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SELF-SIMILARITY OF GRAPHS*

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Abstract. An old problem raised independently by Jacobson and Schönheim seeks to determine the maximum s for which every graph with m edges contains a pair of edge-disjoint isomorphic subgraphs with s edges. In this paper we determine this maximum up to a constant factor. We show that every m-edge graph contains a pair of edge-disjoint isomorphic subgraphs with at least $c(m \log m)^{2/3}$ edges for some absolute constant c, and find graphs where this estimate is off only by a multiplicative constant. Our results improve bounds of Erdős, Pach, and Pyber from 1987.

Key words. isomorphic factorization, isomorphic subgraphs, divisible graphs

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1. Introduction.

The decomposition of a given graph into smaller subgraphs is an old problem in graph theory that has been studied from numerous perspectives. A celebrated result of Wilson [16] asserts that given any fixed graph H, the edge set of any sufficiently large complete graph K_n can be partitioned into edge-disjoint copies of H, as long as the obvious necessary divisibility conditions $e(H) \mid \binom{n}{2}$ and $g \mid n-1$ (where g is the greatest common divisor of the degrees of H) are satisfied.

A *factor* of a graph is a spanning subgraph, and a *factorization* is a partition of its edges into factors. A series of papers by Graham, Harary, Robinson, Wallis, and Wormald (see, e.g., [7, 9, 10, 11, 15]) introduced the systematic study of *isomorphic factorizations*, in which the resulting factors are required to be isomorphic to each other as graphs. In this literature, a graph G is said to be *divisible* by an integer t, or t-*divisible*, if G admits an isomorphic factorization into t parts, although the analogy with the number-theoretic notion of divisibility is only syntactical. The notion of 2-divisibility has also been termed *bisectable*, with some authors tagging on the extra condition that the resulting factors were also connected graphs.

The earliest work concerned the divisibility of the complete graph. Extending a partial result of Guidotti [8], Harary, Robinson, and Wormald [10] proved that the complete graph K_n is divisible by any integer t which satisfies the obvious necessary condition $t \mid \binom{n}{2}$. Most other existing research on divisibility concentrates on trees and forests, perhaps because their simple structure appears more tractable. Algorithmically, Graham and Robinson proved in [7] that it is NP-hard to decide whether a tree is 2-divisible, while Harary and Robinson [9] discovered a polynomial-time algorithm to decide whether a tree admits an isomorphic factorization into two connected graphs. The best general result on trees is due to Alon, Caro, and Krasikov [1], who showed that every m-edge tree can be made 2-divisible by deleting only $O(m/\log\log m)$ edges.

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Once one considers general graphs, however, it becomes essentially impossible to hope for 2-divisibility or even closeness to 2-divisibility. It is therefore natural to ask what is the largest 2-divisible subgraph which must exist in a given graph. This problem (stated below in generality for hypergraphs) was originally raised independently by Jacobson and Schönheim.

PROBLEM 1.1. *Let the* self-similarity *of an r-uniform hypergraph G, denoted $\iota(G)$, be the largest integer s for which G contains a pair of edge-disjoint isomorphic subhypergraphs with s edges each. For each positive integer m, let $\iota_r(m)$ be the minimum of $\iota(G)$ over all r-uniform hypergraphs with m edges. Determine $\iota_r(m)$.*

Remark. This paper focuses on graphs ($r = 2$), so we will write $\iota(m)$ instead of $\iota_2(m)$ throughout.

The first main result in this area was due to Erdős, Pach, and Pyber [4]. Specifically, they proved that there were absolute constants c_r and C_r for which

$$c_r m^{2/(2r-1)} \leq \iota_r(m) \leq C_r m^{2/(r+1)} \cdot \frac{\log m}{\log \log m}\,.$$

Their upper bound construction is based on an appropriately chosen random r-uniform hypergraph. For graphs ($r = 2$), the powers of m coincide at $m^{2/3}$, so their lower bound deviated only by a logarithmic factor from their upper bound construction, which was essentially the Erdős–Rényi random graph. At around the same time, similar results were obtained independently by Alon and Krasikov (unpublished) and by Gould and Rödl. The latter group determined in [6] that for 3-uniform hypergraphs, $\iota_3(m) \geq \frac{1}{23}\sqrt{m}$, which matched the upper bound exponent, but again fell short by a logarithmic factor. Very recently, Horn, Koubek, and Rödl [13] announced lower bounds for $\iota_4(m)$, $\iota_5(m)$, and $\iota_6(m)$ which also came within polylogarithmic factors of the corresponding upper bounds derived from random hypergraphs.

The main result of our paper completely solves the graph case, determining the asymptotic rate of growth of $\iota(m) = \iota_2(m)$.

THEOREM 1.2. *There are absolute constants c and C for which*

$$c(m \log m)^{2/3} < \iota(m) < C(m \log m)^{2/3}\,.$$

The key idea is to exploit rare large deviation events through a constructive algorithm, rather than to attempt to erase them with union bounds. Incidentally, our upper bound construction is still based on a random graph, but with a slightly modified edge probability.

Inspired by the asymptotic optimality of random graphs in the problem of Jacobson and Schönheim, our next result explicitly studies the self-similarity of random graphs. The Erdős–Rényi random graph $G_{n,p}$ is constructed on the vertex set $[n] = \{1, \ldots, n\}$ by taking each potential edge independently with probability p. We say that $G_{n,p}$ possesses a graph property $\mathcal{P}$ *asymptotically almost surely*, or a.a.s. for brevity, if the probability that $G_{n,p}$ possesses $\mathcal{P}$ tends to 1 as n grows to infinity. Since its first appearance in the 1960s, this beautiful object has been a central topic of study in graph theory. Surprisingly, many problems about random graphs arose from research in various other areas of mathematics and theoretical computer science. Yet despite the great amount of work devoted to this topic over the past fifty years, many interesting unresolved questions still remain to be answered. For more on random graphs, we refer the reader to the books [3, 14].

When $p < \frac{0.99}{n}$, it is well known that a.a.s. all connected components of $G_{n,p}$ are either trees or unicyclic (trees with a single additional edge). Applying the previously mentioned result of Alon, Caro, and Krasikov, or even Proposition 2.3 below, it is then easy to see that the self-similarity of $G_{n,p}$ in that regime is $\Theta(m)$ a.a.s., where m is the number of edges. Our second result asymptotically determines $\iota(G_{n,p})$ for the remaining range of p.

THEOREM 1.3.

(i) *If* $\frac{1}{2n} \le p(n) \le \frac{1}{e^6}\sqrt{\frac{\log n}{n}}$, *then* $\iota(G_{n,p}) = \Theta\big(n \cdot \frac{\log n}{\log \gamma(n)}\big)$ *a.a.s., where* $\gamma(n) = \frac{1}{p}\sqrt{\frac{\log n}{n}}$.

