q$. We have $\alpha_q = 1/2$, $\delta_i = (\frac{\epsilon \alpha_i}{4})^{2k}$, and $\alpha_i = \delta_{i+1}\alpha_{i+1}$. Explicitly, $\delta_{q-i} = (\frac{\epsilon}{8})^{2k(2k+1)^i}$, and for $i \geq 1$, $\alpha_{q-i} = \frac{1}{2}(\frac{\epsilon}{8})^{(2k+1)^i - 1}$. Let $\Delta_0 = 1$ and $\Delta_i = \delta_i \Delta_{i-1}^2$ for $1 \leq i \leq q$. Let $\Delta = \Delta_q$. We have from the explicit formula for δ_{q-i} that

$$\Delta = \delta_q \Delta_{q-1}^2 = \delta_q \delta_{q-1}^2 \Delta_{q-2}^4 = \cdots$$
$$= \prod_{i=0}^{q-1} \delta_{q-i}^{2^i} = \prod_{i=0}^{q-1} \left(\frac{\epsilon}{8}\right)^{2k(4k+2)^i} \geq \left(\frac{\epsilon}{8}\right)^{(4k+2)^q} \geq \left(\frac{\epsilon}{8}\right)^{(k+2)^{2q}-2}.$$

Let $\beta_q = 1/12$. For each i, let $\eta_i = \beta_i \Delta$, and $\gamma_i = \eta_i^{k+1}$, and, if $i < q$, $\beta_i = \gamma_{i+1}\beta_{i+1}$. Explicitly, $\beta_{q-i} = \frac{1}{12}(\frac{\Delta}{12})^{(k+2)^i - 1}$, $\eta_{q-i} = (\frac{\Delta}{12})^{(k+2)^i}$, and $\gamma_{q-i} = (\frac{\Delta}{12})^{(k+1)(k+2)^i}$. Finally, let $\Gamma_0 = 1$ and $\Gamma_i = \gamma_i \Gamma_{i-1}$ for $1 \leq i \leq q$. Let $\Gamma = \Gamma_q$. We have

$$\Gamma = \prod_{i=1}^q \gamma_i = \left(\frac{\Delta}{12}\right)^{(k+2)^q - 1} \geq \left(\frac{1}{12}\left(\frac{\epsilon}{8}\right)^{(k+2)^{2q}-2}\right)^{(k+2)^q - 1}$$
$$\geq \left(\left(\frac{\epsilon}{8}\right)^{(k+2)^{2q}}\right)^{(k+2)^q} = \left(\frac{\epsilon}{8}\right)^{(k+2)^{3q}} \geq \left(\frac{\epsilon}{8}\right)^{k^{6q}}.$$

We will next define a sequence of subsets $S_0 \supset S_1 \supset \cdots \supset S_q$ and a sequence of intervals $I_0 \supset I_1 \supset \cdots \supset I_q$ such that for each i, $1 \leq i \leq q$, we have
- $S_i \subset I_i$,
- $d_{I_i}(S_i) \geq \delta_i d_{I_{i-1}}(S_{i-1}) \geq \Delta_i$,
- $|S_i| \geq \gamma_i |S_{i-1}| \geq \Gamma_i n$, and
- there is no separated pair (T_1, T_2) with $T_1, T_2 \subset S_i$ and intervals J_1, J_2 such that, for $j = 1, 2$, $T_j \subset J_j$, $d_{J_j}(T_j) \geq \alpha_i d_{I_i}(S_i)$, $|J_j| \geq \beta_i |S_i|$, and the graph in color i has edge density at least p between T_1 and T_2.

We next show how to pick S_i and I_i, having already picked S_{i-1} and I_{i-1}. Since the graph in color i does not contain a clique of type π, it also does not contain a clique of type (ϕ, η_i, r_i) with $r_i = |I_{i-1}|$. We now wish to apply Lemma 4.3 with $S = S_{i-1}$ to conclude that the graph in color i is not $(\alpha_i, \beta_i, \gamma_i, \delta_i, p)$-heavy with respect to S_{i-1}. To do this, we must verify the assumptions of the lemma.

