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Note

Small subgraphs of random regular graphs

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Abstract

The main aim of this short paper is to answer the following question. Given a fixed graph H, for which values of the degree d does a random d-regular graph on n vertices contain a copy of H with probability close to one?

Keywords: Random regular graphs; Small subgraphs; Thresholds

1. Introduction

Forty years ago Erdős and Rényi wrote a fundamental paper [5] which has become the starting point of the theory of random graphs. One of the main problems studied in this paper was the problem of finding the threshold for the appearance of a small subgraph. Let H be a fixed graph (a triangle or a K_4, say). The problem is to find the smallest value of p such that the random graph $G(n, p)$ almost surely contains H as a subgraph (not necessarily induced). In the literature, this problem is frequently referred to as the *small subgraph* problem.

Let us briefly recall the definition of the Erdős–Rényi random graph $G(n, p)$. A random graph $G(n, p)$ is obtained by connecting every pair $i < j$ of the vertex set $\{1, \dots, n\}$ randomly and independently with probability p. Here and later p can be a function of n, and the asymptotic notation is used under the assumption that $n \to \infty$. For readers who are not familiar with the theory of random graphs, we highly recommend the monographs by Bollobás [4] and by Janson et al. [7], both of which contain an entire section discussing the small subgraph problem.

Let H be a fixed graph with v_H vertices and e_H edges. We call $\rho(H) = e_H/v_H$ *the density* of H. The critical parameter for the small subgraph problem is $m(H)$, the density in the densest subgraph of H, i.e.,

$$m(H) = \max\{\rho(H') | H' \subseteq H, v_{H'} > 0\}.$$

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If for every $H' \subseteq H$ we have $\rho(H') \leqslant \rho(H)$, we call H *balanced*, and we call it *strictly balanced* if every proper subgraph H' has $\rho(H') < \rho(H)$. Throughout the paper the notation $f(n) \gg g(n)$ means that $f/g \to \infty$ together with n. Also for integer-valued random variables $X_1, X_2, \ldots$ and Z we say that X_n converges in distribution to Z if $\mathbb{P}(X_n = k) \to \mathbb{P}(Z = k)$ for every integer k.

In their original paper [5], Erdős and Rényi solved the small subgraph problem for balanced graphs. The general problem was settled by Bollobás [3] in 1981.

Theorem 1.1. *Let H be a fixed graph with $m(H) \geqslant 1$. Then*

$$\lim_{n\to\infty} \Pr(H \subseteq G(n,p)) = \begin{cases} 0 & \text{if } p \ll n^{-1/m(H)}, \\ 1 & \text{if } p \gg n^{-1/m(H)}. \end{cases}$$

The next question is to study the distribution of the number of copies of H in $G(n, p)$ where p is inside the threshold interval, i.e, $p = cn^{-1/m(H)}$ for some positive constant c. In this case, the probability that $G(n, p)$ contains a copy of H is a constant strictly between 0 and 1. Bollobás [3] and Karoński and Ruciński [8], independently showed:

Theorem 1.2. *Let H be a strictly balanced graph with v vertices and $e \geqslant v$ edges. Let* $\mathrm{aut}(H)$ *be the number of automorphisms of H and let X be the number of copies of H in a random graph $G(n, p)$. If $pn^{v/e} \to c$ for some positive constant c, then X converges to* $\mathrm{Po}(\lambda)$, *the Poisson distribution with mean $\lambda = c^e/\mathrm{aut}(H)$.*

Beside the Erdős–Rényi's $G(n, p)$ model, another model of random graphs which also draws lots of attention is the model of random regular graphs. Let $1 \leqslant d \leqslant n-1$ be two positive integers, a random regular graph $G_{n,d}$ is obtained by sampling uniformly at random over the set of all simple d-regular graphs on a fixed set of n vertices. We refer the readers to Wormald's survey [15] for more information (both historical and technical) about this model. The goal of this paper is to establish the analogs of Theorems 1.1 and 1.2 for the random regular model. Here are our main theorems.