(ii) *If* $p(n) > \frac{1}{e^6}\sqrt{\frac{\log n}{n}}$, *then* $\iota(G_{n,p}) = \Theta(n^2p^2)$ *a.a.s.*

We will prove this theorem in the next section. Its proof illustrates the main ideas of the argument for Theorem 1.2, which follows in section 3.

Notation. Let G be a graph with vertex set V. For a subset of vertices $X \subset V$, let $G[X]$ be the subgraph of G induced by X. For a vertex $v \in V$, we use $N(v)$ to denote the set of neighbors of v. Given a bijection $f : V \to V'$, let $f(G)$ be the graph with vertex set V', where $x', y' \in V'$ are adjacent if and only if there exist two adjacent vertices $x, y \in V$ such that $f(x) = x'$ and $f(y) = y'$. For two graphs G_1 and G_2 defined on the same vertex set, let $G_1 \cup G_2$ be the graph obtained by taking the union of the edge sets of the two graphs, and let $G_1 \cap G_2$ be the graph obtained by taking the intersection of the edge sets of the two graphs.

The following standard asymptotic notation will be utilized extensively. For two functions $f(n)$ and $g(n)$, we write $f(n) = o(g(n))$ if $\lim_{n\to\infty} f(n)/g(n) = 0$, and $f(n) = O(g(n))$ or $g(n) = \Omega(f(n))$ if there exists a constant M such that $|f(n)| \le M|g(n)|$ for all sufficiently large n. We also write $f(n) = \Theta(g(n))$ if both $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ are satisfied. All logarithms will be in base $e \approx 2.718$.

2. Random graphs.

We will use the following well-known concentration result, which is a consequence of Theorems A.1.11 and A.1.13 in the book [2]. Let $\mathrm{Bin}(n, p)$ denote the binomial random variable with parameters n and p.

THEOREM 2.1. *If* $X \sim \mathit{Bin}(n, p)$ *and* $\lambda \le np$, *then*

$$\mathbb{P}\left[|X - np| \ge \lambda\right] \le e^{-\frac{\lambda^2}{15np}}.$$

We begin by analyzing the self-similarity of random graphs. In addition to being an interesting question in its own right, this investigation also suggests good intuition for general graphs. The upper bounds on $\iota(G_{n,p})$ follow from relatively straightforward union bounds.

Proof of upper bound in Theorem 1.3. Suppose that we are seeking a pair of edge-disjoint isomorphic subgraphs with t edges. This task is equivalent to finding subgraphs H' with $2t$ edges that can be partitioned into the union $H \cup \pi(H)$, for some t-edge subgraph H and a permutation π of the vertex set. The expected number of such subgraphs H' in $G_{n,p}$ is at most

$$\binom{\binom{n}{2}}{t} \cdot n! \cdot p^{2t} < \left(\frac{en^2p^2}{t}\right)^t e^{n \log n}, \tag{1}$$

where the first binomial coefficient counts the number of ways to select t edges for H out of all $\binom{n}{2}$ available, and the $n!$ bounds the number of permutations π of the vertex set. Together, these choices determine the $2t$ edges which make up H', which appear with probability p^{2t}. Thus, if we select a value of t for which the right-hand side of (1) becomes $o(1)$, we will establish that the number of such H' is zero a.a.s., and hence $\iota(G_{n,p}) < t$ a.a.s.

We separately specify suitable choices for t for the two regimes of p that we consider in this theorem. For part (i), where $\frac{1}{2n} \le p \le \frac{1}{e^6}\sqrt{\frac{\log n}{n}}$, we use $t = \frac{n \log n}{\log \gamma}$, where $\gamma = \frac{1}{p}\sqrt{\frac{\log n}{n}}$. Note that in this range we have $e^6 \le \gamma \le 2\sqrt{n \log n}$. Then the right-hand side of (1) becomes

$$\left(\frac{en^2 \cdot \frac{\log n}{\gamma^2 n}}{\frac{n \log n}{\log \gamma}} \right)^{\frac{n \log n}{\log \gamma}} e^{n \log n} = \left(\frac{e \log \gamma}{\gamma^2} \right)^{\frac{n \log n}{\log \gamma}} e^{n \log n} = e^{-\frac{n \log n}{\log \gamma} \cdot \log\left(\frac{\gamma^2}{e \log \gamma}\right)} \cdot e^{n \log n}.$$

Since $\gamma \ge e^6$, we have $\log(\frac{\gamma^2}{e \log \gamma}) > \frac{3}{2} \log \gamma$, and hence the right-hand side of (1) is at most $e^{-\frac{3}{2} n \log n} \cdot e^{n \log n} = o(1)$.

For part (ii), where $p \ge \frac{1}{e^6}\sqrt{\frac{\log n}{n}}$, we specify $t = e^{12} n^2 p^2$. The right-hand side of (1) then becomes

$$\left(\frac{1}{e^{11}} \right)^{e^{12} n^2 p^2} e^{n \log n} \le \left(\frac{1}{e^{11}} \right)^{n \log n} e^{n \log n} = o(1)\,. \qquad \square$$

The remainder of this section is devoted to constructing large self-similar subgraphs in $G_{n,p}$. The structure given in the following definition turns out to be extremely useful (both for this section and the next section).

Definition 2.2. *Let d and k be positive integers.*

(i) *A* d-star *is a graph consisting of $d+1$ vertices and d edges, where one of the vertices has degree d. We sometimes simply refer to these graphs as* stars.

(ii) *A* (d,k)-star-forest *is a collection of k vertex-disjoint d-stars. We denote a (d,k)-star-forest by the set of pairs $\{(v, N_v) : v \in B\}$, where B is a set of k vertices, and for each v, the set $N_v \subset N(v)$ is a disjoint set of d neighbors of v.*

The following two propositions were the key ideas in [4]. We include their proofs for completeness, as well as to illuminate the points at which we introduce our new arguments. The first claim asserts that the self-similarity of a graph is large if there are many nonisolated vertices.

Proposition 2.3. *Let G be a graph on n vertices with no isolated vertices. Then $\iota(G) \ge \frac{n-2}{4}$.*

Proof. We first prove that G contains vertex-disjoint stars that cover all the vertices of the graph. Given a graph G, iteratively remove edges that connect two vertices of degree at least two (in an arbitrary order). Clearly, this process never creates isolated vertices, and the final graph consists only of stars because all remaining vertices of degree two or more are nonadjacent.

