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The Largest Eigenvalue of Sparse Random Graphs

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We prove that, for all values of the edge probability $p(n)$, the largest eigenvalue of the random graph $G(n,p)$ satisfies almost surely $\lambda_1(G) = (1+o(1))\max\{\sqrt{\Delta}, np\}$, where Δ is the maximum degree of G, and the $o(1)$ term tends to zero as $\max\{\sqrt{\Delta}, np\}$ tends to infinity.

1. Introduction

Let $G = (V,E)$ be a graph with vertex set $V(G) = \{1,\dots,n\}$. The *adjacency matrix* of G, denoted by $A = A(G)$, is the n-by-n 0,1-matrix whose entry A_{ij} is one if $(i,j) \in E(G)$, and is zero otherwise. It is immediate that $A(G)$ is a real symmetric matrix. We thus let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ denote the eigenvalues of A, which are usually also called the eigenvalues of the graph G itself. The family $\{\lambda_1,\dots,\lambda_n\}$ is called the *spectrum* of G.

Spectral techniques play an increasingly important role in modern graph theory. A serious effort has been invested in establishing connections between a graph's spectral characteristics and its other parameters. The interested reader may consult the monographs [6] and [5] for a detailed account of known results. The ability to compute graph eigenvalues efficiently (both from theoretical and practical points of view), combined with results from spectral graph theory, has provided a basis for quite a few graph algorithms. A survey of applications of spectral techniques in algorithmic graph theory by Alon can be found in [1].

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In this paper we study eigenvalues of random graphs. The *random graph* $G(n,p)$ is the discrete probability space composed of all labelled graphs on the vertices $\{1,\ldots,n\}$, where each edge (i,j), $1 \leqslant i < j \leqslant n$, appears randomly and independently with probability $p = p(n)$. Sometimes, with some abuse of notation, we will refer to the random graph $G(n,p)$ as a graph on n vertices generated according to the distribution $G(n,p)$ described above. Usually, asymptotic properties of random graphs are of interest. We say that a graph property $\mathscr{A}$ holds *almost surely*, or a.s. for brevity, in $G(n,p)$ if the probability that $G(n,p)$ has $\mathscr{A}$ tends to one as the number of vertices n tends to infinity. Necessary background information on random graphs may be found in [4] and [8]. It is important to observe that the adjacency matrix of the random graph $G(n,p)$ can be viewed as a random symmetric matrix whose diagonal entries are zeroes and whose entries above the diagonal are i.i.d. random variables, each taking value 1 with probability p and value 0 with probability $1-p$. This allows us to bridge between random graphs and the extensively developed theory of random real symmetric matrices and their spectra (see, *e.g.*, [13]).

The subject of this paper is the asymptotic behaviour of the largest eigenvalue $\lambda_1(G(n,p))$ of random graphs. Notice that, owing to the Perron–Frobenius theorem, for every graph G on n vertices, $\lambda_1(G) \geqslant |\lambda_i(G)|$ for all $i = 2,\ldots,n$. Thus $\lambda_1(G)$ is equal to the spectral norm or the spectral radius of $A(G)$. It is easy to observe that, for every graph $G = (V,E)$, its largest eigenvalue $\lambda_1(G)$ is always squeezed between the average degree of G, $\bar{d} = \sum_{v\in V} d_G(v)/|V|$ and its maximum degree $\Delta(G) = \max_{v\in V} d_G(v)$. Since, for all $p(n) \gg \log n$, the last two quantities are both asymptotically equal to np, it follows that in this range of edge probabilities a.s. $\lambda_1(G(n,p)) = (1+o(1))np$. In fact, much more is known for sufficiently large values of $p(n)$: Füredi and Komlós proved in [7] that, for a constant p, $\lambda_1(G(n,p))$ has a normal distribution asymptotically, with expectation $(n-1)p+(1-p)$ and variance $2p(1-p)$.

In contrast, not much appears to be known for the case of sparse random graphs, that is, when $p(n) = O(\log n)$. Khorunzhy and Vengerovsky [11] and Khorunzhy [10] mainly consider the case $p(n) = 1/n$, and show that in this case the spectral norm of $A(G(n,p))$ a.s. tends to infinity with n. Moreover, it is stated in [11] that the mathematical expectation of the number of eigenvalues that tend to infinity is of order $\Theta(n)$.