Theorem 1.3. *Let H be a fixed graph with $m(H) \geqslant 1$. Then*

$$\lim_{n\to\infty} \Pr(H \subseteq G_{n,d}) = \begin{cases} 0 & \text{if } d \ll n^{1-1/m(H)}, \\ 1 & \text{if } d \gg n^{1-1/m(H)}. \end{cases}$$

Theorem 1.4. *Let H be a strictly balanced graph with v vertices and $e \geqslant v$ edges. Let* $\mathrm{aut}(H)$ *be the number of automorphisms of H and let X be the number of copies of H in a random d-regular graph $G_{n,d}$. If $(d-1)n^{-1+v/e} \to c$ for some positive constant c, then X converges to* $\mathrm{Po}(\lambda)$, *the Poisson distribution with mean $\lambda = c^e/\mathrm{aut}(H)$.*

For a special case when H is a cycle, the statement of Theorem 1.4 was proved earlier by Bollobás [2] and Wormald [14].

One may wonder, given the popularity of Theorems 1.1 and 1.2 and the long history of random regular graphs, why Theorems 1.3 and 1.4 were not proved long time ago. One of the main reasons is, perhaps, that in most situations, arguments used for the Erdős–Rényi model cannot be repeated for the random regular model. For instance, it is very easy to count the expectation of the number of triangles in $G(n, p)$: there are $\binom{n}{3}$ possible triangles and each of them appears with probability p^3, so the expectation is $\binom{n}{3} p^3$. This argument, however, does not make sense in the random regular model, as we do not have independence between the potential edges. Thus, problems concerning random regular graphs are often much more complex than their counterparts concerning the $G(n, p)$ model. Moreover, the notion of thresholds does not really exist for random regular graphs. If for some reason we know that $G_{n,d}$ almost surely has some monotone increasing property $\mathscr{P}$ ($\mathscr{P}$ is monotone increasing if it is preserved under edge addition), it does not follow automatically that $G_{n,d'}$ almost surely has the same property, for $d' > d$.

The rest of the paper is organized as follows. In the next section, we prove a key lemma. This key lemma will enable us to deal with random regular graphs as conveniently as with Erdős–Rényi random graphs. The proofs of Theorems 1.3 and 1.4 are presented in Sections 3 and 4, respectively. The last section, Section 5, contains a few concluding remarks and open questions.

2. The key lemma

In this section we prove our key lemma (Lemma 2.1 below) and its corollary which shows that the expectation of the number of copies of a graph H in $G_{n,d}$ is asymptotically the same as the expectation of the number of copies of H in $G(n, p)$, where $p = d/n$. In fact, we are going to present this lemma in a slightly more general form. A *degree sequence* of a graph is the non-decreasing sequence of degrees of its vertices. Given a sequence of positive integers (d_i), $1 \leqslant i \leqslant n$, we call a graph with degree sequence (d_i) a *(d_i)-graph*. The *random (d_i)-graph*, which we denote by $G_{n,(d_i)}$, is obtained by sampling uniformly at random from the set of all possible simple (d_i)-graphs on n vertices.

Lemma 2.1. *Let (d_i), $1 \leqslant i \leqslant n$ be a degree sequence such that every $d_i = (1 + o(1))d$, d tends to infinity with n and $d = o(n)$. Let $\mathscr{E}$ be a fixed collection of edges on the vertex set $[n]$ of constant size t. Then*

$$\Pr[\mathscr{E} \subseteq G_{n,(d_i)}] = (1 + o(1))(d/n)^t.$$

Corollary 2.2. *Let F be a fixed graph with v vertices and e edges. Let $\mathrm{aut}(F)$ be the number of automorphisms of F and let X be the number of copies of F in a random d-regular graph $G_{n,d}$. If $d \to \infty$ and $d = o(n)$ then*

$$\mathbb{E}[X] = (1 + o(1))\frac{\binom{n}{v}\, v!}{\mathrm{aut}(F)}(d/n)^e = \Theta(n^{v-e}d^e).$$

Proof of Corollary 2.2 via Lemma 2.1. There are exactly $\binom{n}{v}\, v!/\mathrm{aut}(F)$ copies of F in the complete graph K_n. For each copy F' of F in K_n define the indicator random variable $X_{F'} = 1$ if and only if $F' \subset G_{n,d}$. Note that by Lemma 2.1, $\Pr[F' \subset G_{n,d}] = (1 + o(1))(d/n)^e$. Therefore

$$\mathbb{E}[X] = \sum_{F'} \mathbb{E}[X_{F'}] = (1 + o(1))\frac{\binom{n}{v}\, v!}{\mathrm{aut}(F)}(d/n)^e = \Theta(n^{v-e}d^e). \qquad \square$$

The original proofs of Theorems 1.1 and 1.2 used high moment arguments, which, in turn, rely on expectation estimates. Lemma 2.1 and Corollary 2.2 enable us to repeat these moment arguments for the random regular model.