It remains to show that any n-vertex star forest contains two large edge-disjoint isomorphic subgraphs G_1 and G_2. We consider the stars in the forest by their type. Note that 1-stars are nothing more than single edges, so for every two 1-stars, we can put one of them in G_1 and the other in G_2. We account for this as a contribution of $+1$ toward $\iota(G)$ from the four vertices in the two 1-stars. On the other hand, for $d \ge 2$, we can split the edges of every d-star into two sets of size $\lfloor \frac{d}{2} \rfloor$, possibly with one edge left over. By adding one part to G_1 and the other to G_2, we see that the $d+1$ vertices of each d-star contribute $+\lfloor \frac{d}{2} \rfloor$ to $\iota(G)$. Accumulating the contributions from all vertices, except possibly for at most two vertices from a single unpaired 1-star, we find that

$$\iota(G) \ge (n-2) \cdot \min\left\{ \frac{1}{4},\ \min_{d \ge 2} \left\{ \frac{\lfloor d/2 \rfloor}{d+1} \right\} \right\} = \frac{n-2}{4}\,. \qquad \square$$

Although our problem considers the self-similarity within a single graph, our lower bound argument first separates the given graph into two disjoint subgraphs, and constructs a suitable mapping between them which overlaps many edges.

DEFINITION 2.4. *Let G_1 and G_2 be two edge-disjoint graphs, on possibly overlapping vertex sets V_1 and V_2 of the same cardinality. Let their* similarity $\iota(G_1, G_2)$ *be the maximum integer s such that there exists a bijection $f : V_1 \to V_2$ for which $f(G_1) \cap G_2$ contains s edges.*

The next proposition uses a random mapping as the input in Definition 2.4, in order to measure similarity of two random bipartite graphs.

PROPOSITION 2.5. *For $i = 1, 2$, let G_i be edge-disjoint bipartite graphs with parts A_i and B_i, where $|A_1| = |A_2| = n_1$ and $|B_1| = |B_2| = n_2$. Suppose that $A_1 \cup A_2$ and $B_1 \cup B_2$ are disjoint, but A_1 may intersect A_2 and B_1 may intersect B_2. Then $\iota(G_1, G_2) \geq \frac{|E(G_1)||E(G_2)|}{n_1 n_2}$.*

Proof. Independently sample uniformly random bijections from A_1 to A_2 and from B_1 to B_2, and let f be their combination. For each pair of edges $e_1 \in E(G_1)$ and $e_2 \in E(G_2)$, the probability that e_1 gets mapped to e_2 by f is exactly $\frac{1}{n_1 n_2}$. Such a situation contributes $+1$ to the intersection size $f(G_1) \cap G_2$. Therefore, by linearity of expectation, the expected number of edges in $f(G_1) \cap G_2$ is at least $\frac{|E(G_1)||E(G_2)|}{n_1 n_2}$, and there exists a suitable f which achieves that bound. ∎

COROLLARY 2.6. *Let G be a bipartite graph with parts A and B such that $|E(G)| \geq 10$. Then $\iota(G) \geq \frac{|E(G)|^2}{5|A||B|}$.*

Proof. Arbitrarily partition G into two edge-disjoint subgraphs $G_1 \cup G_2$ with $\lfloor \frac{1}{2}|E(G)| \rfloor \geq \frac{|E(G)|-1}{2} \geq \frac{9|E(G)|}{20}$ edges, and apply Proposition 2.5. ∎

COROLLARY 2.7. *Let G be a graph with n vertices and m edges, where $m \geq 20$. Then $\iota(G) \geq \frac{m^2}{5n^2}$.*

Proof. Let $A \cup B$ be a bipartition of the vertex set of G chosen uniformly at random. The probability of a single edge intersecting both parts is exactly $\frac{1}{2}$, and thus by averaging, there exists a bipartition $A \cup B$ for which the bipartite graph H between A and B contains at least $\frac{m}{2}$ edges. Since $|A||B| \leq \frac{n^2}{4}$ and $m/2 \geq 10$, by Corollary 2.6, we have $\iota(G) \geq \frac{(m/2)^2}{5(n^2/4)} = \frac{m^2}{5n^2}$. ∎

To prove Proposition 2.5, we considered a random bijection between the two vertex sets, as there exists a map such that the resulting number of overlapping edges is at least its expectation. This strategy turns out to be strong enough when the graph is dense. On the other hand, for sparse graphs, Proposition 2.3 produces a reasonable bound. These were the key steps used by Erdős, Pach, and Pyber in [4]. In order to establish Theorem 1.3, however, we need something slightly more powerful for the intermediate edge density regime.

The key new ingredient is to design a vertex permutation that performs better than a uniformly random one. To sketch our argument, consider the illustrative case $p = n^{-1/2}$, which represents the most delicate range. We first randomly split the vertices into four parts A_1, A_2, B_1, B_2 of equal size, and let G_i be the bipartite graph formed by the edges between A_i and B_i. We discard all other edges and bound only the similarity between G_1 and G_2. Rather than searching for an unstructured permutation of the whole vertex set, we build a favorable bijection $f : A_1 \cup B_1 \to A_2 \cup B_2$ which sends A_1 to A_2 and B_1 to B_2 with many overlapping edges. Note that if we let f be a uniformly random bijection from $A_1 \cup B_1$ to $A_2 \cup B_2$, then we essentially recover Proposition 2.5, thus producing a lower bound of order only $\Theta(n)$, which falls short of Theorem 1.3 by a logarithmic factor.

We start with a uniformly random bijection from B_1 to B_2, and carefully extend it from A_1 to A_2 as follows. Consider a fixed vertex v_1 in A_1 and a fixed vertex v_2 in A_2. If we mapped v_1 to v_2, we would increase the number of overlapping edges by exactly $|f(N(v_1)) \cap N(v_2)|$, where $N(v_i)$ represents the set of neighbors of v_i in B_i. (Recall that we discarded all other edges, so the v_i only have neighbors in their corresponding B_i.) Since we have $p = n^{-1/2}$, if v_2 is chosen uniformly at random, the expected size of the set $f(N(v_1)) \cap N(v_2)$ is some constant λ, and this observation led to the $\Theta(n)$ lower bound when considering a uniformly random bijection.

The crucial observation is that for each individual pair of v_i, the overlap $|f(N(v_1)) \cap N(v_2)|$ asymptotically has the Poisson distribution with mean λ. Therefore, with probability at least $n^{-\varepsilon}$, it will be of size at least $\varepsilon' \frac{\log n}{\log \log n}$ for some small constants ε and ε'. Since A_2 has $\frac{n}{4}$ vertices, the expected number of vertices $v_2 \in A_2$ that will give this high gain together with v_1 is $\Omega(n^{1-\varepsilon})$. In particular, it is very likely that there exists a suitable vertex v_2 for v_1 such that $|f(N(v_1)) \cap N(v_2)| \geq \varepsilon' \frac{\log n}{\log \log n}$, and we will map v_1 to v_2 in such a situation. By repeating this for a constant proportion of vertices in A_1, we will obtain $\iota(G_{n,p}) \geq \Omega(n \cdot \frac{\log n}{\log \log n})$. Since $\gamma = \sqrt{\log n}$, this gives $\iota(G_{n,p}) \geq \Omega(n \cdot \frac{\log n}{\log \gamma(n)})$ for our choice of p. Our next two lemmas formalize this intuition.