Here we determine the asymptotic value of the largest eigenvalue of sparse random graphs. To better grasp the result, observe that, if Δ denotes the maximum degree of a graph G, then G contains a star S_Δ and therefore $\lambda_1(G) \geqslant \lambda_1(S_\Delta) = \sqrt{\Delta}$. Also, as mentioned above, $\lambda_1(G)$ is at least as large as the average degree of G. Since, for all values of $p(n) \gg 1/n^2$, a.s. $|E(G(n,p))| = (1+o(1))(n^2p/2)$, we get that a.s. $\lambda_1(G(n,p)) \geqslant (1+o(1))np$. Combining the above lower bounds, we get that a.s. $\lambda_1(G(n,p)) \geqslant (1+o(1))\max\{\sqrt{\Delta}, np\}$. As it turns out, this lower bound can be matched by an upper bound of the same asymptotic value, as stated by the following theorem.

Theorem 1.1. *Let $G = G(n,p)$ be a random graph and let Δ be the maximum degree of G. Then almost surely the largest eigenvalue of the adjacency matrix of G satisfies*

$$\lambda_1(G) = (1+o(1))\max\{\sqrt{\Delta}, np\},$$

where the $o(1)$ term tends to zero as $\max\{\sqrt{\Delta}, np\}$ tends to infinity.

As the asymptotic value of the maximum degree of $G(n,p)$ is known for all values of $p(n)$ (see Lemma 2.2 below), the above theorem enables us to estimate the asymptotic value of $\lambda_1(G(n,p))$ for all relevant values of p. In particular, for the case $p = c/n$ we get the following corollary.

Corollary 1.2. *For any constant $c > 0$, a.s. $\lambda_1(G(n,c/n)) = (1+o(1))\sqrt{\frac{\log n}{\log\log n}}$.* □

The rest of the paper is organized as follows. In the next section we gather necessary technical information about random graphs, used later in the proof of the main result. The main theorem, Theorem 1.1, is proved in Section 3. Section 4, the last section of the paper, is devoted to concluding remarks and discussion of related open problems.

Throughout the paper we systematically omit floor and ceiling signs, for the sake of clarity of presentation. All logarithms are natural. We will frequently use the inequality $\binom{a}{b} \leqslant \left(\frac{ea}{b}\right)^b$.

2. Some properties of sparse random graphs

In this section we show some properties of sparse random graphs which we will use later to prove Theorem 1.1. First we need the following definition. Let $G(n,p)$ be a random graph and let

$$\Delta_p = \max\left\{k : n\binom{n-1}{k}p^k(1-p)^{n-k} \geqslant 1\right\}.$$

In words, Δ_p is the maximal k for which the expectation of the number of vertices of degree k in $G(n,p)$ is still at least one. The following lemma summarizes properties of Δ_p that we will need later.

Lemma 2.1.

(i) *If $p \leqslant e^{-(\log\log n)^2}/n$, then $\Delta_p = o(\log n)$.*
(ii) *If $\Delta_p \to \infty$ and $p \leqslant e^{-(\log\log n)^2}/n$, then $n(enp)^{\Delta_p+1} \leqslant O\big((\Delta_p+1)^{\Delta_p+2}\big)$.*
(iii) *If $p \geqslant \log^{1/2} n/n$, then $\Delta_p = o\big((np)^2\big)$.*
(iv) *If $p \geqslant e^{-(\log\log n)^2}/n$, then $\Delta_p \geqslant \Omega\big(\log n/(\log\log n)^2\big)$.*

Proof. (i) Let $p \leqslant e^{-(\log\log n)^2}/n$ and let $k \geqslant \log n/(\log\log n)$; then

$$n\binom{n-1}{k}p^k(1-p)^{n-k} \leqslant nn^kp^k = n(np)^k \leqslant ne^{-\log n(\log\log n)} = nn^{-\log\log n} \ll 1.$$

Therefore, by definition, $\Delta_p \leqslant \log n/(\log\log n) = o(\log n)$.

(ii) Since $\Delta_p \to \infty$, then, by Stirling's formula,

$$(\Delta_p+1)! = (1+o(1))\sqrt{2\pi(\Delta_p+1)}\left(\frac{\Delta_p+1}{e}\right)^{\Delta_p+1}.$$

By definition of Δ_p, we have that

$$1 \geqslant n\binom{n-1}{\Delta_p+1}p^{\Delta_p+1}(1-p)^{n-\Delta_p-1} = (1+o(1))n\frac{(np)^{\Delta_p+1}}{(\Delta_p+1)!}$$
$$= (1+o(1))\frac{n(enp)^{\Delta_p+1}}{\sqrt{2\pi(\Delta_p+1)}(\Delta_p+1)^{\Delta_p+1}}.$$

Therefore $n(enp)^{\Delta_p+1} \leqslant O\big(\sqrt{\Delta_p+1}\,(\Delta_p+1)^{\Delta_p+1}\big) \leqslant O\big((\Delta_p+1)^{\Delta_p+2}\big)$.