Now we are going to prove Lemma 2.1. The most important step is the following technical lemma.

Lemma 2.3. *Let (d_i), $1 \leqslant i \leqslant n$, be a degree sequence such that every $d_i = (1 + o(1))d$, d tends to infinity with n and $d = o(n)$. Let $\mathscr{E}$ be a fixed collection of edges on the vertex set $[n]$ of constant size and let uw be an edge in $\mathscr{E}$. Then*

$$\Pr[\mathscr{E} \subseteq G_{n,(d_i)}] = (1 + o(1))(d/n)\ \Pr[\mathscr{E}\backslash\{uw\} \subseteq G_{n,(d_i)}].$$

Proof of Lemma 2.3. We use the so-called *edge switching* technique introduced by McKay and Wormald in [13]. This technique is very useful and plays an important role in the proofs of several conjectures concerning random regular graphs (see, for instance [9,12]). We would also like to call the reader's attention to the fact that this technique is entirely combinatorial and is very different (in nature) from the probabilistic techniques used to study the Erdős–Rényi model $G(n, p)$.

Let $\mathscr{C}_1$ be the set of all (d_i)-graphs on n vertices containing $\mathscr{E}$ and let $\mathscr{C}_0$ be the set of all (d_i)-graphs containing $\mathscr{E}\backslash\{uw\}$ but not containing edge uw. By definition, we have that

$$\frac{\Pr[\mathscr{E} \subseteq G_{n,(d_i)}]}{\Pr[\mathscr{E}\backslash\{uw\} \subseteq G_{n,(d_i)}]} = \frac{|\mathscr{C}_1|}{|\mathscr{C}_0| + |\mathscr{C}_1|}.$$

Since $d = o(n)$, to prove the lemma it is enough to show that $|\mathscr{C}_1|/|\mathscr{C}_0| = (1 + o(1))d/n$. Then $|\mathscr{C}_1|/|\mathscr{C}_0| = o(1)$ and therefore

$$\frac{|\mathscr{C}_1|}{|\mathscr{C}_0| + |\mathscr{C}_1|} = \frac{|\mathscr{C}_1|/|\mathscr{C}_0|}{1 + |\mathscr{C}_1|/|\mathscr{C}_0|} = (1 + o(1))\frac{|\mathscr{C}_1|}{|\mathscr{C}_0|} = (1 + o(1))d/n.$$

Given a graph $G \in \mathscr{C}_1$, we define an operation called *forward switching* as follows. Choose two edges u_1w_1 and u_2w_2 of $G\backslash\mathscr{E}$, deleting these edges together with the edge uw and inserting new edges wu_1, w_1u_2, w_2u. We will allow only choices of u_1w_1 and u_2w_2 such that all the six endpoints of the edges are distinct and in addition wu_1, w_1u_2, w_2u are not edges of G. Then it is easy to see that the graph obtained from G by a forward switching belongs to $\mathscr{C}_0$. For counting purposes, we think about the edges as being oriented, i.e., edges u_1w_1 and w_1u_1 are different. It is simple to check that the number of choices for u_1w_1 and u_2w_2 (after u_1w_1 is chosen) are $(1+o(1))\,dn - O(d^2)$ (keep in mind that $\mathscr{E}$ has constant size). Since $d = o(n)$ we obtain that the number of possible forward switchings is

$$((1+o(1))\,dn - O(d^2))((1+o(1))\,dn - O(d^2)) = (1+o(1))\,d^2n^2.$$

A *reverse switching* is applied to $G' \in \mathscr{C}_0$ by deleting edges wu_1, uw_2, u_2w_1 of $G'\backslash\mathscr{E}$ and inserting edges uw, u_1w_1, u_2w_2. We again allow only switchings for which all six vertices are distinct and u_1w_1, u_2w_2 are not edges of G'. Note that this procedure produces a graph which belongs to $\mathscr{C}_1$. Since $d \to \infty$, the number of choices for wu_1 and uw_2 are $(1+o(1))d - O(1)$. Moreover, the number of choices for u_2w_1 is $(1+o(1))\,dn - O(d^2)$. It follows that the number of reverse switchings is

$$((1+o(1))d - O(1))((1+o(1))d - O(1))((1+o(1))\,dn - O(d^2)) = (1+o(1))\,d^3n.$$

An easy double counting argument shows that the ratio between the number of reverse switchings and the number of forward switchings (asymptotically) equals the ratio between $|\mathscr{C}_1|$ and $|\mathscr{C}_0|$. Thus, we have

$$\frac{|\mathscr{C}_1|}{|\mathscr{C}_0|} = \frac{(1+o(1))\,d^3n}{(1+o(1))\,d^2n^2} = (1+o(1))\frac{d}{n},$$

completing the proof of the lemma. □

Lemma 2.1 follows by applying Lemma 2.3 recursively.