LEMMA 2.8. *Let n and p satisfy $n^{-\frac{21}{40}} \leq p \leq \frac{1}{e^6}\sqrt{\frac{\log n}{n}}$, and define $\gamma = \frac{1}{p}\sqrt{\frac{\log n}{n}}$. Let $N_1, \ldots, N_s \subset B$ be $s \geq n^{1/3}$ disjoint sets of size $\frac{np}{16}$, and consider the random set B_p, where we take each element of B independently with probability p. Then with probability at least $1 - e^{-\Omega(n^{1/12})}$, there is an index i such that $|B_p \cap N_i| \geq \frac{\log n}{20 \log \gamma}$.*

Proof. Let $t = \left\lceil \frac{\log n}{20 \log \gamma} \right\rceil$. In our range of p, we always have $2 \leq t \leq \left\lceil \frac{\log n}{120} \right\rceil$, so in particular $t \leq \frac{\log n}{10 \log \gamma}$. For a fixed index i, the probability that $|B_p \cap N_i| \geq \frac{\log n}{20 \log \gamma}$ is at least $\binom{|N_i|}{t} p^t (1-p)^{|N_i|-t}$. Using the bounds $\binom{n}{k} \geq \left(\frac{n}{k}\right)^k$ and $(1-p) \geq e^{-\frac{16}{15}p}$ for small p, we have

$$\begin{aligned}
\binom{|N_i|}{t} p^t (1-p)^{|N_i|-t} &\geq \left(\frac{np^2}{16t}\right)^t e^{-np^2/15} = \left(\frac{\log n}{16\gamma^2 t}\right)^t e^{-np^2/15} \\
&\geq \left(\frac{10 \log \gamma}{16\gamma^2}\right)^{\log n/(10 \log \gamma)} \cdot n^{-1/(15e^{12})} \\
&= e^{-\frac{\log n}{10 \log \gamma} \cdot \log\left(\frac{16\gamma^2}{10 \log \gamma}\right)} \cdot n^{-1/(15e^{12})},
\end{aligned}$$

which by $\log\left(\frac{16\gamma^2}{10 \log \gamma}\right) \leq 2 \log \gamma$ (deduced from $\gamma \geq e^6$), is at least

$$e^{-(\log n/5)} \cdot n^{-1/(15e^{12})} \geq n^{-1/4}.$$

Hence the expected number of indices i such that $|B_p \cap N_i| \geq \frac{\log n}{20 \log \gamma}$ is at least $s \cdot n^{-1/4} \geq n^{1/12}$. Since the sets N_i are disjoint, the above events for different choices of i are mutually independent. Therefore, by Chernoff's inequality, with probability at least $1 - e^{-\Omega(n^{1/12})}$, we can find an index i (indeed, several) for which $|B_p \cap N_i| \geq \frac{\log n}{20 \log \gamma}$. ∎

The previous estimate enables us to bound the similarity between random bipartite graphs.

LEMMA 2.9. *Let n and p satisfy $n^{-\frac{21}{40}} \leq p \leq \frac{1}{e^6}\sqrt{\frac{\log n}{n}}$, and let $\gamma = \frac{1}{p}\sqrt{\frac{\log n}{n}}$. Let A_1, B_1, A_2, B_2 be disjoint sets of size $\frac{n}{4}$ each, and for each $i = 1, 2$, let G_i be a* *random bipartite graph with parts A_i and B_i, where each edge appears independently with probability p. Then $\iota(G_1, G_2) \geq \frac{n \log n}{160 \log \gamma}$ a.a.s.*

Proof. Start with a uniformly random bijection f from B_1 to B_2, and also expose all edges in the random bipartite graph G_2. Since $p \geq n^{-\frac{21}{40}}$, Chernoff's inequality and a union bound establish that a.a.s., all degrees in G_2 are between $\frac{np}{8}$ and np. Condition on this event. We expose the edges in the bipartite graph G_1 by iterating over the vertices in A_1, exposing each vertex's incident edges in turn. Consider the following greedy algorithm for finding a bijection between A_1 and A_2. Let A_1' be the set of vertices in A_1 whose edges have been exposed, and suppose that we have an injective map $f : A_1' \to A_2$ such that for all $x \in A_1'$, $f(N(x))$ and $N(f(x))$ intersect in at least $\frac{\log n}{20 \log \gamma}$ vertices. Let $A_2' = f(A_1')$, and let $A_i'' = A_i \setminus A_i'$ for $i = 1, 2$. Suppose that $|A_1''| = |A_2''| \geq \frac{|A_1|}{2}$ at some point of the process.

We first prove that the graph $A_2'' \cup B_2$ contains a $(\frac{np}{16}, n^{1/3})$-star-forest. Indeed, let k be the largest integer such that there exists a $(\frac{np}{16}, k)$-star-forest $\{(x, N_x) : x \in X\}$ for some set $X \subset A_2''$ of size $|X| = k$, and suppose that $k < n^{1/3}$. Let $N(X)$ be the union of all neighborhoods of vertices in X. We know that for every vertex $w \in A_2'' \setminus X$, we have $|N(w) \cap N(X)| \geq \left(\frac{1}{8} - \frac{1}{16}\right) np \geq \frac{np}{16}$, as otherwise we find a $(\frac{np}{16}, k+1)$-star-forest, contradicting maximality. Therefore, there are at least $\frac{np}{16} \cdot (|A_2''| - |X|) \geq \frac{n^2 p}{128}$ edges between the sets $A_2'' \setminus X$ and $N(X)$, and in particular, the set $N(X)$ has at least $\frac{n^2 p}{128}$ incident edges in G_2. Note that $|N(X)| \leq knp \leq n^{4/3} p$, since we conditioned on all degrees in G_2 being at most np, and by the same reason, the number of edges incident to $N(X)$ must be at most $n^{7/3} p^2 < \frac{n^2 p}{128}$, contradiction. Therefore, we have $k \geq n^{1/3}$, as claimed.

Now take any vertex $v_1 \in A_1''$ and expose its edges to B_1. Its neighborhood $N(v_1)$ is a random subset of B_1, where each vertex of B_1 appears independently with probability p. Since the bijection $f : B_1 \to B_2$ was fixed from the outset, the image of the neighborhood $f(N(v_1))$ is also a random subset of B_2 with the same product distribution. By Lemma 2.8, with probability at least $1 - e^{-\Omega(n^{1/12})}$, we can find a vertex $v_2 \in X \subset A_2''$ such that $|f(N(v_1)) \cap N_{v_2}| \geq \frac{\log n}{20 \log \gamma}$, where X and N_{v_2} were from the star forest constructed above. Define $f(v_1) = v_2$ and repeat the procedure. Since the probability of success at each round is $1 - o(n^{-1})$, we can successfully iterate $\frac{|A_1|}{2}$ times a.a.s., and then finish by extending f by an arbitrary bijection between the nonmapped vertices of A_1 and A_2. In this way, we obtain a bijection f such that the number of edges in $f(G_1) \cap G_2$ is at least $\frac{|A_1|}{2} \cdot \frac{\log n}{20 \log \gamma} = \frac{n \log n}{160 \log \gamma}$, as desired. ∎

We are now ready to prove the lower bounds of Theorem 1.3.

