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A few remarks on Ramsey–Turán-type problems

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Abstract

Let H be a fixed *forbidden* graph and let f be a function of n. Denote by $\mathbf{RT}(n, H, f(n))$ the maximum number of edges a graph G on n vertices can have without containing H as a subgraph and also without having at least $f(n)$ independent vertices. The problem of estimating $\mathbf{RT}(n, H, f(n))$ is one of the central questions of so-called Ramsey–Turán theory.

In their recent paper (Discrete Math. 229 (2001) 293–340), Simonovits and Sós gave an excellent survey of this theory and mentioned some old and new interesting open questions. In this short paper we obtain some new bounds for Ramsey–Turán-type problems. These results give partial answers for some of the questions.

1. Introduction

Let H be a fixed so-called *forbidden* graph and let f be a function of n. Denote by $\mathbf{RT}(n, H, f(n))$ the maximum number of edges a graph G on n vertices can have without containing H as a subgraph and also without having at least $f(n)$ independent vertices. This problem was motivated by the classical Ramsey and Turán theorems and attracted a lot of attention during the last 30 years, see e.g., the recent survey [10] of Simonovits and Sós. First, we want to recall some open questions which were mentioned in [10].

An early and probably one of the most celebrated results in Ramsey–Turán theory claims that

$$\mathbf{RT}(n, K_4, o(n)) = (1 + o(1))\frac{n^2}{8},$$

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where K_4 is a complete graph on four vertices. The upper bound was obtained by Szemerédi [11] and the lower bound was proved by Bollobás and Erdős [2]. This result is quite surprising, since it seems to be more plausible to suspect that there are no K_4-free graphs on n vertices with a quadratic number of edges and with maximum independent set of size $o(n)$. Roughly speaking, the graph of Bollobás and Erdős consists of two disjoint copies of order $n/2$ of the Borsuk graph with a dense bipartite graph in between. The fact that this graph has independence number $o(n)$ was proved by applying an isoperimetric theorem for the high dimensional sphere. On the other hand, one can easily see that the independence number of the Borsuk graph is rather large. So replacing $o(n)$ by slightly smaller functions perhaps one could get smaller upper bounds on the number of edges. This natural question was posed in [4] and also repeated in [10], more precisely they asked the following:

Problem 1.1. *Is it true that for some* $c > 0$,

$$\mathbf{RT}\left(n, K_4, \frac{n}{\ln n}\right) < \left(\frac{1}{8} - c\right)n^2?$$

Similarly, what happens if $o(n)$ *is replaced by* $O(n^{1-\varepsilon})$ *for some fixed but small constant* $\varepsilon > 0$?

Another more general question asked by Simonovits and Sós [10] is to characterize forbidden graphs H for which replacing the $o(n)$ condition by $O(n^{1-\varepsilon})$ can change significantly the corresponding Ramsey–Turán numbers.

Problem 1.2. *Under which conditions on the forbidden graph* H *do there exist two positive constants* c, c_1 *for which*

$$\mathbf{RT}(n, H, o(n)) - \mathbf{RT}(n, H, f(n)) > c_1 n^2 \quad \textit{for every } f(n) = O(n^{1-c})?$$

Denote by $K_s(t_1, \ldots, t_s)$ a complete s-partite graph with parts of size $t_1, \ldots, t_s$. In [6] Erdős, Hajnal, Sós and Szemerédi developed an interesting method, based on a modified version of arboricity, which allows one to determine $\mathbf{RT}(n, K_s(t_1, \ldots, t_s), o(n))$ for a large family of complete multi-partite graphs. One of the first graphs which cannot be handled by this technique is $K_3(2,2,2)$. Finding the Ramsey–Turán number of this graph remains an intriguing open problem. Even the following simpler question is unsolved (see, e.g., [4,6,10]).

Problem 1.3. *Decide if* $\mathbf{RT}(n, K_3(2,2,2), o(n)) = o(n^2)$ *or not.*

Finally, we want to mention an additional open question which appeared in [5]. In this paper the authors studied the variant of Ramsey–Turán-type problems where instead of imposing a bound on the size of the maximum independent set they considered what happens if one forbids large K_p-free sets. Let the K_p-independence number $\alpha_p(G)$ be the maximum order of an induced subgraph in G which contains no copy of K_p.