Let $\lambda_i = (\epsilon \alpha_i / 4)^{2k}$ and $\kappa_i = \lambda_i \beta_i d_{I_{i-1}}(S_{i-1})^2 \eta_i^k r_i$. Note that $\delta_i = \lambda_i$ and

$$\eta_i = \beta_i \Delta \le \beta_i \Delta_i = \beta_i \delta_i \Delta_{i-1}^2 \le \beta_i \lambda_i d_{I_{i-1}}(S_{i-1})^2.$$

We also have

$$\gamma_i |S_{i-1}| = \eta_i^{k+1} |S_{i-1}| \le \beta_i \lambda_i d_{I_{i-1}}(S_{i-1})^2 \eta_i^k |S_{i-1}| \le \lambda_i \beta_i d_{I_{i-1}}(S_{i-1})^2 \eta_i^k r_i = \kappa_i.$$

Finally, since $\gamma_i |S_{i-1}| \ge \Gamma_i n \ge \Gamma n$ and $n = 2^{k^{20q}}$, we have

$$\kappa_i \ge \Gamma n \ge n \left(\frac{\epsilon}{8} \right)^{k^{6q}} = n \left(\frac{q^{-2\sqrt{k \log_q n}}}{16} \right)^{k^{6q}} \ge n (2^{-k^{12q}})^{k^{6q}} = n 2^{-k^{18q}} \ge k.$$

Here we used the fact that $q^{-2\sqrt{k \log_q n}} \ge 2^{-2k^{10q+1}\sqrt{\log q}} \ge 2^{-k^{12q}+4}$.

We may therefore apply Lemma 4.3. Hence, there is a subset $S_i \subset S_{i-1}$ and an interval $I_i \subset I_{i-1}$ satisfying the four desired properties itemized above.

However, by Lemma 4.1, S_q contains a separated pair (T_1, T_2) and intervals J_1, J_2 such that, for $j = 1, 2$, $T_j \subset J_j$, $d_{J_j}(T_j) \ge d_{I_q}(S_q)/2$, and $|J_j| \ge |S_q|/12$. By the pigeonhole principle, for some i, $1 \le i \le q$, the density across T_1, T_2 in color i is at least $1/q = p$. But

$$\frac{1}{2} d_{I_q}(S_q) = \alpha_q d_{I_q}(S_q) \ge \alpha_q \delta_q d_{I_{q-1}}(S_{q-1})$$

$$= \alpha_{q-1} d_{I_{q-1}}(S_{q-1}) \ge \alpha_{q-2} d_{I_{q-2}}(S_{q-2}) \ge \cdots \ge \alpha_i d_{I_i}(S_i)$$

and, similarly, $|S_q|/12 \ge \beta_i |S_i|$, contradicting that S_i contains no such separated pair. $\qquad \square$

5. Further remarks

5.1. Asymptotics of maximum weight monochromatic cliques

A well-known conjecture of Erdős states that the limit $\lim_{n \to \infty} \frac{\log r(n)}{n}$ exists. If this limit exists, denote it by c_0. We will assume the conjecture that c_0 exists. The bounds of Erdős and Erdős–Szekeres on Ramsey numbers imply that $1/2 \le c_0 \le 2$.

Recall that the weight of a set S of integers greater than 1 is the sum of $1/\log s$ over all $s \in S$ and that $f(n)$ is the maximum real number for which any red-blue edge-coloring of K_n contains a monochromatic clique of weight at least $f(n)$. Theorem 1.1

shows that $f(n)$ is within a constant factor of $\log \log \log n$. We further conjecture the constant factor.

CONJECTURE 5.1
We have

$$f(n) = \left(c_0^{-2} + o(1)\right) \log \log \log n,$$

where $c_0 = \lim_{n \to \infty} \frac{\log r(n)}{n}$.