Proof of lower bound in Theorem 1.3. Part (i) has two subcases. First, for $\frac{1}{2n} \leq p \leq n^{-21/40}$, note that $\gamma = \frac{1}{p}\sqrt{\frac{\log n}{n}} \geq n^{1/40}\sqrt{\log n}$, so the desired lower bound is of order $n \cdot \frac{\log n}{\log \gamma} = \Theta(n)$. In this range, the number of nonisolated vertices is $\Theta(n)$ a.a.s., so Proposition 2.3 completes this case. For the next range $n^{-\frac{21}{40}} \leq p \leq \frac{1}{e^6}\sqrt{\frac{\log n}{n}}$, we apply Lemma 2.9 after splitting the vertex set into four parts. Part (ii) follows directly from Corollary 2.6. ∎

3. Self-similarity of general graphs.

Although general graphs are not intrinsically random, we apply probabilistic techniques to find large edge-disjoint isomorphic subgraphs. The outline of our proof for general graphs is similar to that for random graphs (see the discussion following Corollary 2.7 in the previous section). The key idea is to exploit tail events in the Poisson distribution. However, establishing this was somewhat easier for random graphs since we had independence and could expose edges in a controlled manner. For general graphs, there are no random edges to expose. Instead, we turn to star forests, which were also an important component in the proof of Lemma 2.9.

Let G be a given graph on n vertices with average degree d. As before, we begin by randomly splitting the vertices into four parts A_1, A_2, B_1, B_2, and consider the bipartite graphs G_i formed by the edges between A_i and B_i. We attempt to find a total of $\Omega(n^{1-\alpha})$ many $(\frac{d}{8}, n^{\alpha})$-star-forests $S_{i,j} = \{(v, N_v) : v \in X_{i,j}\}$ for $i = 1, 2$, $1 \leq j \leq \Omega(n^{1-\alpha})$, where the sets $X_{i,j} \subset A_i$ are disjoint for different indices. Note that $\bigcup X_{i,j}$ then cover a constant fraction of each A_i, and hence the edges in these star forests constitute a constant fraction of the edges in the entire graph G. If we fail to find such star forests, then we will be able to pass to a subgraph where we can find even larger isomorphic subgraphs. On the other hand, once we find such star forests, we take a random bijection f_B from B_1 to B_2, and extend it by independent bijections from $X_{1,j}$ to $X_{2,j}$. To this end, we declare f_B to be *good* for the index j if it can be extended to a bijection between the sets $B_1 \cup X_{1,j}$ and $B_2 \cup X_{2,j}$ so that the two star forests overlap in $\Omega\big(|X_{1,j}| \cdot \frac{\log n}{\log(\frac{n \log n}{d^2})}\big)$ edges under the map. If some bijection f_B happens to be good for a constant proportion of indices j, then we can extend the bijection f_B to the sets $X_{1,j}$ for these indices, and thereby construct a map f that overlaps many edges of G_1 and G_2.

To begin this program, our first lemma establishes the tail probability of the main random variable in our setting. It is the analogue of Lemma 2.8.

LEMMA 3.1. *Let $\alpha < \frac{1}{2}$ be a fixed positive real number, and let d and n satisfy $n^{\frac{1}{2}-\frac{\alpha}{16}} \leq d \leq \sqrt{\alpha n \log n}$. Let $N_1, \ldots, N_s \subset [n]$ be fixed disjoint sets of size $\frac{d}{2}$ for some $s \geq \frac{1}{5} n^{\alpha}$, and let N be a uniformly random subset of $[n]$ with exactly d elements. Then with probability at least $1 - e^{-\Omega(n^{\alpha/4})}$, there exists an index i such that $|N \cap N_i| \geq \frac{\alpha \log n}{8 \log(\frac{n \log n}{d^2})}$.*

Proof. Let N' be a random subset of $[n]$ obtained by independently taking each element with probability $\frac{d}{2n}$. The distribution of N' conditioned on the event $|N'| \leq d$ can be coupled with the random variable N, so that $N' \subset N$ (given N', let N be a set of size d containing N' chosen uniformly at random). By Chernoff's bound, the probability of $|N'| > d$ is at most $e^{-\Omega(d)} < e^{-\Omega(n^{\alpha/4})}$, since $d \geq n^{\frac{1}{2}-\frac{\alpha}{16}}$ and $\alpha < \frac{1}{2}$. Therefore, in order to prove our lemma, it suffices to show that with probability at least $1 - e^{-\Omega(n^{\alpha/4})}$, there exists an index i such that $|N' \cap N_i| \geq \frac{\alpha \log n}{8 \log(\frac{n \log n}{d^2})}$. Define

$$\gamma = \frac{n \log n}{d^2} \quad \text{and} \quad t = \left\lceil \frac{\alpha \log n}{8 \log \gamma} \right\rceil .$$

Since $n^{\frac{1}{2}-\frac{\alpha}{16}} \leq d \leq \sqrt{\alpha n \log n}$, we have

$$2 < \frac{1}{\alpha} \leq \gamma \leq n^{\frac{\alpha}{8}} \log n,$$

from which it follows that

$$t \geq \frac{\alpha \log n}{8 \log \gamma} \geq \frac{\alpha \log n}{8 \log(n^{\frac{\alpha}{8}} \log n)} = \frac{\alpha \log n}{\alpha \log n + 8 \log \log n} \geq \frac{1}{2} \tag{2}$$

for sufficiently large n. Therefore, the rounding effect in the definition of t at most doubles the value, and we have $1 \leq t \leq \frac{\alpha \log n}{4 \log \gamma}$.

For each index i, let E_i be the event that $|N' \cap N_i| \geq t$. As $|N' \cap N_i|$ is binomially distributed, just as in the proof of Lemma 2.8, we may use the bounds $\binom{n}{k} \geq \left(\frac{n}{k}\right)^k$ and $1 - p > e^{-2p}$ (for small p) to find

$$\mathbb{P}[E_i] \geq \binom{|N_i|}{t} \left(\frac{d}{2n}\right)^t \left(1 - \frac{d}{2n}\right)^{|N_i|-t} \geq \left(\frac{d/2}{t}\right)^t \left(\frac{d}{2n}\right)^t \left(e^{-\frac{d}{n}}\right)^{\frac{d}{2}} = \left(\frac{d^2}{4nt}\right)^t e^{-\frac{d^2}{2n}}.$$

Substitute $t \leq \frac{\alpha \log n}{4 \log \gamma}$ to get

$$\mathbb{P}[E_i] \geq \left(\frac{d^2}{4n} \cdot \frac{4 \log \gamma}{\alpha \log n}\right)^t e^{-\frac{d^2}{2n}} = \left(\frac{\log \gamma}{\alpha \gamma}\right)^t e^{-\frac{\log n}{2\gamma}}.$$