Problem 1.4. *Is it true that if G is a K_5-free graph on n vertices and $\alpha_3(G) = o(n)$, then the number of edges in G is $o(n^2)$?*

Motivated by all these problems, in this short paper we obtain a few new bounds for the Ramsey–Turán numbers. These results give partial answers for some of the questions. The rest of this note is organized as follows. In Section 2 we prove our main lemma, which we think is of independent interest. We apply this lemma in Section 3 to obtain various bounds for some Ramsey–Turán-type problems. Section 4 of the paper is devoted to concluding remarks.

We close this section with some conventions and notation. Given a graph $G = (V, E)$ and a subset $W \subset V$, we denote by $N(W)$ the set of vertices of G adjacent to all the vertices in W. A graph G is *d-degenerate* if any subgraph of it contains a vertex of degree at most d. Obviously, such a graph contains an independent set of size $|V(G)|/(d+1)$. We denote by ln the natural logarithm. Throughout the paper, we omit the floor and ceilings signs for the sake of convenience.

2. Key lemma

In this section we prove our main lemma which we think is of independent interest. When this paper was written we learned that independently and before us a similar statement was proved by Kostochka and Rödl [9]. The proof we present here is simpler and gives slightly improved constants. It is based on probabilistic arguments and was influenced by Gowers [7].

Lemma 2.1. *Let $0 < c < 1/2$ and let t, k, m and n be positive integers satisfying the following two inequalities*:

$$(2c)^t n \geqslant 2m \quad \text{and} \quad n^k \left(\frac{m}{n}\right)^t \leqslant k! m. \tag{1}$$

Then every graph $G = (V, E)$ on n vertices with $|E| \geqslant cn^2$ contains a set $U \subset V$ of size at least m with the property that any subset W of U of size k has $|N(W)| \geqslant m$.

Proof. Let $x_1, \ldots, x_t$ be a collection of t, not necessarily distinct vertices of G, which we pick uniformly at random. Denote by A the set of common neighbors of $x_1, \ldots, x_t$ in G. Note that the size of A is a random variable and that for any $v \in A$ all x_i should belong to $N(v)$. Denote by $d(v)$ the degree of vertex v. Then, using Jensen's inequality, we can estimate the expected size of A.

$$\begin{aligned}\mathbf{E}(|A|) &= \sum_{v \in V} \Pr(v \in A) = \sum_{v \in V} \left(\frac{|N(v)|}{n}\right)^t = \frac{\sum_{v \in V} (d(v))^t}{n^t} \geqslant \frac{n\left(\frac{\sum_{v \in V} d(v)}{n}\right)^t}{n^t} \\ &= \frac{n(2|E(G)|/n)^t}{n^t} \geqslant (2c)^t n \geqslant 2m.\end{aligned}$$

On the other hand, by definition, the probability that a given subset of vertices W belongs to A equals $(|N(W)|/n)^t$. Denote by Y the number of subsets W of A of size k which satisfy $|N(W)| < m$. Then by (1) the expected value of Y is at most

$$\mathbf{E}(Y) = \sum_{|W|=k,\ |N(W)|<m} \Pr(W \subset A) \leqslant \binom{n}{k} \left(\frac{m}{n}\right)^t \leqslant \frac{n^k}{k!} \left(\frac{m}{n}\right)^t \leqslant m.$$

Therefore by linearity of expectation there exists a choice of $x_1, \ldots, x_t$ for which $|A| - Y \geqslant m$. Fix such A and delete an arbitrary vertex from every subset W of A of size k which has $|N(W)| < m$. This produces the set U guaranteed by the assertion of the lemma. $\square$

Using this lemma we immediately obtain the following corollary which we will use in the next section to derive results on Ramsey–Turán-type problems.