The construction of Rödl described in the introduction can easily be modified to obtain

$$f(n) \leq \left(c_0^{-2} + o(1)\right) \log \log \log n.$$

Indeed, let $a = 1 + \epsilon$ with $\epsilon \to 0$ slowly as $n \to \infty$ (picking $\epsilon = 1/\log \log \log n$ will do). Cover $[2, n]$ by intervals, where the ith interval is $[2^{a^{i-1}}, 2^{a^i})$ and has largest element less than $n_i := 2^{a^i}$. The number of intervals is $d = \lceil \frac{1}{\log a} \log \log n \rceil = O(\epsilon^{-1} \log \log n)$. Note that the logarithm of any two numbers in the same interval is within a factor $a = 1 + \epsilon$ of each other. We red-blue edge-color the complete graph on each of these intervals so as to minimize the order of the largest monochromatic clique in the interval. Then the weight of any monochromatic clique in the ith interval is at most $(1/\log n_i)(c_0^{-1} + o(1)) \log n_i = c_0^{-1} + o(1)$, where the $o(1)$ term goes to 0 as n_i increases. We color between intervals monochromatically so as to minimize the order of the largest monochromatic clique with vertices in distinct intervals. The order of this monochromatic clique with vertices in distinct intervals is $(c_0^{-1} + o(1)) \log d = (c_0^{-1} + o(1)) \log \log \log n$. Hence, $f(n) \leq (c_0^{-1} + o(1))(c_0^{-1} + o(1)) \log \log \log n = (c_0^{-2} + o(1)) \log \log \log n$.

In the other direction, a simple modification of the proof of Theorem 1.1 with a careful analysis gives the lower bound

$$f(n) \geq \left(\frac{1}{4} - o(1)\right) \log \log \log n,$$

which would be sharp if the exponential constant in the upper bound for diagonal Ramsey numbers is best possible (i.e., if $c_0 = 2$). We next give a rough sketch of how to achieve this.

One first constructs $d = (\log \log n)^{1-o(1)}$ intervals S_i of the form $[n_i, 2n_i)$ with $n_i = i(\log \log n)^{o(1)} + (1/2) \log \log n$, where the $o(1)$ term slowly goes to 0 as n tends to infinity. After going through the proof, we obtain in each S_i a red clique R_i and a blue clique B_i, such that for each $i < j$, the complete bipartite graph between $R_i \cup B_i$ and $R_j \cup B_j$ is monochromatic. The monochromatic cliques R_i and B_i are chosen

to be the largest monochromatic cliques of each color in a particular subset $T_{i,d} \subset S_i$ with $|T_{i,d}| = |S_i|^{1-o(1)}$. By the Erdős–Szekeres estimate, we have $|R_i| \geq (r_i - o(1)) \log n_i$ and $|B_i| \geq (b_i - o(1)) \log n_i$, where b_i and r_i (asymptotically) satisfy $(b_i + r_i) \log \frac{b_i + r_i}{r_i} - b_i \log \frac{b_i}{r_i} = 1$.

Consider the induced red-blue edge-coloring of the complete graph with one vertex v_i from each $R_i \cup B_i$. Assign vertex v_i red weight r_i and blue weight b_i. An appropriate variant of Lemma 3.2, the weighted version of Ramsey's theorem, tells us that there is a monochromatic clique $v_{i_1}, v_{i_2}, \ldots, v_{i_s}$ of large weight. Assuming without loss of generality that this clique is red, the tailored variant of Lemma 3.2 then tells us that the red weight of the clique is asymptotically at least $(1/4) \log d = (1/4 + o(1)) \log \log \log n$. This is obtained when for each i, $b_i = r_i = 1/2 + o(1)$ and the clique has size $(1/2) \log d$. Let S be the union of the R_{i_j} with $1 \leq j \leq s$. As, for each $i < j$, the complete bipartite graph between $R_i \cup B_i$ and $R_j \cup B_j$ is monochromatic red, the set S forms a monochromatic clique of weight

$$\sum_{j \in S} \frac{1}{\log j} \geq \left(\frac{1}{4} + o(1) \right) \log \log \log n.$$

The proof sketched above uses an application of both Ramsey's theorem and its weighted variant, so that the asymptotics of the lower bound on $f(n)$ are dictated by the bounds in these theorems. We believe that the optimal bounds should always follow, as above, from the diagonal case, in which case Conjecture 5.1 would follow.