Since $\alpha < \frac{1}{2}$, $\log \gamma > \log 2$, and $t \leq \frac{\alpha \log n}{4 \log \gamma}$, this is at least

$$\left(\frac{1}{\gamma}\right)^{\frac{\alpha \log n}{4 \log \gamma}} e^{-\frac{\log n}{2\gamma}} = n^{-\frac{\alpha}{4}} n^{-\frac{1}{2\gamma}} \geq n^{-\frac{\alpha}{4}} n^{-\frac{\alpha}{2}} = n^{-\frac{3\alpha}{4}}.$$

The E_i are independent because the N_i are disjoint. Therefore the number of E_i that occur stochastically dominates a binomial random variable with mean $sn^{-3\alpha/4} \geq \frac{1}{5} n^{\alpha/4}$, and we conclude by the Chernoff bound that at least one E_i (indeed, several) occurs with probability $1 - e^{-\Omega(n^{\alpha/4})}$, as desired. ▯

In the previous section, in Lemma 2.9, we exploited the fact that the given graph was random and the edges were independent. This trick is too restrictive to be applied to general graphs. However, the next lemma says that for star forests, one can obtain a lemma similar to Lemma 2.9.

LEMMA 3.2. *Let $\alpha < \frac{1}{2}$ be a fixed positive real number, and suppose that n and d satisfy $n^{\frac{1}{2} - \frac{\alpha}{16}} \leq d \leq \sqrt{\alpha n \log n}$ and are sufficiently large. For $i = 1, 2$, let G_i be a (d, n^α)-star-forest $\{(v, N_v) : v \in X_i\}$ in the vertex set $X_i \cup B_i$, where $|X_i| = n^\alpha$ and $|B_i| = n$. The bijection f_B from B_1 to B_2 chosen uniformly at random satisfies the following property with probability at least $1 - e^{-\Omega(n^{\alpha/4})}$: f_B can be extended to $X_1 \cup B_1$ so that the graph $f_B(G_1) \cap G_2$ has at least $|X_1| \cdot \frac{\alpha \log n}{36 \log\left(\frac{n \log n}{d^2}\right)}$ edges.*

Proof. Consider a uniformly random bijection f_B from B_1 to B_2. As in the proof of Lemma 2.9, we will pick vertices of X_1 one at a time, mapping each one to some vertex in X_2 in such a way that their neighbors intersect in at least $\frac{\alpha \log n}{9 \log\left(\frac{n \log n}{d^2}\right)}$ vertices under the map f_B. By repeating this for $|X_1|/4$ steps, we then extend f_B to form a total of at least $\frac{|X_1|}{4} \cdot \frac{\alpha \log n}{9 \log\left(\frac{n \log n}{d^2}\right)}$ overlapping edges, as required.

To this end, suppose that we have already embedded some set $X_1' \subset X_1$ of size less than $|X_1|/4$, and let X_2' be the image of X_1'. Further suppose that we have only exposed the outcome of f_B on the neighbors of X_1'. Let $B_1' = \bigcup_{x \in X_1'} N_x$ and B_2' be its image (which is already fully determined by our partial exposure). The unexposed remainder of f_B, conditioned on the previous outcome, is a random uniform bijection from $B_1 \setminus B_1'$ to $B_2 \setminus B_2'$. Choose an arbitrary vertex $x_1 \in X_1 \setminus X_1'$. Call a vertex $x_2 \in X_2 \setminus X_2'$ *available* if $|N_{x_2} \setminus B_2'| \geq \frac{d}{2}$, or equivalently, $|N_{x_2} \cap B_2'| \leq \frac{d}{2}$. Since each unavailable vertex accounts for at least $\frac{d}{2}$ vertices of $|B_2'|$, and those sets are disjoint for different unavailable vertices (because G_2 is a star forest), we conclude that the number of unavailable vertices is at most

$$\frac{|B_2'|}{d/2} = \frac{d|X_2'|}{d/2} = 2|X_2'| \leq \frac{|X_1|}{2},$$

and hence the number of available vertices in $X_2 \setminus X_2'$ is at least $|X_1|/4$.

We now expose the images of the d neighbors of x_1. This is a uniformly random d-element subset of $B_2 \setminus B_2'$, where

$$(1-o(1))n = n - d|X_1'| \le |B_2 \setminus B_2'| \le n\,.$$

For each available vertex x_2, its (deterministically known) neighborhood in $B_2 \setminus B_2'$ has size at least $d/2$, and there are at least $|X_1|/4 = n^\alpha/4$ such neighborhoods, all disjoint, coming from different available vertices. We are therefore in the setting of Lemma 3.1 (with $(1-o(1))n$ instead of n), and so we conclude that with probability $1-e^{-\Omega(n^{\alpha/4})}$, there is an available vertex x_2 such that $|f_B(N_{x_1}) \cap N_{x_2}| \ge \frac{\alpha \log n}{9 \log\left(\frac{n \log n}{d^2}\right)}$. Furthermore, we only need to expose the outcome of f_B on N_{x_1}. We can continue the process for at least $\frac{|X_1|}{4}$ times, with probability at least $1 - \frac{|X_1|}{4} \cdot e^{-\Omega(n^{\alpha/4})} = 1 - e^{-\Omega(n^{\alpha/4})}$. This proves the lemma. ∎

Our next proposition bounds the self-similarity of a graph in terms of its median degree. To prove the proposition, we will find many star forests in our graph and apply Lemma 3.2 several times.

PROPOSITION 3.3. *Let $\alpha \le \frac{1}{25}$ be a fixed positive real number. Then for every sufficiently large n and d satisfying $6n^{\frac{1}{2}-\frac{\alpha}{16}} \le d \le \sqrt{\alpha n \log n}$, every n-vertex graph G with at least $\frac{n}{2}$ vertices of degree at least d has $\iota(G) > \frac{\alpha n \log n}{2592 \log\left(\frac{n \log n}{d^2}\right)}$.*

Proof. Take a uniformly random partition $A_1 \cup A_2 \cup B_1 \cup B_2$ of the vertex set, where $|A_1| = |A_2| = |B_1| = |B_2| = \frac{n}{4}$. For $i = 1, 2$, let G_i be the bipartite graph formed by the edges between A_i and B_i. Since $d > n^{1/3}$, by the concentration of the hypergeometric distribution (see, e.g., Theorem 2.10 of [14]) and a union bound, one can see that a.a.s. each A_i contains at least $\frac{n}{9}$ vertices that have at least $\frac{d}{5}$ neighbors in B_i in the graph G_i. Condition on this event.