(iii) Let $p \geqslant \log^{1/2} n/n$ and let $k \geqslant (np)^2/\sqrt{\log\log n}$; then

$$n\binom{n-1}{k}p^k(1-p)^{n-k} \leqslant n\left(\frac{en}{k}\right)^k p^k = n\left(\frac{enp}{k}\right)^k \leqslant n\left(\frac{e\sqrt{\log\log n}}{\log^{1/2} n}\right)^{\log n/\sqrt{\log\log n}}$$
$$= ne^{-(1/2+o(1))\log n\sqrt{\log\log n}} = nn^{-(1/2+o(1))\sqrt{\log\log n}} \ll 1.$$

Therefore, by definition, $\Delta_p \leqslant (np)^2/\sqrt{\log\log n} = o\big((np)^2\big)$.

(iv) Let $p = e^{-(\log\log n)^2}/n$; then it is easy to check that, for $k = \log n/(4(\log\log n)^2)$, we have that $n\binom{n-1}{k}p^k(1-p)^{n-k} > 1$. Therefore $\Delta_p \geqslant \Omega\big(\log n/(\log\log n)^2\big)$. Since Δ_p is easily seen to be a non-decreasing function of p, we get the required estimate. □

Lemma 2.2. *Let $G = G(n,p)$ be a random graph. Then we have the following conditions.*

(i) *The maximum degree of G almost surely satisfies $\Delta(G) = (1+o(1))\Delta_p$, where the $o(1)$ term tends to zero when Δ_p tends to infinity.*
(ii) *If $np \to 0$ then almost surely G is a forest.*
(iii) *If $p \leqslant e^{-(\log\log n)^2}/n$, then almost surely all connected components of G are of size at most $(1+o(1))\Delta_p$, where $o(1) \to 0$ when $\Delta_p \to \infty$.*
(iv) *If $p \leqslant \log^{1/2} n/n$, then almost surely every vertex of G is contained in at most one cycle of length $\leqslant 4$.*

Proof. Parts (i) and (ii) are well known and can be found, *e.g.*, in the monograph by Bollobás [4]. To prove (iii) it is sufficient to bound from above the expectation of the number Y of labelled trees on $t = (1+1/\log\log n)\Delta_p + 2$ vertices, contained in $G(n,p)$ as subgraphs. Obviously this expectation is equal to

$$EY = \binom{n}{t}t^{t-2}p^{t-1} \leqslant \frac{n^t}{t!}t^{t-2}p^{t-1} \leqslant \frac{n^t}{(t/e)^t}t^{t-2}p^{t-1}$$
$$= \frac{en}{t^2}\big(enp\big)^{t-1} = \frac{e}{t^2}\Big(n\big(enp\big)^{\Delta_p+1}\Big)\big(enp\big)^{t-2-\Delta_p}.$$

From Lemma 2.1 we have that $n\big(enp\big)^{\Delta_p+1} \leqslant O((\Delta_p+1)^{\Delta_p+2})$ and $\Delta_p = o(\log n)$. Therefore, using that $p \leqslant e^{-(\log\log n)^2}/n$ and $t > \Delta_p$, we conclude

$$EY \leqslant O\left(\frac{e}{t^2}(\Delta_p+1)^{\Delta_p+2}\big(enp\big)^{\Delta_p/\log\log n}\right) \leqslant O\left(\left(\frac{e(\Delta_p+1)}{\log n}\right)^{\Delta_p}\right) = o(1).$$

Now (iii) follows from Markov's inequality. Finally, the expected number of pairs of intersecting cycles of length $s, t \leqslant 4$ in the graph G is at most $O(n^s n^{t-1} p^{s+t}) \leqslant O(\log^4 n/n) = o(1)$. This, by Markov's inequality, implies (iv). $\square$

Next we show that the set of vertices of relatively high degree in $G(n,p)$ spans a graph with small maximum degree and with no cycles. More precisely, the following stronger statement is true.

Lemma 2.3. *Let $p \geqslant e^{-(\log\log n)^2}/n$ and let X be the set of vertices of the random graph $G = G(n,p)$ with degree larger than $np(1 + 1/\log\log n) + \Delta_p^{1/3}$. Then we have the following conditions.*

(i) *Almost surely every cycle of G of length k intersects X in fewer than $k/2$ vertices.*
(ii) *Almost surely every vertex in G has fewer than $\Delta_p^{7/8}$ neighbours in X.*