3. Proof of Theorem 1.3

First assume that $d \ll n^{1-1/m(H)}$. If $m(H) = 1$ then there is nothing to prove. Let H_0 be a subgraph of H for which $e_{H_0}/v_{H_0} = m(H) > 1$. If d is constant, then it is well known (see, e.g. [15]) that almost surely $G_{n,d}$ contains no copy of H_0, and thus, no copy of H. Therefore, we can assume that $d \to \infty$. Then by Corollary 2.2 we get that the expected number of copies of H_0 in $G_{n,d}$ is o(1) and by Markov's inequality

$$\Pr[H \subset G_{n,d}] \leqslant \Pr[H_0 \subset G_{n,d}] = o(1).$$

Next we consider the case that $d \gg n^{1-1/m(H)}$. If $d = \Theta(n)$, then it was proved by Krivelevich et al. [12] that almost surely the second largest (in the absolute value) eigenvalue of $G_{n,d}$ is $O(d^{3/4})$. This implies that almost surely $G_{n,d}$ has very strong pseudo-random properties. For such a d-regular graph, the number of copies of any fixed graph H in it equals (see, e.g., [11, Theorem 4.10])

$$(1+o(1))\frac{\binom{n}{v_H} v_H!}{\operatorname{aut}(H)}(d/n)^{e_H} \to \infty.$$

It remains to deal with the case when $d = o(n)$. Let X be the number of copies of H in $G_{n,d}$ and let H has v vertices and e edges. By Corollary 2.2 we know that $\mathbb{E}[X] = \Theta(n^{v-e}d^e) \to \infty$. We next estimate the variance of X. For each copy H' of H in K_n define the indicator random variable $X_{H'} = 1$ iff $H' \subset G_{n,d}$. Then $X = \sum_{H'} X_{H'}$ and

$$\begin{aligned} \mathrm{VAR}[X] &= \sum_{H',H''} \mathrm{COV}(X_{H'}, X_{H''}) = \sum_{H',H''} (\mathbb{E}[X_{H'}X_{H''}] - \mathbb{E}[X_{H'}]\mathbb{E}[X_{H''}]) \\ &\leqslant \sum_{E(H')\cap E(H'')=\emptyset} (\mathbb{E}[X_{H'}X_{H''}] - \mathbb{E}[X_{H'}]E[X_{H''}]) + \sum_{E(H')\cap E(H'')\neq\emptyset} \mathbb{E}[X_{H'}X_{H''}]. \end{aligned} \tag{1}$$

Note that even when the sets of edges of two copies H' and H'' are disjoint, we still cannot claim that the corresponding random variables $X_{H'}$, $X_{H''}$ are independent. (This is a crucial difference from the Erdős–Rényi model.) On the other hand, using Lemma 2.1 the computation for the variance can still be done roughly in the same way as for $G(n, p)$.

Indeed, if H' and H'' are disjoint then the edge set of the union $H' \cup H''$ has exactly $2e$ edges and by Lemma 2.1 we get that

$$\mathbb{E}[X_{H'}X_{H''}] = \Pr[H' \cup H'' \subset G_{n,d}] = (1+o(1))(d/n)^{2e} = (1+o(1))\mathbb{E}[X_{H'}]\mathbb{E}[X_{H''}].$$

Therefore,

$$\sum_{E(H')\cap E(H'')=\emptyset} (\mathbb{E}[X_{H'}X_{H''}] - \mathbb{E}[X_{H'}]\mathbb{E}[X_{H''}]) \leqslant \sum_{H',H''} o((d/n)^{2e}) = o(n^{2v}(d/n)^{2e}) = o(\mathbb{E}[X]^2).$$