5.2. Weighted cliques with alternative weight functions

One question which arises naturally is whether we can also find cliques of large weight for other weight functions. Let $w(i)$ be a weight function defined on all positive integers $n \geq a$, and let $f(n, w)$ be the minimum over all 2-colorings of $[a, n]$ of the maximum weight of a monochromatic clique. In particular, if $w_1(i) = 1/\log i$ and $a = 2$, then $f(n, w_1) = f(n)$.

The next interesting case is when $w_2(i) = 1/\log i \log \log \log i$, since, for any function $u(i)$ which tends to infinity with i, Theorem 1.1 implies that $f(n, u') \to \infty$, where $u'(i) = u(i)/\log i \log \log \log i$. We may show also that $f(n, w_2) \to \infty$.

Sketch of the proof

Suppose that we are using the weight function w_2. We consider the intervals $I_j = [n_j, 2n_j)$ for which $2n_j \leq n$ with $\log \log n_j = 10j \log \log \log n$. The number d of such intervals is $\log \log n / 10 \log \log \log n$. By applying the methods used in the proof of Theorem 3.1, we may find d sets $T_1, T_2, \ldots, T_d$, with $T_j \subset I_j$, such that the collection of edges between T_i and T_j is monochromatic for every $i \neq j$, and each T_j is the union of a red clique of size roughly $r_j \log n_j$ and a blue clique of size $b_j \log n_j$.

Here r_j and b_j are chosen to satisfy the balancing condition stipulated by Lemma 3.1. Any vertex in T_j will have weight about $1/\log n_j \log \log \log n_j$, the full contribution of the red clique is $\Omega(r_j/\log \log \log n_j) = \Omega(r_j/(\log j + \log \log \log \log n)) = \Omega(r_j/\log \max(j, \log \log \log n))$, and the blue clique is $\Omega(b_j/\log \max(j, \log \log \log n))$.

We may now treat the T_j as though they were vertices with two weights in a graph whose edges have been 2-colored. For $j \geq \log \log \log n$, the red weight is $r_j/\log j$ and the blue weight is $b_j/\log j$. For smaller j, the red weight is $r_j/\log \log \log \log n$ and the blue weight is $b_j/\log \log \log \log n$. However, there are so few such smaller j that we will be able to safely ignore such vertices. We would like to repeat the argument above with this new graph on d vertices. To begin, we consider $e \approx \log \log d/10 \log \log \log d$ intervals $S_1, \ldots, S_e$ in $[d]$, each of the form $[d_i, 2d_i)$ with $\log \log d_i = 10i \log \log \log d$. For the rest of the argument we only consider vertices j in one of these intervals, so that $j \geq d_1 \geq \log \log \log n$ and j has red weight $r_j/\log j$ and blue weight $b_j/\log j$. We may assume that r_j and b_j are each less than $(\log j)^2$, as otherwise the vertex j, or rather the red or blue subset of T_j, would be a monochromatic clique of weight $\Omega(\log j) = \Omega(\log \log \log \log n)$. By Lemma 3.1, this also implies that all r_j and b_j are at least $1/(16 \log \log j)$. Therefore the ratio between any two of r_j and any two of b_j is at most $16 \log^2 j \log \log \log j \leq (\log j)^3$ and hence we may split each S_i into $h_i = 6 \log \log d_i$ subsets, so that the r_j and b_j are within a factor 2 of each other within each piece. That is, we are decomposing the interval S_i into $S_{i,1}, \ldots, S_{i,h_i}$ so that within any $S_{i,\ell}$ all r_j and b_j are essentially the same. Within each S_i, we pass to the largest $S_{i,\ell}$, which we will call U_i. As $|U_i| \geq d_i/(6 \log \log d_i)$, we have $\log |U_i| \approx \log |S_i|$ for each i. We let r_i' and b_i' be the minimum over $j \in U_i$ of r_j and b_j, respectively.