Proof. First we consider the case when $e^{-(\log\log n)^2}/n \leqslant p \leqslant \log^{1/4} n/n$. In this case, from Lemma 2.1, $\Delta_p \geqslant \Omega(\log n/(\log\log n)^2)$ and $np \leqslant \log^{1/4} n$. To prove the lemma we first estimate the probability that all the vertices of a fixed set T of size $|T| = t$ have degrees at least $\log^{1/3} n/\log\log n < \Delta_p^{1/3}$. It is easy to see that, for such a set T, either there are at least $(\log^{1/3} n/\log\log n)t/3$ edges in the cut $(T, V(G) - T)$, or the set T spans at least $(\log^{1/3} n/\log\log n)t/3$ edges of G. Since the number of edges in the cut $(T, V(G) - T)$ is a binomially distributed random variable with parameters $t(n-t)$ and p, we can bound the probability of the first event by

$$\binom{t(n-t)}{\frac{\log^{1/3} n}{3\log\log n}t} p^{\frac{\log^{1/3} n}{3\log\log n}t} \leqslant \left(\frac{3e(n-t)p\log\log n}{\log^{1/3} n}\right)^{\frac{\log^{1/3} n}{3\log\log n}t}$$
$$\leqslant \left(\frac{3e\log^{1/4} n\log\log n}{\log^{1/3} n}\right)^{\frac{\log^{1/3} n}{3\log\log n}t}$$
$$\leqslant e^{-\Omega(t\log^{1/3} n)}.$$

Also, the number of edges spanned by T is a binomially distributed random variable with parameters $t(t-1)/2$ and p. We can thus bound the probability of the second event similarly by

$$\binom{\frac{t(t-1)}{2}}{\frac{\log^{1/3} n}{3\log\log n}t} p^{\frac{\log^{1/3} n}{3\log\log n}t} \leqslant \left(\frac{3e(t-1)p\log\log n}{2\log^{1/3} n}\right)^{\frac{\log^{1/3} n}{3\log\log n}t}$$
$$\leqslant \left(\frac{3e\log^{1/4} n\log\log n}{2\log^{1/3} n}\right)^{\frac{\log^{1/3} n}{3\log\log n}t}$$
$$\leqslant e^{-\Omega(t\log^{1/3} n)}.$$

Therefore, the probability that all the vertices in the given set of size t have degree at least $\Delta_p^{1/3}$ is at most $e^{-\Omega(t\log^{1/3} n)}$. Essentially repeating the above argument shows that conditioning on the presence of any specific set of at most $2t$ edges in G leaves the latter probability still at most $e^{-\Omega(t\log^{1/3} n)}$.

Using this bound we can easily estimate the probability that there exists a cycle of length k with at least $k/2$ vertices inside the set X. Clearly this probability is at most

$$\sum_{k\geqslant 3} n^k p^k \binom{k}{\lceil k/2\rceil} e^{-\Omega((k/2)\log^{1/3} n)} \leqslant \sum_{k\geqslant 3} \left(2npe^{-\Omega(\log^{1/3} n)}\right)^k \leqslant \sum_{k\geqslant 3} \left(2(\log^{1/4} n)e^{-\Omega(\log^{1/3} n)}\right)^k = o(1).$$

(First choose k vertices of a cycle and fix their order, then require that the k edges of the cycle are present in $G(n,p)$, then choose a set T of the cycle vertices of cardinality $|T| = t = \lceil k/2\rceil$, and then require all vertices of T to belong to X, conditioning on the presence of the cycle edges in $G(n,p)$.) This implies claim (i) of the lemma. Similarly, the probability that there exists a vertex with at least $\Delta_p^{7/8}$ neighbours in X is at most

$$n\binom{n}{\Delta_p^{7/8}} p^{\Delta_p^{7/8}} e^{-\Omega(\Delta_p^{7/8}\log^{1/3} n)} \leqslant n\left(npe^{-\Omega(\log^{1/3} n)}\right)^{\Delta_p^{7/8}} \leqslant n\left((\log^{1/4} n)e^{-\Omega(\log^{1/3} n)}\right)^{\Omega\left(\left(\frac{\log n}{(\log\log n)^2}\right)^{7/8}\right)} \leqslant ne^{-\Omega(\log^{13/12} n)} = o(1).$$

This completes the proof of the lemma for $e^{-(\log\log n)^2}/n \leqslant p \leqslant \log^{1/4} n/n$.

Next we consider the case when $p \geqslant \log^{1/4} n/n$. We again start by estimating the probability that all the vertices of a fixed set T of size $t \leqslant n/2$ have degree at least $np(1+1/\log\log n)$. As before, for such a set T, there are at least $t(n-t)p+tnp/(3\log\log n)$ edges in the cut $(T, V(G)-T)$, or the set T spans at least $t(t-1)p/2 + tnp/(3\log\log n)$ edges. By the standard estimates for the binomial distributions (see, *e.g.*, [3, Appendix A]) it follows that the probability of the first event is at most $e^{-\Omega(tnp/(\log\log n)^2)}$. The same estimates can be used to show that if $n/(6\log\log n) \leqslant t \leqslant n/2$ then the probability of the second event is also bounded by $e^{-\Omega(tnp/(\log\log n)^2)}$. On the other hand, if $t \leqslant n/(6\log\log n)$, then this probability can be bounded directly by

$$\binom{\frac{t(t-1)}{2}}{\frac{tnp}{3\log\log n}} p^{\frac{tnp}{3\log\log n}} \leqslant \left(\frac{3e(t-1)p\log\log n}{2np}\right)^{\frac{tnp}{3\log\log n}} \leqslant \left(\frac{e}{4}\right)^{\frac{tnp}{3\log\log n}} \leqslant e^{-\Omega(tnp/(\log\log n)^2)}.$$