Let $d' = \frac{d}{10}$ and $n' = \frac{n}{4}$, and note that since $\alpha \le \frac{1}{25}$, $\frac{2n^\alpha}{3} < (n')^\alpha < n^\alpha$. Let k_1 be the largest integer for which we can find a collection of $(d', (n')^\alpha)$-star-forests $S_{1,j} = \{(v, N_v) : v \in X_{1,j}\}$ in G_1, where the sets $X_{1,j}$ are disjoint subsets of A_1 for $1 \le j \le k_1$. We claim that $k_1 \ge \frac{n^{1-\alpha}}{18}$. Indeed, if not, then there exist over $\frac{n}{9} - k_1(n')^\alpha \ge \frac{n}{18}$ vertices in A_1 that are not covered by the sets of the form $X_{1,j}$ and have degree at least $\frac{d}{5}$ in the set B_1. Let A_1' be the set of these vertices. By our maximality assumption, we know that the graph $G_1[A_1' \cup B_1]$ does not contain a $(d', (n')^\alpha)$-star-forest. Let $S = \{(v, N_v) : v \in X\}$ be a (d', h)-star-forest in $G_1[A_1' \cup B_1]$, where $X \subset A_1'$ and h is as large as possible. By our assumption, we know that $h < (n')^\alpha$. Then all the vertices in $A_1' \setminus X$ have degree at least $\frac{d}{10}$ in the set $N = \bigcup_{v \in X} N_v$. Note that $|N| = d'h < \frac{d}{10} \cdot n^\alpha$ and $|A_1' \setminus X| \ge \frac{n}{18} - h > \frac{n}{19}$. In this case, Corollary 2.6 applied to $G[(A_1' \setminus X) \cup N]$ already gives

$$\iota(G) \ge \iota(G[(A_1' \setminus X) \cup N]) \ge \frac{((d/10) \cdot |A_1' \setminus X|)^2}{5|N| \cdot |A_1' \setminus X|} = \frac{d^2 |A_1' \setminus X|}{500|N|} > \frac{dn^{1-\alpha}}{950} > \frac{n^{4/3}}{950}\,,$$

which for large n is already far more than enough. Therefore, we may assume that $k_1 \ge \frac{n^{1-\alpha}}{18}$. Similarly, there is a collection of $\frac{n^{1-\alpha}}{18}$ many $(d', (n')^\alpha)$-star-forests $S_{2,j} = \{(v, N_v) : v \in X_{2,j}\}$ in G_2, where $X_{2,j}$ are disjoint subsets of A_2.

Let f_B be a bijection from B_1 to B_2 chosen uniformly at random. Our initial conditions on n and d imply that n' and d' satisfy the requirements of Lemma 3.2, so for each fixed j, with probability at least $1 - e^{-\Omega(n^{\alpha/4})}$, f_B can be extended to a bijection between $B_1 \cup X_{1,j}$ and $B_2 \cup X_{2,j}$ such that $f_B(G_1[B_1 \cup X_{1,j}])$ and $G_2[B_2 \cup X_{2,j}]$ overlap in at least

$$|X_{1,j}| \cdot \frac{\alpha \log n'}{36 \log\left(\frac{n' \log(n')}{(d')^2}\right)} > \frac{2n^\alpha}{3} \cdot \frac{\alpha \log\left(\frac{n}{4}\right)}{36 \log\left(\frac{25 n \log n}{d^2}\right)} \geq \frac{n^\alpha \cdot \alpha \log n}{144 \log\left(\frac{n \log n}{d^2}\right)}$$

edges, where we used $\frac{n \log n}{d^2} \geq \frac{1}{\alpha} \geq 25$.

Since the sets $X_{1,j}$ are disjoint for distinct j, and $X_{2,j}$ are also disjoint for distinct j, a union bound shows that we can independently extend the bijection f_B by each $X_{1,j} \to X_{2,j}$ to construct a map $f : A \to B$ which establishes

$$\iota(G_1, G_2) > \frac{n^{1-\alpha}}{18} \cdot \frac{n^\alpha \cdot \alpha \log n}{144 \log\left(\frac{n \log n}{d^2}\right)} = \frac{\alpha n \log n}{2592 \log\left(\frac{n \log n}{d^2}\right)},$$

completing the proof. ∎

We are now ready to prove Theorem 1.2 and establish the correct order of magnitude of the function $\iota(m)$.

Proof of Theorem 1.2. Consider the random graph $G_{n,p}$ with $p = \sqrt{\frac{\log n}{n}}$. For $m = \frac{1}{2} n^{3/2} \sqrt{\log n}$, we a.a.s. have $e(G_{n,p}) = (1 + o(1))m$, and by Theorem 1.3, $\iota(G_{n,p}) = \Theta(n \log n) = \Theta((m \log m)^{2/3})$. Since the function ι is monotone, this shows that $\iota(m) \leq O((m \log m)^{2/3})$ and establishes the upper bound. In the remainder of the proof, we focus on proving the lower bound.

Let G be the given graph with n vertices and m edges. Without loss of generality, we may assume that G contains no isolated vertices. Let $n_0 = n$, $m_0 = m$, $G_0 = G$, and let V_0 be the vertex set of G_0. Let $n_0 = 2^{a_0} \frac{m_0^{2/3}}{(\log m_0)^{1/3}}$ for some real a_0. Let $t = 1$ in the beginning and consider the following iterative process. At each step t, we will either find two large isomorphic edge-disjoint subgraphs or an induced subgraph G_t on the vertex set V_t such that for $n_t = |V_t|$, $m_t = |E(G_t)|$ and a_t satisfying $n_t = 2^{a_t} \frac{m_t^{2/3}}{(\log m_t)^{1/3}}$, we have the following properties:

(i) G_t has no isolated vertex,

(ii) $m_0 \geq m_t \geq \left(1 - \sum_{i=0}^{t-1} 2^{-a_i}\right) m_0 > \frac{m}{3}$, and

(iii) $a_t \leq a_{t-1} - \frac{1}{3}$ for $t \geq 1$.

Note that the properties indeed hold for $t = 0$. Suppose that we are given parameters as above for some $t \geq 0$. If $n_t \geq (m_t \log m_t)^{2/3}$, then by Proposition 2.3, we have $\iota(G) \geq \frac{(m_t \log m_t)^{2/3} - 2}{4} = \Omega((m \log m)^{2/3})$. On the other hand, if $n_t \leq \frac{8 m_t^{2/3}}{(\log m_t)^{1/3}}$, then by Corollary 2.7, we have

$$\iota(G) \geq \frac{m_t^2}{5 n_t^2} \geq \Omega((m \log m)^{2/3}).$$

Therefore, we may assume that

$$\frac{8 m_t^{2/3}}{(\log m_t)^{1/3}} < n_t < (m_t \log m_t)^{2/3}, \tag{3}$$

from which it follows that $3 < a_t < \log_2 \log m_t$. Define

$$d_t = \frac{m_t}{2^{a_t} \cdot n_t} = 2^{-2a_t} (m_t \log m_t)^{1/3},$$

and let V_t' be the subset of vertices which have degree at least d_t in the graph G_t. Using the upper bound of (3) together with $a_t < \log_2 \log m_t$, one can see that

$$d_t > \frac{n_t^{3/2}/\log m_t}{2^{a_t} \cdot n_t} > \frac{n_t^{1/2}}{(\log m_t)^2} > 6n_t^{1/2-\alpha/16}$$

for $\alpha = \frac{1}{25}$. The lower bound of (3) gives $m_t < (\frac{n_t}{8})^{3/2}(\log m_t)^{1/2}$, so using $a_t > 3$, we find that

$$d_t \le \frac{(n_t/8)^{3/2}\sqrt{\log m_t}}{2^{a_t} \cdot n_t} < \frac{1}{147}\sqrt{n_t \log n_t} < \sqrt{\alpha n_t \log n_t}\,.$$

