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The oriented size Ramsey number of directed paths

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ABSTRACT

An *oriented graph* is a directed graph with no bi-directed edges, i.e. if xy is an edge then yx is not an edge. The *oriented size Ramsey number* of an oriented graph H , denoted by $\vec{r}(H)$, is the minimum m for which there exists an oriented graph G with m edges, such that every 2-colouring of G contains a monochromatic copy of H .

In this paper we prove that the oriented size Ramsey number of the directed paths on n vertices satisfies $\vec{r}(\vec{P}_n) = \Omega(n^2 \log n)$. This improves a lower bound by Ben-Eliezer, Krivelevich and Sudakov. It also matches an upper bound by Bucić and the authors, thus establishing an asymptotically tight bound on $\vec{r}(\vec{P}_n)$.

We also discuss how our methods can be used to improve the best known lower bound of the k -colour version of $\vec{r}(\vec{P}_n)$.

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1. Introduction

Given graphs G and H , we write $G \rightarrow H$ if there is a monochromatic copy of H in every 2-edge-colouring of G . The *size Ramsey number* of a graph H , denoted by $r(H)$, is the minimum number of edges in G over graphs G satisfying $G \rightarrow H$. The concept of size Ramsey numbers was introduced

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by Erdős, Faudree, Rousseau and Schelp [5] in 1972, and has received considerable attention since. A notable example is the size Ramsey number of a path on n vertices, which was shown by Beck [1] to be linear in n , thus disproving a conjecture of Erdős [4].

Here we consider an analogue of size Ramsey numbers for oriented graphs (recall that an oriented graph is a directed graph where at most one of xy and yx is an edge for every two vertices x and y). The *oriented size Ramsey number* of an oriented graph H , denoted by $\vec{r}(H)$, is the minimum number of edges of G over oriented graphs G satisfying $G \rightarrow H$.

In this note, we focus on the oriented size Ramsey number of the directed path on n vertices, denoted by \vec{P}_n . Unlike the undirected case, it turns out that $\vec{r}(\vec{P}_n)$ is not linear in n , as shown by Ben-Eliezer, Krivelevich and Sudakov [2], who established the following bounds (where c_1 and c_2 are positive absolute constants).

$$\frac{c_1 n^2 \log n}{(\log \log n)^3} \leq \vec{r}(\vec{P}_n) \leq c_2 n^2 (\log n)^2.$$

Recently, Bucić and the authors [3] improved the upper bound to $\vec{r}(\vec{P}_n) \leq c_3 n^2 \log n$, by establishing a lower bound on the longest monochromatic path in 2-coloured random tournaments, thereby bringing the upper and lower bounds very close together. The main aim of this note is to obtain a matching lower bound on $\vec{r}(\vec{P}_n)$, thus showing that

$$\vec{r}(\vec{P}_n) = \Theta(n^2 \log n).$$

We achieve our aim in the following theorem.

Theorem 1. *Let G be a directed graph with at most $n^2 \log n$ edges. Then G can be 2-coloured such that all monochromatic directed paths have length at most $169n$.*

We prove [Theorem 1](#) in the next section and conclude the paper in [Section 3](#) with a discussion of a generalisation to more colours. Throughout the paper we omit floor and ceiling signs whenever they are not crucial, and all logarithms are in base 2.

2. The proof

In our proof of [Theorem 1](#), we follow the footsteps of Ben-Eliezer, Krivelevich and Sudakov [2]. The main difference is an improvement on their main tool in their proof of the lower bound, presented in [Lemma 2](#).

Before stating the lemma we make a definition. We call a set U of vertices in a directed graph *k-special* if it is acyclic and its components (in the underlying graph) have order at most k .

In [2], the authors proved that an oriented graph on n vertices with at most εn^2 edges contains an acyclic subset of size at least $\frac{c \log n}{\varepsilon \log(1/\varepsilon)}$. It turns out that the proof of this statement, which is a directed version of a lemma by Erdős and Szemerédi [6], can be adapted to give the following stronger statement which ensures the existence of a large special (and, in particular, acyclic) subset.

Lemma 2. *Let G be an oriented graph with n vertices and at most εn^2 edges, where $\varepsilon > 1/n$. Then there is a $(\log n)$ -special set of size at least $\frac{\log n}{20\varepsilon \log(1/\varepsilon)}$.*

In the proof of [Lemma 2](#) we shall need the following simple lemma. Its proof follows by induction from the fact that every oriented m -vertex graph has a vertex with out-degree at most $m/2$.

Lemma 3. *Every oriented graph on m vertices contains an acyclic subset of size at least $\log m$.*

Proof of [Lemma 2](#). We may assume that $\varepsilon < 1/8$, otherwise the proof follows from [Lemma 3](#). Remove all vertices whose degree (in the underlying graph) is at least $4\varepsilon n$, and denote the resulting graph by G' ; note that G' has at least $n - \frac{2\varepsilon n^2}{4\varepsilon n} = n/2$ vertices.