Therefore, the probability that all the degrees of the vertices in a given set of size t are at least $np(1+1/\log\log n)$ is at most $e^{-\Omega(tnp/(\log\log n)^2)}$. Again, conditioning on the presence of any specific set of at most $2t$ edges does not change the order of the exponent in the above estimate.

Using this bound together with the fact that $np \geqslant \log^{1/4} n$, we can estimate the probability that there exists a cycle of length k with at least $k/2$ vertices inside set X. Clearly this probability is at most

$$\sum_{k\geqslant 3} n^k p^k \binom{k}{\lceil k/2\rceil} e^{-\Omega((k/2)np/(\log\log n)^2)} \leqslant \sum_{k\geqslant 3} \left(2npe^{-\Omega(np/(\log\log n)^2)}\right)^k \leqslant \sum_{k\geqslant 3} e^{-\Omega\left(k\frac{\log^{1/4} n}{(\log\log n)^2}\right)} = o(1).$$

This implies claim (i). Similarly, the probability that there exists a vertex with at least $\Delta_p^{7/8}$ neighbours in X is at most

$$n\binom{n}{\Delta_p^{7/8}} p^{\Delta_p^{7/8}} e^{-\Omega(\Delta_p^{7/8} np/(\log\log n)^2)} \leqslant n\big(npe^{-\Omega(np/(\log\log n)^2)}\big)^{\Delta_p^{7/8}}$$
$$\leqslant ne^{-\Omega\left(\frac{\log^{1/4} n}{(\log\log n)^2}\left(\frac{\log n}{(\log\log n)^2}\right)^{7/8}\right)}$$
$$\leqslant ne^{-\Omega(\log^{17/16} n)} = o(1).$$

This implies claim (ii) and completes the proof of the lemma. □

Finally we need one additional lemma.

Lemma 2.4. *Let $G = G(n,p)$ be a random graph with $e^{-(\log\log n)^2}/n \leqslant p \leqslant \log^{1/2} n/n$. Then a.s. G contains no vertex that has at least $\Delta_p^{1/3}$ other vertices of G with degree $\geqslant \Delta_p^{3/4}$ within distance at most two.*

Proof. Let v be a vertex of $G(n,p)$ and let $u_i, i = 1, \dots, \Delta_p^{1/3}$ be vertices with degree at least $\Delta_p^{3/4}$ which are within distance at most two from v. Let T be the set of vertices of the smallest connected subgraph of G which contains v together with all the vertices u_i. Since the shortest path from v to u_i may contain only one vertex distinct from v and u_i, then the size of T satisfies $\Delta_p^{1/3} + 1 \leqslant |T| = t \leqslant 2\Delta_p^{1/3} + 1$. In addition, each u_i has at least $\Delta_p^{3/4} - t \geqslant \frac{1}{2}\Delta_p^{3/4}$ neighbours outside set T. Therefore there are at least $\frac{1}{2}\Delta_p^{3/4} \cdot \Delta_p^{1/3} = \frac{1}{2}\Delta_p^{13/12}$ edges of G between T and $V(G) - T$. Since the number of edges in the cut $(T, V(G) - T)$ is a binomially distributed random variable with parameters $t(n-t)$ and p, we can bound the probability of this event for a fixed set T of size $|T| = t$ by

$$\binom{t(n-t)}{\frac{1}{2}\Delta_p^{13/12}} p^{\frac{1}{2}\Delta_p^{13/12}} \leqslant \left(\frac{2et(n-t)p}{\Delta_p^{13/12}}\right)^{\frac{1}{2}\Delta_p^{13/12}} \leqslant \left(\frac{5e\log^{1/2} n}{\Delta_p^{3/4}}\right)^{\frac{1}{2}\Delta_p^{13/12}} \leqslant e^{-\log^{25/24} n}.$$

Here we used the facts that, by Lemma 2.1, for $p \geqslant e^{-(\log\log n)^2}/n$,

$$\Delta_p \geqslant \Omega(\log n/(\log\log n)^2),$$

and that $np \leqslant \log^{1/2} n$ and $t \leqslant 2\Delta_p^{1/3} + 1$.