To estimate the second term in the left hand side of (1), observe that for any subgraph $F \subseteq H$ there are $\Theta(n^{v_F} n^{2(v-v_F)}) = \Theta(n^{2v-v_F})$ pairs H', H'' of copies of H in K_n such that $H' \cap H''$ is isomorphic to F. Moreover, if $H' \cap H''$ is isomorphic to F then $H' \cup H''$ has $2e - e_F$ edges and by Lemma 2.1 we get

$$\mathbb{E}[X_{H'}X_{H''}] = \Pr[H' \cup H'' \subset G_{n,d}] = (1+o(1))(d/n)^{2e-e_F}.$$

By the definition of $m(H)$, $d \gg n^{1-1/m(H)} \geqslant n^{1-v_F/e_F}$ for any subgraph $F \subseteq H$ and hence $n^{e_F-v_F}/d^{e_F} = o(1)$. Using all the above and the fact that the number of subgraphs of H is constant we conclude that

$$\sum_{E(H')\cap E(H'')\neq\emptyset} \mathbb{E}[X_{H'}X_{H''}] \leqslant \Theta\left(\sum_{F\subseteq H, e_F>0} n^{2v-v_F}(d/n)^{2e-e_F}\right) = \Theta(n^{2(v-e)}d^{2e}) \sum_{F\subseteq H, e_F>0} \frac{n^{e_F-v_F}}{d^{e_F}}$$
$$\leqslant \Theta(\mathbb{E}[X]^2) \sum_{F\subseteq H, e_F>0} \frac{n^{e_F-v_F}}{d^{e_F}} = o(\mathbb{E}[X]^2).$$

Therefore $\mathrm{VAR}(X) = o(\mathbb{E}[X]^2)$. Now by Chebyshev's inequality

$$\Pr[H \not\subset G_{n,d}] = \Pr[X=0] \leqslant \frac{\mathrm{VAR}(X)}{\mathbb{E}[X]^2} = o(1),$$

completing the proof.

Note that this proof also gives the following stronger result, which asserts that above the threshold, the number of copies of H is almost always asymptotically its expectation.

Corollary 3.1. *Let H be a fixed graph with v vertices, e edges and $m(H) \geqslant 1$. Let X be the number of copies of H in the random graph $G_{n,d}$. If $d \gg n^{1-1/m(H)}$, then almost surely*

$$X = (1+o(1))\mathbb{E}[X] = (1+o(1))\frac{\binom{n}{v}\, v!}{\mathrm{aut}(H)}(d/n)^e.$$

4. Proof of Theorem 1.4

This proof is a bit sketchy. The leading idea is to combine the arguments from the proof of Theorem 1.2 (see [7]) with Lemma 2.1 and Corollary 2.2.

The only strictly balanced graphs with $e = v$ are cycles and this case has already been proved by Bollobás [2] and Wormald [14]. They showed that for a constant d, the number of cycles of fixed length i in $G_{n,d}$ is asymptotically Poisson random variable with mean $(d-1)^i/2i$.

From now on we assume that $e > v$. Then by the condition $(d-1)n^{-1+v/e} \to c$ we have that $d \to \infty$ and $d = o(n)$. This guarantees the condition of Lemma 2.1. As usual, let X be the number of copies of H in $G_{n,d}$. Consider the kth factorial moment of X, defined as $\mathbb{E}(X)_k = \mathbb{E}[X(X-1)\cdots(X-k+1)]$. We have that for every constant $k \geqslant 1$,

$$\mathbb{E}(X)_k = \sum_{H_1,\ldots,H_k} \Pr[H_1, \ldots, H_k \subset G_{n,d}],$$

where the summation extends over all ordered k-tuples of distinct copies of H in K_n.

Next we decompose $\sum_{H_1,\ldots,H_k} \Pr[H_1, \ldots, H_k \subset G_{n,d}]$ into two parts

$$\sum_{H_1,\ldots,H_k} \Pr[H_1, \ldots, H_k \subset G_{n,d}] = S'_k + S''_k,$$

where S'_k is the partial sum where the copies in the same k-tuple are mutually vertex disjoint. By the properties of the Poisson's distribution (see [7, Corollary 6.8]), it suffices to show that $\mathbb{E}(X)_k \to (c^e/\mathrm{aut}(H))^k$. It is easy to see that there are $(1 + o(1))(n^v/\mathrm{aut}(H))^k$ ordered mutually vertex disjoint k-tuples of distinct copies of H in K_n. In such a k-tuple, the copies together have ke edges and by Lemma 2.1 the probability that $G_{n,d}$ contains all of them is $(1+o(1))(d/n)^{ke}$. Therefore,