Let U be a maximum $(\log n)$ -special set in G' (from now on, we call such sets *special*). We may assume that $|U| < \frac{\log n}{20\varepsilon \log(1/\varepsilon)}$, and, since $\varepsilon > 1/n$, also $|U| < n/20$.

The number of edges between U and $V(G') \setminus U$ is at most $4\epsilon n|U|$. Hence, the number of vertices in $V(G') \setminus U$ which have at least $10\epsilon|U|$ neighbours in U , is at most $2n/5$. Therefore, the set W , of vertices outside of U that have fewer than $10\epsilon|U|$ neighbours in U , has size at least $n/2 - |U| - 2n/5 \geq n/20$.

For each vertex $w \in W$, let S_w be a subset of U of size exactly $10\epsilon|U|$ that contains all the neighbours of w in U . Note that the number of possible sets S_w is at most the following (using $\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$).

$$\binom{|U|}{10\epsilon|U|} \leq \left(\frac{e}{10\epsilon}\right)^{10\epsilon|U|} \leq 2^{2\frac{\log n}{\log(1/\epsilon)} \log(e/10\epsilon)} \leq \sqrt{n}.$$

Hence, there is a subset W' of W of size at least $|W|/\sqrt{n} \geq \sqrt{n}/20$, for which S_w is the same for all $w \in W'$. By Lemma 3, there is an acyclic subset W'' of W' whose size is at least $\frac{1}{2} \log n$. Write $U' = (U \setminus S) \cup W''$, where $S = S_w$ for some $w \in W'$. There are no edges between $U' \setminus S$ and W'' , hence, since U and W'' are acyclic, so is U' . Furthermore, the components in U' are contained in either U or in W'' , thus they have order at most $\log n$. It follows that U' is special. Finally, we have the following (note that $\epsilon < 1/2$).

$$|U'| \geq |U| - 10\epsilon|U| + \frac{\log n}{2} \geq |U| - \frac{\log n}{2 \log 1/\epsilon} + \frac{\log n}{2} > |U|.$$

This is a contradiction to the choice of U as a maximum special set. The lemma follows. \square

Before turning to the proof of Theorem 1, we mention the following lemma.

Lemma 4. *Every acyclic graph on m vertices can be 2-edge-coloured such that every monochromatic directed path has length at most \sqrt{m} .*

Proof. Let G be an acyclic graph on m vertices. Let v_1, \dots, v_m be an ordering of the vertices of G such that there are no edges $v_i v_j$ with $i > j$. Define $U_k = \{v_{1+(k-1)\sqrt{m}}, \dots, v_{k\sqrt{m}}\}$ for $k \in [\sqrt{m}]$ (we assume that \sqrt{m} is integer for simplicity). Colour edges inside the U_i 's red and edges between the U_i 's blue. It is easy to see that every monochromatic directed path has length at most \sqrt{m} . \square

The following corollary easily follows: colour each of the components of a k -special set using the colouring described in Lemma 4.

Corollary 5. *The edges of a k -special set can be 2-coloured such that monochromatic paths have length at most \sqrt{k} .*

We are now ready for the proof of Theorem 1.

Proof of Theorem 1. Our aim is to partition G into not too many special sets (i.e. $(\log n)$ -special) and a small remainder. The main tool is the following claim.

Lemma 6. *Let G be an oriented graph with at most $n^2 \log n$ edges. Then the vertices of G can be partitioned into at most $\frac{160n}{\sqrt{\log n}}$ special sets, and a remainder of at most $8n\sqrt{\log n}$ vertices.*

Proof. Our plan is very simple: we remove, one by one, a special set of maximum size, until we remain with at most $8n\sqrt{\log n}$ vertices. In order to show that the number of special sets removed in such a process is not too large, we divide the process into stages.

Let α be such that the number of vertices of G is $\alpha n\sqrt{\log n}$; note that we may assume $\alpha > 8$, as otherwise we are done trivially. Write $\alpha_i = \alpha/2^i$, for $0 \leq i \leq I$, where I is smallest for which $\alpha_i \leq 8$ holds (so $\alpha_i \geq 4$). The first stage is the first part of the process described above, where special sets of maximum size are removed one by one, run until the first time when the number of vertices drops below $\alpha_1 n\sqrt{\log n}$. Similarly, the i th stage consists of the part of the process which starts right after the end of the $(i - 1)$ th stage, and lasts until the number of vertices drops below $\alpha_i n\sqrt{\log n}$.