As we explained in the previous paragraph, the probability that there exists a vertex that violates the assertion of the lemma is bounded by the probability that there exists a connected subgraph on $|T| = t \leqslant 2\Delta_p^{1/3} + 1$ vertices such that the number of edges in the cut $(T, V(G) - T)$ is at least $\frac{1}{2}\Delta_p^{13/12}$. Using the fact that, for $p \leqslant \log^{1/2} n/n$, by definition, $\Delta_p = o(\log n)$, we can bound this probability by

$$\sum_{t \leqslant 2\Delta_p^{1/3}+1} \binom{n}{t} t^{t-2} p^{t-1} e^{-\log^{25/24} n} \leqslant \sum_{t \leqslant 2\Delta_p^{1/3}+1} \frac{en}{t^2}\big(enp\big)^{t-1} e^{-\log^{25/24} n}$$
$$\leqslant 3\Delta_p^{1/3} n\big(enp\big)^{2\Delta_p^{1/3}} e^{-\log^{25/24} n}$$
$$\leqslant n\log^{1/3} n\big(e\log^{1/2} n\big)^{\log^{1/3} n} e^{-\log^{25/24} n} = o(1).$$

This completes the proof. □

3. Proof of the main result

In this section we prove Theorem 1.1. We start by stating some simple properties of the largest eigenvalue of a graph that we will need later.

Proposition 3.1. *Let G be a graph on n vertices and m edges and with maximum degree Δ. Let $\lambda_1(G)$ be the largest eigenvalue of the adjacency matrix of G. Then the following properties hold.*

(i) $\max\left(\sqrt{\Delta}, \frac{2m}{n}\right) \leqslant \lambda_1(G) \leqslant \Delta$.

(ii) *If $E(G) = \bigcup_i E(G_i)$ then $\lambda_1(G) \leqslant \sum_i \lambda_1(G_i)$. If in addition the graphs G_i are vertex-disjoint, then $\lambda_1(G) = \max_i \lambda_1(G_i)$.*

(iii) *If G is a forest then $\lambda_1(G) \leqslant \min\left(2\sqrt{\Delta - 1}, \sqrt{n-1}\right)$. In particular if G is a star on $\Delta + 1$ vertices then $\lambda_1(G) = \sqrt{\Delta}$.*

(iv) *If G is a bipartite graph such that degrees on both sides of bipartition are bounded by Δ_1 and Δ_2 respectively, then $\lambda_1(G) \leqslant \sqrt{\Delta_1\Delta_2}$.*

Proof. Most of these easy statements can be found in Chapter 11 of the book by Lovász [12]. Here we sketch the proof of few remaining ones for the sake of completeness.

(iii) Let A be the adjacency matrix of G and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be its eigenvalues. Since G is a forest on n vertices, it is easy to see that the trace of A^2 satisfies

$$\sum_i \lambda_i^2 = tr(A^2) = \sum_v d_v \leqslant 2(n-1).$$

On the other hand $\lambda_1 = -\lambda_n$, because G is bipartite. Therefore we can conclude that $2\lambda_1^2 \leqslant 2(n-1)$ and hence $\lambda_1 \leqslant \sqrt{n-1}$. For the proof of the rest of statement (iii) see, *e.g.*, [12].

(iv) Let A be the adjacency matrix of G. Then by definition it is easy to see that A^2 is the adjacency matrix of a multigraph with maximum degree $\Delta_1\Delta_2$. Therefore by (i) we have that $\lambda_1(A^2) = \lambda_1^2(G) \leqslant \Delta_1\Delta_2$ and hence $\lambda_1 \leqslant \sqrt{\Delta_1\Delta_2}$. □

Having finished all the necessary preparations, we are now ready to complete the proof of our main theorem.

Proof of Theorem 1.1. We start with the easy case when the random graph is very sparse. If $p \leqslant e^{-(\log\log n)^2}/n$, then by Lemma 2.2 a.s. $G = G(n,p)$ is a disjoint union of trees of size at most $(1+o(1))\Delta_p$. Therefore, by claims (ii) and (iii) of Proposition 3.1, we have that $\lambda_1(G) \leqslant (1+o(1))\sqrt{\Delta_p}$. On the other hand, by Lemma 2.2, the maximum degree of G is almost surely at least $(1+o(1))\Delta_p$, and thus claim (i) of Proposition 3.1 implies that $\lambda_1(G) \geqslant (1+o(1))\sqrt{\Delta_p}$. Since the value of the edge probability satisfies $np = o(1) < \sqrt{\Delta_p}$, we obtain that $\lambda_1(G) = (1+o(1))\sqrt{\Delta_p} = (1+o(1))\max\left(\sqrt{\Delta(G)}, np\right)$.