Corollary 2.2. *Let c be a positive constant and k be a fixed non-negative integer. Let G be a graph on n vertices with at least cn^2 edges and let $\omega(n)$ be any function which tends to infinity arbitrarily slowly with n. Then, for sufficiently large n, G contains a subset of vertices U of size $ne^{-\omega(n)\sqrt{\ln n}}$ such that any $W \subset U$ of size k has $|N(W)| \geqslant ne^{-\omega(n)\sqrt{\ln n}}$.*

Proof. Define $t = \sqrt{\ln n}$ and let $m = ne^{-\omega(n)\sqrt{\ln n}}$. Then it is easy to check that

$$(2c)^t n = ne^{-\ln(1/2c)\sqrt{\ln n}} \geqslant 2ne^{-\omega(n)\sqrt{\ln n}} = 2m$$

and also

$$n^k \left(\frac{m}{n}\right)^t = n^k \left(e^{-\omega(n)\sqrt{\ln n}}\right)^{\sqrt{\ln n}} = n^{k-\omega(n)} = o(1) \ll m.$$

Now the corollary follows from Lemma 2.1. $\square$

3. Applications

In this section we show how to apply Corollary 2.2 to the Ramsey–Turán-type problems. Our first two results were motivated by Problems 1.1–1.3. These results give a precise characterization of the forbidden graphs H for which $\mathbf{RT}(n, H, n^{1-\varepsilon}) = o(n^2)$, for any fixed $\varepsilon > 0$.

Theorem 3.1. *Let $H = (V, E)$ be a fixed graph such that there exists a partition $V = V_1 \cup V_2$ of the vertices of H with the property that the induced subgraph $H[V_i]$, $i = 1, 2$ is acyclic. Then the Ramsey–Turán number of H satisfies*

$$\mathbf{RT}(n, H, f(n) = ne^{-\omega(n)\sqrt{\ln n}}) = o(n^2),$$

where $\omega(n) \to \infty$ arbitrarily slowly with n.

Proof. Let $c>0$ be a constant and let G be a graph on n vertices with cn^2 edges which contains no copy of H. Then to prove the theorem it is enough to show that $\alpha(G) \geqslant ne^{-\omega(n)\sqrt{\ln n}}$. Denote by k the order of the graph H and set $\omega'(n) = \omega(n) - 1$. Since k is a constant, then by Corollary 2.2 G contains a subset of vertices U of size $ne^{-\omega'(n)\sqrt{\ln n}} \geqslant kne^{-\omega(n)\sqrt{\ln n}}$ such that any $W \subset U$ of size k has $|N(W)| \geqslant ne^{-\omega'(n)\sqrt{\ln n}} \geqslant kne^{-\omega(n)\sqrt{\ln n}}$. Consider $G[U]$, the subgraph of G induced by the set U. If $G[U]$ is $(k-1)$-degenerate then it contains an independent set of size $|U|/k = ne^{-\omega(n)\sqrt{\ln n}}$ and we are done. Else it contains a subgraph with minimal degree k. Then by a well known folklore result such a subgraph contains any forest on k vertices and in particular a copy of the graph $H[V_1]$. Denote by $W_1 \subset U$ a vertex set of this copy. Clearly the size of W_1 is at most k and therefore we have that $|N(W_1)| \geqslant kne^{-\omega(n)\sqrt{\ln n}}$. Next consider a subgraph of G induced by the set $N(W_1)$. If this graph contains a copy of $H[V_2]$, denote by $W_2 \subset N(W_1)$ a vertex set of this copy. Since every vertex in W_1 is adjacent to every vertex in W_2 and $H[V_i] \subset G[W_i], i = 1, 2$, then by definition G contains a copy of H on the vertex set $W_1 \cup W_2$, a contradiction. Therefore $G[N(W_1)]$ has no copy of $H[V_2]$. Then similarly as above we conclude that this graph is $(k-1)$-degenerate and hence it contains an independent set of size $|N(W_1)|/k \geqslant ne^{-\omega(n)\sqrt{\ln n}}$. This completes the proof of the theorem. □

Note that if we partition K_4 into two parts of size two, then each part is just an edge. Therefore we immediately obtain that $\mathbf{RT}(n, K_4, ne^{-\omega(n)\sqrt{\ln n}}) = o(n^2)$, for any $\omega(n) \to \infty$. This answers the second part of Problem 1.1 and shows that it is enough to reduce $o(n)$ condition a little bit (not even to $n^{1-\varepsilon}$) in order to reduce significantly the Ramsey–Turán numbers of K_4.