Write $\varepsilon_i = \frac{n^2 \log n}{(\alpha_i n \sqrt{\log n})^2} = 1/\alpha_i^2$. By Lemma 2, the special sets removed in the i th stage have size at least $\frac{\log n}{20\varepsilon_i \log(1/\varepsilon_i)} = \frac{\alpha_i^2 \log n}{40 \log \alpha_i}$. Since the number of vertices removed is at most $\alpha_{i-1} n \sqrt{\log n}$, the number of special sets removed in the i th stage, where $i \leq I$, is at most the following.

$$\frac{\alpha_{i-1} n \sqrt{\log n}}{\frac{\alpha_i^2 \log n}{40 \log(\alpha_i)}} = \frac{80 \log \alpha_i}{\alpha_i} \cdot \frac{n}{\sqrt{\log n}} \tag{1}$$

Note that $\alpha_i = \alpha_i \cdot 2^{l-i} \geq 4 \cdot 2^{l-i} = 2^{2+l-i}$, as $\alpha_i \geq 4$. Since $\frac{\log x}{x}$ is decreasing for $x \geq e$, we have that $\frac{\log(\alpha_i)}{\alpha_i} \leq \frac{\log(2^{2+l-i})}{2^{2+l-i}} = \frac{2+l-i}{2^{2+l-i}}$ for $0 \leq i \leq I$. Hence,

$$\begin{aligned} \sum_{0 \leq i \leq I} \frac{80 \log \alpha_i}{\alpha_i} &\leq \sum_{2 \leq i \leq I+2} \frac{80i}{2^i} \leq 40 \sum_{i \geq 0} (i+1)2^{-i} = 40 \sum_{i \geq 0} \sum_{j \leq i} 2^{-i} \\ &= 40 \left(\sum_{i \geq 0} 2^{-i} \right)^2 = 160. \end{aligned} \tag{2}$$

It follows from (1) and (2) that the total number of special sets removed in this process is at most $\frac{160n}{\sqrt{\log n}}$. Also, by definition of α_i , the number of vertices remaining in the graph is at most $8n\sqrt{\log n}$, as required. \square

By Lemma 6, there is a partition of the vertices of G into at most $\frac{160n}{\sqrt{\log n}}$ special sets and a remainder W of at most $8n \log n$ vertices. By iterating Lemma 3 we can partition W into at most $\frac{8n}{\sqrt{\log n}}$ acyclic sets of size $\log n$ and a remainder of at most n vertices. As these acyclic sets are special, we thus obtain a partition $\{U_1, \dots, U_l, W'\}$ of the vertices of G where U_i is special for $i \in [l]$, $l \leq \frac{168n}{\sqrt{\log n}}$, and $|W'| \leq n$.

We colour the edges inside the U_i 's with red and blue in such a way that monochromatic paths inside the U_i 's have length at most $\sqrt{\log n}$; this is possible due to Corollary 5. We then colour edges between U_i and U_j red if $i < j$ and blue if $i > j$. Finally, we colour edges into W' red, edges from W' blue, and colour the edges inside W' arbitrarily. Any monochromatic path in this colouring contains at most $\sqrt{\log n}$ vertices from each U_i and at most n vertices from W' , hence it has length at most $169n$, as required. \square

3. Conclusion

We conclude this note with a discussion of a generalisation to a k -coloured setting. We shall consider $\vec{r}(\vec{P}_n, k+1)$, the $(k+1)$ -colour oriented size Ramsey number of \vec{P}_n . The following bounds follow from [2] and [3].

$$\frac{c_1 n^{2k} (\log n)^{1/k}}{(\log \log n)^{(k+2)/k}} \leq \vec{r}(\vec{P}_n, k+1) \leq c_2 n^{2k} \log n.$$

Using our approach, it is possible to improve the lower bound and obtain the following.

$$\vec{r}(\vec{P}_n, k+1) \geq c_1 n^{2k} (\log n)^{2/(k+1)}.$$

We give only a sketch of the proof. As in the proof of Theorem 1, we may partition a graph with at most $n^{2k} (\log n)^{2/(k+1)}$ edges into at most $\frac{cn^k (\log n)^{1/(k+1)}}{\log n}$ special sets and a remainder of at most n vertices. Using an analogue of Lemma 4, we may colour the edges of special sets with $k+1$ colours such that monochromatic paths have length at most $(\log n)^{1/(k+1)}$. In order to colour the edges between special sets, we use the fact (see, e.g. [2]) that a directed graph on m vertices can be $(k+1)$ -coloured (where, if both xy and yx are edges, we may use a separate colour for each of them) such that monochromatic paths have length at most $cm^{1/k}$ and all colour classes are acyclic. We thus obtain a $(k+1)$ -colouring in which monochromatic paths have length at most $c_2 n$.

Finally, we note that the upper bound in [3] was obtained by showing that random tournaments have, with high probability, monochromatic paths of the required length in every 2-colouring of their edges. It seems plausible that a similar statement holds for $(k + 1)$ -colourings of random tournaments, and perhaps a matching upper bound on $\vec{\mathcal{R}}(\vec{P}_n, k + 1)$ can be proved using the methods from [3].

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