Another relatively simple case is when $p \geqslant \log^{1/2} n/n$. Then, by Lemma 2.1 we have that $\Delta_p = o\left((np)^2\right)$, and hence it is sufficient to prove that

$$\lambda_1(G) = (1+o(1))\max\left(\sqrt{\Delta_p}, np\right) = (1+o(1))np.$$

To get a lower bound on the largest eigenvalue note that the standard Chernoff estimates for the binomial distributions (see, *e.g.*, [3, Appendix A]) imply that the number of edges in $G(n,p)$ is a.s. $(1+o(1))n^2p/2$. Therefore, by claim (i) of Proposition 3.1, the largest eigenvalue of $G(n,p)$ is almost surely at least $(1+o(1))n^2p/n = (1+o(1))np$.

To get an upper bound, let X denote the set of vertices of a random graph $G = G(n,p)$ with degree larger than $np(1+1/\log\log n)+\Delta_p^{1/3}$. Let G_1 be a subgraph of G induced by the set X, let G_2 be a subgraph of G induced by the set $V(G)-X$, and finally let G_3 be a bipartite subgraph of G containing all the edges between X and $V(G)-X$. By definition, $G = \bigcup_i G_i$ and thus by claim (ii) of Proposition 3.1 we obtain that $\lambda_1(G) \leqslant \sum_{i=1}^3 \lambda_1(G_i)$. Since the maximum degree of graph G_2 is $np(1+1/\log\log n)+\Delta_p^{1/3} = (1+o(1))np$, then by claim (i) of Proposition 3.1 it follows that $\lambda_1(G_2) \leqslant (1+o(1))np$. Also note that, by our construction, any cycle in the graphs G_1 or G_3 should have fewer than half of its vertices in the set X. Therefore, from Lemma 2.3 we get that almost surely G_1 and G_3 contains no cycles. In addition, by Lemma 2.2, the maximum degree of these two forests is bounded by $(1+o(1))\Delta_p$. Then, using claim (iii) of Proposition 3.1, we obtain that $\lambda_1(G_i) \leqslant (2+o(1))\sqrt{\Delta_p}$, $i = 1, 3$. This implies that

$$\lambda_1(G) \leqslant \lambda_1(G_1)+\lambda_1(G_2)+\lambda_1(G_3) \leqslant (1+o(1))np + (4+o(1))\sqrt{\Delta_p} = (1+o(1))np.$$

Finally we treat the remaining case when $e^{-(\log\log n)^2}/n \leqslant p \leqslant \log^{1/2} n/n$. As before, we have that a.s. the maximum degree of $G = G(n,p)$ is $(1+o(1))\Delta_p$ and the total number of edges in G is $(1+o(1))n^2p/2$. Therefore claim (i) of Proposition 3.1 implies that

$$\lambda_1(G) \geqslant (1+o(1))\max\big(\sqrt{\Delta_p}, n^2p/n\big) = (1+o(1))\max\big(\sqrt{\Delta(G)}, np\big).$$

To handle the upper bound on λ_1, we again use a partition of G into smaller subgraphs, whose largest eigenvalue is easier to estimate.

Let X_1 denote the set of vertices of G with degree at least $\Delta_p^{3/4}$, and let X_2 denote the set of vertices with degrees larger than $np(1+1/\log\log n)+\Delta_p^{1/3}$ but less than $\Delta_p^{3/4}$. Let $X = X_1 \cup X_2$, and let Y_1 contain all vertices of $V(G)-X$ with at least one neighbour in X_1. Finally, let Y_2 be the set $V(G)-X\cup Y_1$. Note that by definition there are no edges between X_1 and Y_2.

We consider the following subgraphs of G. Let G_1 be the subgraph of G induced by the set X. Then, by Lemma 2.3, G_1 contains no cycles and has maximum degree at most $\Delta_p^{7/8}$. Therefore, by claim (iii) of Proposition 3.1 we get that

$$\lambda_1(G_1) \leqslant 2\sqrt{\Delta_p^{7/8}} = o\big(\sqrt{\Delta_p}\big).$$

Our second graph G_2 consists of all edges between X_2 and $V(G)-X$. Note that, by definition, any cycle in G_2 has exactly half of its vertices in $X_2 \subset X$. Thus, by Lemma 2.3, almost surely G_2 is a forest. In addition, the maximum degree in G_2 is bounded by the maximal possible degree of a vertex from the set $V(G)-X_1$, which is $\Delta_p^{3/4}$. Using claim (iii) of Proposition 3.1 we get that

$$\lambda_1(G_2) \leqslant 2\sqrt{\Delta_p^{3/4}} = o\big(\sqrt{\Delta_p}\big).$$