If we again apply the method of Theorem 3.1, we will find a collection of sets $T_i' \subset S_i$ such that the graph is monochromatic between any two sets and T_i' contains a red clique of size $\hat{r}_i \log |U_i| \approx \hat{r}_i \log |S_i|$ and a blue clique of size roughly $\hat{b}_i \log |S_i|$. The red clique will have red weight $\Omega(\hat{r}_i r_i')$ and the blue clique will have blue weight $\Omega(\hat{b}_i b_i')$. Treating the T_i' as though they were the vertices in a graph, we see that the vertex i will have red weight $\Omega(\hat{r}_i r_i')$ and blue weight $\Omega(\hat{b}_i b_i')$, where $\hat{r}_i$ and $\hat{b}_i$ as well as r_i' and b_i' satisfy, up to a constant factor, the balancing criterion stipulated by Lemma 3.1. It is now easy to verify that the weight functions $\hat{r}_i r_i'$ and $\hat{b}_i b_i'$ satisfy the requirements of Lemma 3.3 with $c > 0$ an appropriately chosen absolute constant. Hence, we will be able to find a monochromatic clique of weight $\Omega(\log e) = \Omega(\log \log \log d) = \Omega(\log \log \log \log \log n)$. This yields a clique of the same weight in the original graph. $\qquad\square$

It is not hard to show that this bound is tight up to the constant. Color the interval $I_j = [2^{2^{j-1}}, 2^{2^j})$ so that the largest clique has size at most 2^{j+1}. Then the contri-

bution of the jth interval will be $O(1/\log j)$. We now treat I_j as though it were a vertex of weight $O(1/\log j)$ and, blowing up Rödl's coloring, color monochromatically between the different I_j so that the largest weight of any monochromatic clique is $O(\log\log\log d) = O(\log\log\log\log\log n)$.

On the other hand, by using Rödl's coloring, we can show that if $w_1'(i) = 1/(\log i)^{1+\epsilon}$, for any fixed $\epsilon > 0$, then $f(n, w_1')$ converges. By using the coloring from the previous paragraph, we may improve this to show that if $w_2'(i) = 1/\log i (\log\log\log i)^{1+\epsilon}$, then $f(n, w_2')$ also converges.

More generally, we have the following theorem. Here $\log_{(i)}(x)$ is the iterated logarithm given by $\log_{(0)}(x) = x$ and, for $i \geq 1$, $\log_{(i)}(x) = \log(\log_{(i-1)}(x))$.

THEOREM 5.1
Let $w_s(i) = 1/\prod_{j=1}^{s}\log_{(2j-1)}i$. *Then* $f(n, w_s) = \Theta(\log_{(2s+1)}n)$. *However, letting* $w_s'(x) = w_s(x)/(\log_{(2s-1)}i)^{\epsilon}$ *for any fixed* $\epsilon > 0$, *then* $f(n, w_s')$ *converges.*

That is, the sequence of functions w_s forms a natural boundary below which $f(n, \cdot)$ converges.

5.3. A counterexample to finding skewed cliques in hypergraphs

For 3-uniform hypergraphs, the Ramsey number $r_3(t)$ is defined to be the smallest natural number n such that in any 2-coloring of the edges of $K_n^{(3)}$ there is a monochromatic copy of $K_t^{(3)}$. It is known (see [3], [8], [9]) that

$$2^{ct^2} \leq r_3(t) \leq 2^{2^{c't}}$$

and the upper bound is widely conjectured to be correct. Phrased differently, we know that every 2-coloring of the edges of $K_n^{(3)}$ contains a monochromatic clique of size at least $\Omega(\log\log n)$ and that there are 2-colorings of $K_n^{(3)}$ which contain no monochromatic clique of size $O(\sqrt{\log n})$.

Let $\rho_3(n)$ be the function which gives the minimum size of the largest monochromatic clique taken over every 2-coloring of $K_n^{(3)}$. Note that this function is increasing and that $\rho_3(r_3(t)) = t$. In keeping with Erdős's conjecture for graphs, we can give a weight of $1/\rho_3(i)$ to vertex i and let the weight of a set S be $\sum_{i \in S} 1/\rho_3(i)$. We then ask for the minimum over all 2-colorings of the edges of the complete 3-uniform hypergraph on vertex set $[n]$ of the maximum weight of a monochromatic clique.