Another interesting corollary of the above theorem deals with case when $H = K_3(2, t, t)$, for any fixed integer $t \geqslant 2$. Indeed, it is easy to check that $K_3(2, t, t)$ has a partition into two parts such that each part is a star of size $t+1$. Thus we can conclude that $\mathbf{RT}(n, K_3(2, t, t), ne^{-\omega(n)\sqrt{\ln n}}) = o(n^2)$. This result shows that if there is a construction of a graph G which implies that the answer to Problem 1.3 is no, then the size of the maximum independent set in such a construction should be almost linear. Next we present a simple example which shows that the result of Theorem 3.1 is tight in the following sense.

Proposition 3.2. *Let $H = (V, E)$ be a fixed graph such that for any partition $V = V_1 \cup V_2$ of the vertices of H at least one of the induced graphs $H[V_i]$, $i = 1, 2$ contains a cycle. Then there exists a constant $\varepsilon = \varepsilon(H) > 0$ such that for any large n there exists a graph G on n vertices with at least $n^2/4$ edges which contains no copy of H and has independence number at most $n^{1-\varepsilon}$.*

Proof. Denote by k the order of H. By the celebrated result of Erdős [3] there exists a constant $\varepsilon > 0$ such that for any large enough n there exists a graph G' on $n/2$ vertices with the following properties. G' contains no cycles of length shorter than $k+1$ and has independence number at most $n^{1-\varepsilon}$. Take two disjoint copies of G' and add the complete bipartite graph between them. Then we obtain a graph G on n vertices with at least $n^2/4$ edges and with independence number at most $n^{1-\varepsilon}$. In addition G contains no copy of H, since in any partition of H into two parts, one part will contain a cycle whose length is at most k. This completes the proof. □

Finally, motivated by Problem 1.4 we obtain a bound on the size of the maximum K_p-free subset in a graph on n vertices which contains no copy of K_{2p} and has $\Omega(n^2)$ edges.

Theorem 3.3. *Let $p \geqslant 3$ be an integer and let G be a graph on n vertices such that G contains no copy of K_{2p}. If in addition $\alpha_p(G) < ne^{-\omega(n)\sqrt{\ln n}}$, where $\omega(n) \to \infty$ arbitrarily slowly with n, then the number of edges in G is $o(n^2)$.*

Proof. Let $c > 0$ be a constant and let G be a graph on n vertices with cn^2 edges which contains no copy of K_{2p}. Then to prove the theorem it is enough to show that $\alpha_p(G) \geqslant ne^{-\omega(n)\sqrt{\ln n}}$. By Corollary 2.2 G contains a subset of vertices U of size $ne^{-\omega(n)\sqrt{\ln n}}$ such that any $W \subset U$ of size p has $|N(W)| \geqslant ne^{-\omega(n)\sqrt{\ln n}}$. Consider $G[U]$, a subgraph of G induced by the set U. If $G[U]$ contains no K_p, then U is a K_p-independent set and we are done. So suppose it contains a copy of K_p and denote by $W' \subset U$ a vertex set of this copy. Clearly the size of W' is p and therefore we have that $|N(W')| \geqslant ne^{-\omega(n)\sqrt{\ln n}}$. If $N(W')$ also contains a copy of K_p then together with the vertices in W' we obtain a complete subgraph of G on $2p$ vertices, a contradiction. Thus $N(W')$ is a K_p-independent set of size at least $ne^{-\omega(n)\sqrt{\ln n}}$. This completes the proof of the theorem. □

Since a graph without copy of K_5 obviously contains no K_6, this theorem implies that any graph G on n vertices with no K_5 subgraph and with $\Omega(n^2)$ edges has a triangle-free subset of size $ne^{-\omega(n)\sqrt{\ln n}} \geqslant n^{1-\varepsilon}$ for any fixed $\varepsilon > 0$. This implies that in Problem 1.4 if we restrict the size of $\alpha_3(G)$ to be slightly smaller than just $o(n)$, then the number of edges in the graph G will be definitely $o(n^2)$. It is also very interesting to compare Theorem 3.3 with the results of [5], where for every $p \geqslant 3$ the authors constructed graphs on n vertices which contain no K_{2p}, have at least $(1 + o(1))n^2/8$ edges and their maximum K_p-free subset is only of order $o(n)$. Our result shows that the value $o(n)$ cannot be reduced significantly in these examples, without a dramatic drop in the number of edges.