Consequently, if $|V_t'| \ge \frac{|V_t|}{2}$, then by Proposition 3.3 we have

$$\iota(G) > \frac{\alpha n_t \log n_t}{2592 \log\left(\frac{n_t \log n_t}{d_t^2}\right)} = \frac{n_t \log n_t}{64800 \log\left(\frac{n_t \log n_t}{d_t^2}\right)}\,.$$

Since $\frac{n_t}{d_t^2} = \frac{2^{5a_t}}{\log m_t}$ and $\log m_t > \log n_t = a_t \log 2 + \frac{2}{3}\log m_t - \frac{1}{3}\log\log m_t > \frac{1}{2}\log m_t$, we have

$$\iota(G) > \frac{n_t \log n_t}{64800 \log\left(\frac{n_t}{d^2}\log n_t\right)} > \frac{n_t \log m_t}{(648000 \log 2)a_t} = \frac{2^{a_t}}{(648000\log 2)a_t} \cdot m_t^{2/3}(\log m_t)^{2/3}.$$

Since $a_t > 3$, we have $\frac{2^{a_t}}{a_t} > 2$, and thus

$$\iota(G) > \frac{1}{324000 \log 2} \cdot m_t^{2/3}(\log m_t)^{2/3} = \Omega((m \log m)^{2/3})\,.$$

Otherwise, we have $|V_t'| < \frac{|V_t|}{2}$. Let V_{t+1} be the set of nonisolated vertices in the induced subgraph $G[V_t']$. Let $n_{t+1} = |V_{t+1}|$ and let m_{t+1} be the number of edges in the induced subgraph $G_{t+1} = G[V_{t+1}]$. Define a_{t+1} so that $n_{t+1} = 2^{a_{t+1}} \frac{m_{t+1}^{2/3}}{(\log m_{t+1})^{1/3}}$. Note that since we only removed vertices whose degree in G_t was less than d_t, our new number of edges is $m_{t+1} > m_t - n_t d_t = (1 - 2^{-a_t})m_t$, and in particular is well above $m_t/2$ because $a_t > 3$. Property (i) follows from the definition. For property (ii), note that

$$m_{t+1} > (1-2^{-a_t})m_t \ge (1-2^{-a_t})\left(1 - \sum_{i=0}^{t-1} 2^{-a_i}\right) m_0 > \left(1 - \sum_{i=0}^{t} 2^{-a_i}\right) m_0\,,$$

and moreover, since $a_i > 3$ and $a_{i+1} \le a_i - \frac{1}{3}$ for all i, we have

$$\left(1 - \sum_{i=0}^{t} 2^{-a_i}\right) m > \left(1 - \sum_{i=0}^{\infty} 2^{-3-\frac{i}{3}}\right) m = \left(1 - \frac{1}{8} \cdot \frac{1}{1-2^{-1/3}}\right) m > \frac{m}{3}\,.$$

Finally, since $m_t/2 < m_{t+1} \le m_t$, we have

$$n_{t+1} < \frac{n_t}{2} = 2^{a_t - 1}\frac{m_t^{2/3}}{(\log m_t)^{1/3}} < 2^{a_t-1}\frac{(2m_{t+1})^{2/3}}{(\log m_{t+1})^{1/3}} = 2^{a_t - \frac{1}{3}}\frac{m_{t+1}^{2/3}}{(\log m_{t+1})^{1/3}},$$

from which property (iii) follows. Note that by property (iii), at some time s we will reach $a_s \le 3$, and will be done by Corollary 2.7, in the middle of the process at time s. ∎

4. Concluding remarks. In this paper, we proved that $\iota(m) = \Theta((m \log m)^{2/3})$. The upper bound followed by considering the random graph $G_{n,p}$ with $p = \sqrt{\frac{\log n}{n}}$. For this range of p, we have $m = \Theta(n^{3/2}(\log n)^{1/2})$, or equivalently $n = \Theta\big(\frac{m^{2/3}}{(\log m)^{1/3}}\big)$. By carefully studying the proof of Theorem 1.2, one can notice that every graph G with $\iota(G) \le O((m \log m)^{2/3})$ has to be somewhat similar to the above random graph. Indeed, by choosing different parameters in the proof, one can see that for every $\varepsilon > 0$, such graphs G must contain a subgraph on $n' = \Theta\big(\frac{m^{2/3}}{(\log m)^{1/3}}\big)$ vertices with at least $(1-\varepsilon)m$ edges, where the degree of at least $(1-\varepsilon)n'$ vertices is $\Omega(d)$, for d being the average degree of the subgraph (thus $d = \Theta((m \log m)^{1/3})$). Moreover, the edges of this subgraph are well-distributed, in the sense that there does not exist a pair of disjoint vertex subsets X, Y satisfying $e(X,Y) \gg d\sqrt{|X||Y|}$ (since in this case we can directly apply Corollary 2.7).

For a positive integer $s \ge 2$, let $\iota_s(G)$ be the maximum t for which G contains an s-divisible subgraph with t edges, and let $\iota_{r,s}(m)$ be the minimum of $\iota_s(G)$ over all r-uniform hypergraphs with m edges (thus we have $\iota_r(m) = \iota_{r,2}(m)$). By slightly adjusting our proof of the bound $\iota(m) = \Theta((m \log m)^{2/3})$, we can also prove for fixed constant s that $\iota_{2,s}(m) = \Theta(m^{\frac{s}{2s-1}}(\log m)^{\frac{2s-2}{2s-1}})$. The upper bound follows by considering the random graph $G_{n,p}$ with $p = \big(\frac{\log n}{n}\big)^{1/s}$. For the lower bound, if $n \le \frac{m^{s/(2s-1)}}{(\log m)^{1/(2s-1)}}$, then we can use an argument similar to that of Corollary 2.7, and if $n \ge m^{\frac{s}{2s-1}}(\log m)^{\frac{2s-2}{2s-1}}$, then we can use an argument similar to that of Proposition 2.3. In the remaining range of parameters, we can proceed as in section 3. The value $\frac{\alpha \log n}{8 \log(\frac{n \log n}{d^2})}$ in Lemma 3.1 will be replaced by $\Omega\big(\frac{\log n}{\log(\frac{n^{s-1}}{d^s} \log n)}\big)$.

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