Split $[n]$ into intervals given by $R_j = [r_3(2^{j-1}), r_3(2^j))$. Within each interval, we color so that the largest monochromatic clique has size at most 2^j. If $i < j$, we color edges containing two vertices from R_i and one vertex from R_j red and edges containing two vertices from R_j and one vertex from R_i blue. We color all other edges arbitrarily.

Suppose now that we have a monochromatic clique S. Then S has at most one vertex in all but one of the sets R_j. Otherwise, if there were two vertices, say u_1 and u_2, in R_i and two vertices, v_1 and v_2, in R_j, the edges $u_1u_2v_1$ and $u_1v_1v_2$ would have opposite color. We may therefore suppose that $S = T_\ell \cup \{s_1, s_2, \dots\}$, where $T_\ell \subset R_\ell$ and s_i is a single vertex from R_i.

Since, for any $i \in R_\ell$, we have $\rho_3(i) \geq \rho_3(r_3(2^{\ell-1})) = 2^{\ell-1}$ and the largest monochromatic clique in R_ℓ has size at most 2^ℓ, the contribution from T_ℓ is at most 2. Similarly, the contribution from s_i is at most 2^{1-i}, so that the total weight of the clique is at most $2 + \sum_{i=1}^{\infty} 2^{1-i} \leq 4$. Therefore, unlike the graph case, there are colorings for which the maximum weight of a monochromatic clique is bounded.

5.4. A simple construction

Here we present a simple explicit construction which beats the random lower bound for Ramsey numbers for a certain prescribed order on the consecutive differences. A sequence $n_1 < n_2 < \cdots < n_k$ is *convex* if $n_2 - n_1 < n_3 - n_2 < \cdots < n_k - n_{k-1}$.

PROPOSITION 5.1
For $i < j$, let $f(i, j) = \lfloor \log(j - i) \rfloor$. Consider the 2-edge-coloring of the complete graph on the first $n = 4^{k-1}$ positive integers where the color of edge (i, j) with $i < j$ is the parity of $f(i, j)$. This coloring has no convex monochromatic clique of order $k + 1$.

Proof
Suppose for contradiction that $a_1 < \cdots < a_{k+1}$ is a convex monochromatic clique of order $k + 1$ in this 2-edge-coloring of the complete graph on n. We claim that for $1 \leq i \leq k - 1$, $f(a_{i+2}, a_{i+1}) \geq f(a_{i+1}, a_i) + 2$. Indeed, as the sequence is convex, $a_{i+1} - a_i < a_{i+2} - a_{i+1}$, and hence $f(a_{i+2}, a_{i+1}) \geq f(a_{i+1}, a_i)$. If the claim does not hold, then for some i, $1 \leq i \leq k - 1$, we have $f(a_{i+2}, a_{i+1}) = f(a_{i+1}, a_i)$ or $f(a_{i+2}, a_{i+1}) = f(a_{i+1}, a_i) + 1$. In the first case, as $a_{i+2} - a_i = (a_{i+2} - a_{i+1}) + (a_{i+1} - a_i)$, we have that $f(a_{i+2}, a_i) = f(a_{i+1}, a_i) + 1$, so the edges (a_i, a_{i+2}) and (a_i, a_{i+1}) are different colors. In the second case, (a_{i+2}, a_{i+1}) and (a_{i+1}, a_i) are different colors. As the clique is monochromatic, this cannot happen, and hence the claim holds. From the claim, we have $f(a_{k+1}, a_k) \geq f(2, 1) + 2(k - 1) \geq 2(k - 1)$. It follows that $a_{k+1} > a_{k+1} - a_k \geq 2^{2(k-1)}$, contradicting $a_{k+1} \leq n = 4^{k-1}$ and completing the proof. $\square$

We actually proved that not only is there no convex monochromatic complete graph on $k + 1$ vertices in the 2-edge-coloring of the complete graph on the first 4^{k-1} positive integers, but also that a much sparser graph on $k + 1$ vertices is forbidden as a

monochromatic subgraph in convex position, namely, the square of the monotone path on $k + 1$ vertices. That is, for this coloring, there is no convex sequence $a_1, \ldots, a_{k+1}$ such that all edges (a_i, a_j) with $|j - i| \leq 2$ are the same color. This is in strong contrast to Ramsey numbers without order, where the Ramsey number of the square of a path or, more generally, any bounded degree graph (see, e.g., [1], [4]) is linear in the number of vertices.