4. Concluding remarks

As we already pointed out, most results in Ramsey–Turán theory deal with the case when the independence number of the graph G is bounded by $o(n)$. In the previous section we see some interesting phenomena when we restrict $\alpha(G)$ to be at most $ne^{-\omega(n)\sqrt{\ln n}}$. So it is a natural question what happens if we go even further when $\alpha(G) \leqslant n^{1-\varepsilon}$ for various fixed values of ε. Here we will make first a very week attempt to address this question. Without too much extra effort, only using ideas from the proof of Theorem 3.1, we can show the following.

Proposition 4.1. *Let $r \geqslant 2$ be an integer, then the Ramsey–Turán numbers of K_4 satisfy*

$$\mathbf{RT}(n, K_4, n^{1-1/r}) < n^{2-1/r(r+1)}.$$

Proof. Let G be a graph on n vertices with $n^{2-1/r(r+1)}$ edges which contains no copy of K_4. It is enough to show that $\alpha(G) \geqslant n^{1-1/r}$. Define $t = r + 1$, $m = n^{1-1/r}$, $k = 2$ and $c = n^{-1/r(r+1)}$. Then it is easy to check that they satisfy the following inequalities

$$(2c)^t n \geqslant 2m \quad \text{and} \quad n^k \left(\frac{m}{n}\right)^t \leqslant k!m.$$

Therefore by Lemma 2.1 G contains a subset of vertices U of size $n^{1-1/r}$ such that any $W \subset U$ of size 2 has $|N(W)| \geqslant n^{1-1/r}$. If U is an independent set, then we are done. Else U contains an edge (u, v). As we already mentioned the common neighborhood $N(u, v)$ has size at least $n^{1-1/r}$. Since every edge in $N(u, v)$ together with vertices u, v forms a copy of K_4 we conclude that $N(u, v)$ is an independent set. This implies that $\alpha(G) \geqslant n^{1-1/r}$ and completes the proof. □

Using some additional ideas we can slightly improve this result and extend it to other values of $f(n)$. We plan to return to this problem in the future.

Using Theorem 3.3 we can obtain an interesting connection between the Ramsey–Turán numbers of K_5 and K_6 and the usual Ramsey number of K_3. Indeed, let G be a graph on n vertices with $\Omega(n^2)$ edges which contains no copy of K_6. Then by Theorem 3.3 G contains a triangle-free set U of size at least $ne^{-\Omega(\sqrt{\ln n})}$. Using well known bounds of Ajtai et al. [1], we obtain that U contains an independent set of size at least $\Omega(\sqrt{|U| \ln |U|}) \geqslant \sqrt{n}e^{-\omega(n)\sqrt{\ln n}} = n^{1/2-o(1)}$, where $\omega(n) \to \infty$ arbitrarily slowly with n. This implies that

$$\mathbf{RT}(n, K_5, \sqrt{n}e^{-\omega(n)\sqrt{\ln n}}) \leqslant \mathbf{RT}(n, K_6, \sqrt{n}e^{-\omega(n)\sqrt{\ln n}}) = o(n^2).$$

On the other hand it is easy to see that this result is nearly tight, since we can take two copies of the triangle-free graph on $n/2$ vertices with independence number at most $O(\sqrt{n \ln n})$ (this graph exists by the result of Kim [8]) and connect these copies by a complete bipartite graph. Obviously, the new graph is K_5-free, has the same independence number and at least $n^2/4$ edges. Therefore we conclude

$$\mathbf{RT}(n, K_5, O(\sqrt{n \ln n})) \geqslant \mathbf{RT}(n, K_6, O(\sqrt{n \ln n})) \geqslant \frac{n^2}{4}.$$

We would like to remark that similar results can be obtained for other graphs like K_{2p} or $K_3(3,3,3)$ (in any partition of $K_3(3,3,3)$ into two parts, one part contains $K_2(2,3)$). But since the exact asymptotic behavior of the Ramsey numbers of any graph other than K_3 is not known, the results one can get are less interesting and we will not discuss them in detail.

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