$$S'_k = (1 + o(1))(n^v(d/n)^e/\mathrm{aut}(H))^k = (c^e/\mathrm{aut}(H))^k.$$

To finish the proof it suffices to show that $S''_k = o(1)$. Let e_t be the minimum number of edges in the union of k not mutually disjoint copies of H, provided that these copies together have exactly t vertices. It is known (see [7, p. 66]) that if H is strictly balanced, $k \geqslant 2$ and $v \leqslant t < kv$ then $e_t > te/v$. Thus by Lemma 2.1 the probability that $G_{n,d}$ contains such a k-tuple is at most $(1 + o(1))(d/n)^{e_t} = o((d/n)^{te/v}) = o(n^{-t})$. On the other hand, the number of k-tuples of copies of H which together have exactly t vertices is $O\binom{n}{t} = O(n^t)$. Therefore $S''_k = o(1)$, completing the proof. □

5. Concluding remarks and open questions

- Our proof of Theorem 1.3 showed that the variance of X is $o(\mathbb{E}[X]^2)$. It would be of interest to compute the right order of magnitude of this variance and other fixed moments of $(X - \mathbb{E}[X])$.
- Our results and proofs (especially the main lemma) show that with respect to the small subgraph problem, $G_{n,d}$ behaves (asymptotically) the same way as $G(n, p)$ where $p = d/n$. This similarity between the two models has been discovered in many other problems such as coloring, independent sets, connectivity, etc. (see [12]). Recently, Kim and Vu [10] proved a general theorem which states that in a certain range of d, the two models $G_{n,d}$ and $G(n, d/n)$ are asymptotically the same with respect to most natural problems. In particular, their theorem implies that for $d \leqslant n^{1/3-o(1)}$, a random regular graph $G_{n,d}$ contains a random graph $G(n, p)$ inside it, for some $p \approx d/n$. Via this result one can obtain another proof of Theorem 1.3 in the case where H is fairly sparse, $m(H) < \frac{3}{2}$.
- Another well-known problem concerning small graphs is the problem of estimating the probability that a random graph (with density well above the threshold) does not contain a copy of a fixed graph H. Answering a question of Erdős and Rényi, Janson et al. [6] showed that this probability is at most $e^{-\Theta(\Phi_H)}$ where

  $$\Phi_H = \min\{\mathbb{E}[X(F)] | F \subseteq H, e_F > 0\}$$

  and $X(F)$ is the number of copies of F in the random graph $G(n, p)$. The above mentioned result of Kim and Vu implies that this bound holds also for random d-regular graphs but only in a certain range of d and provided that H is sufficiently sparse. Therefore, it would be interesting to obtain such a bound in the general case.
- Although we cannot get the strong bound shown in the previous paragraph, using Lemma 2.1 we can get an exponentially small estimate on the probability that $G_{n,d}$ with $d \geqslant n^{1-1/m(H)+\varepsilon}$ does not contain a copy of a fixed graph H. This proof is little bit sketchy and the reader is invited to fill in the details. Let $\omega(n)$ be a function which tends to infinity arbitrarily slowly with n and let $k = d/(\omega(n)n^{1-1/m(H)})$. Consider a random partition of the vertex set of $G_{n,d}$ into k almost equal parts $V_1, \ldots, V_k$. For the sake of convenience, we assume that k and $d' = d/k$ are integers. For a vertex $v \in G = G_{n,d}$, let $d_i(v)$ denote the number of neighbors of v in the part V_i. We will condition on the particular sequence $\{d_i(v)\}$. It is easy to see that the probability $d_i(v) \neq (1 + o(1))d'$ for some i and v is exponentially small. Therefore we can consider only the sequences in which $d_i(v) = (1+o(1))d'$ for all pairs i and v. Then every induced subgraph $G_i = G[V_i]$ is a random graph with a fixed degree sequence where every degree in the sequence is $(1 + o(1))d' \geqslant n^{1-1/m(H)}$. Moreover, these random graphs are conditionally independent. Now using Lemma 2.1 as in proof of Theorem 1.3 we can show that probability that each G_i contains no copy of H is $o(1) \leqslant e^{-\omega_1(n)}$ for some function $\omega_1(n)$ going to infinity with n. Therefore probability that $G = G_{n,d}$ contains no H is at most $e^{-k\omega_1(n)}$ which is exponentially small. Following these arguments, it is easy to figure out the optimal value of k.

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