As with ordinary Ramsey numbers, the lower bound for complete Ramsey numbers with order types which comes from considering a random 2-edge-coloring of the complete graph is of the form $2^{k/2+o(k)}$. As the simple constructive coloring in Proposition 5.1 gives a better bound while forbidding a much sparser structure, it suggests that Ramsey's theorem with order types is a substantially different and more intricate problem than Ramsey's theorem.

5.5. Counterexamples to variants of Ramsey's theorem with order types

There are several natural variants of Väänänen's question which have negative answers. For example, the natural hypergraph analogue fails. Indeed, there is a coloring of the complete 3-uniform hypergraph on the positive integers such that every monochromatic set $a_1, \ldots, a_k$ satisfies that the sequence $a_2 - a_1, a_3 - a_2, \ldots, a_k - a_{k-1}$ of consecutive differences is monotone. We color an edge (a_1, a_2, a_3) with $a_1 < a_2 < a_3$ red if $a_3 - a_2 \geq a_2 - a_1$ and blue otherwise. Hence, if $a_1 < a_2 < a_3 < a_4$ are positive integers, (a_1, a_2, a_3) and (a_2, a_3, a_4) are both red or both blue if and only if $a_2 - a_1, a_3 - a_2, a_4 - a_3$ is a monotone sequence.

Another variant which fails to hold is the case of monochromatic cliques where the higher differences have a prescribed order. This was first observed by Erdős, Hajnal, and Pach in [7]. We give such an example forbidding an ordering of the second differences $a_{i+2} - a_i$. Before describing this coloring, we first remark that it is easy to show that any second difference is realizable. That is, for any permutation π of $[k - 2]$, there are (many) sequences $a_1 < \cdots < a_k$ of positive integers satisfying

$$a_{\pi(1)+2} - a_{\pi(1)} > a_{\pi(2)+2} - a_{\pi(2)} > \cdots > a_{\pi(k-2)+2} - a_{\pi(k-2)}.$$

However, for certain π there exist 2-edge-colorings of the complete graph on the positive integers in which none of these sequences form a monochromatic clique. Indeed, consider the 2-edge-coloring of the complete graph on the positive integers, where the color of (i, j) with $i < j$ is given by the parity of $f(i, j) = \lfloor \log(j - i) \rfloor$. In this coloring, no monochromatic clique with vertices $a_1 < a_2 < a_3 < a_4 < a_5 < a_6 < a_7$ satisfies $a_5 - a_3$ is the largest of the second differences and $a_4 - a_2, a_6 - a_4$ are the two smallest second differences. Suppose that such a monochromatic clique exists. By symmetry, we may assume without loss of generality that $a_4 - a_3 \geq a_5 - a_4$. For $a_i < a_j < a_h$, as $a_h - a_i = (a_h - a_j) + (a_j - a_i)$, we have $\max(f(a_i, a_j), f(a_j, a_h)) \leq$

$f(a_i, a_h) \leq \max(f(a_i, a_j), f(a_j, a_h)) + 1$. Since the parity of $f(a, b)$ is the same for any two vertices $a < b$ of the monochromatic clique, we must have $f(a_i, a_h) = \max(f(a_i, a_j), f(a_j, a_h))$. In particular, this implies that $f(a_3, a_5) = f(a_3, a_4)$ and that $f(a_1, a_5) = f(a_3, a_5)$. Since $a_3 - a_1 \geq a_4 - a_2$ (by minimality of $a_4 - a_2$), we must have $a_2 - a_1 \geq a_4 - a_3$ and hence $f(a_3, a_5) \geq f(a_1, a_3) \geq f(a_1, a_2) \geq f(a_3, a_4) = f(a_3, a_5)$, where the first inequality comes from the fact that $a_5 - a_3$ is the largest second difference. But if $f(a_1, a_3) = f(a_3, a_5)$, then $f(a_1, a_5) > f(a_3, a_5)$, contradicting the equality deduced earlier.

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