Next consider the graph G_3, induced by the set of vertices Y_1. Let $v \in V(G) - X$ be a vertex with at least $\Delta_p^{1/3} + 1$ neighbours in Y_1. Since, by definition, every neighbour of v in Y_1 is also a neighbour of some vertex in X_1, we obtain that there are at least $\Delta_p^{1/3} + 1$ paths of length two from v to the set X_1. On the other hand, by Lemma 2.2, v is almost surely contained in at most one cycle of length 4. This implies that all but at most one of the endpoints of these paths in X_1 are different. Therefore vertex v has at least $\Delta_p^{1/3}$ distinct vertices of X_1 within distance two. Now from Lemma 2.4 it follows that a.s. there is no vertex with this property. Hence every vertex $v \in V(G) - X$ has almost surely at most $\Delta_p^{1/3}$ neighbours in Y_1. In particular, the maximum degree of G_3 is bounded by $\Delta_p^{1/3}$, which implies that $\lambda_1(G_3) \leqslant \Delta_p^{1/3} = o\big(\sqrt{\Delta_p}\big)$.

Let G_4 be the bipartite subgraph consisting of all the edges of G between Y_1 and Y_2. By definition, the degree of every vertex in Y_1 is at most $np(1 + 1/\log\log n) + \Delta_p^{1/3}$, and we proved in the previous paragraph that the degree of every vertex from Y_2 in this graph is at most $\Delta_p^{1/3}$. Therefore, using claim (iv) of Proposition 3.1 together with the facts that $np \leqslant \log^{1/2} n$ and $\Delta_p \geqslant \Omega\big(\log n/(\log\log n)^2\big)$, we obtain

$$\lambda_1(G_4) \leqslant \sqrt{\Delta_p^{1/3}\big(np(1 + 1/\log\log n) + \Delta_p^{1/3}\big)} \leqslant \Delta_p^{1/3} + (1 + o(1))\Delta_p^{1/6}\sqrt{np} = o\big(\sqrt{\Delta_p}\big).$$

Finally we define G_5 to be the subgraph of G induced by the set Y_2, and G_6 to be a bipartite graph containing all the edges of G between X_1 and Y_1. Since there are no edges crossing from X_1 to Y_2, it is easy to check that $E(G) = \bigcup_{i=1}^{6} E(G_i)$. Also, since the graphs G_5 and G_6 are vertex-disjoint, then by claim (ii) of Proposition 3.1 we obtain that $\lambda_1(G_5 \cup G_6) = \max\big(\lambda_1(G_5), \lambda_1(G_6)\big)$ and almost surely

$$\lambda_1(G) \leqslant \lambda_1(G_1) + \cdots + \lambda_1(G_4) + \lambda_1(G_5 \cup G_6) = \max\big(\lambda_1(G_5), \lambda_1(G_6)\big) + o\big(\sqrt{\Delta_p}\big).$$

By definition, the maximum degree of G_5 is bounded by $(1 + o(1))np + \Delta_p^{1/3}$, which implies that $\lambda_1(G_5) \leqslant (1 + o(1))np + \Delta_p^{1/3}$. Hence, to finish the proof it remains to bound $\lambda_1(G_6)$.

Consider the graph G_6. Let T be the set of vertices from Y_1 with degrees greater than one in G_6, and let $u \in X_1$ be a vertex with at least $\Delta_p^{1/3} + 1$ neighbours in T. By definition, every neighbour of u in T also has an additional neighbour in X_1 which is distinct from u. Therefore we obtain that there are at least $\Delta_p^{1/3} + 1$ simple paths of length two from u to the set X_1. On the other hand, by Lemma 2.2, u is almost surely contained in at most one cycle of length 4. This implies that all but at most one of the endpoints of these paths in X_1 are different. Therefore vertex u has at least $\Delta_p^{1/3}$ distinct vertices of X_1 within distance two. Now from Lemma 2.4 it follows that a.s. there is no vertex with this property. In addition it follows that every vertex from Y_1 has degree at most $\Delta_p^{1/3}$ in G_6. Let H be the subgraph of G_6 containing all the edges from X_1 to T. Then, by the above discussion, its maximum degree is bounded by $\Delta_p^{1/3}$ and therefore $\lambda_1(H) \leqslant \Delta_p^{1/3}$. On the other hand, since the degree of every vertex in $Y_1 - T$ in G_6 is at most one and the graph is bipartite, we obtain that $G_6 - H$ is a union of vertex-disjoint stars. The size of each star is at most the maximum degree of G. Then, by claims (ii) and (iii) of Proposition 3.1, we get that

$$\lambda_1(G_6) \leqslant \lambda_1(H) + \lambda_1(G_6 - H) \leqslant \Delta_p^{1/3} + (1 + o(1))\sqrt{\Delta_p}.$$

This implies the desired upper bound on $\lambda_1(G)$, since

$$\begin{aligned}\lambda_1(G) &\leqslant \max\big(\lambda_1(G_5), \lambda_1(G_6)\big) + o\big(\sqrt{\Delta_p}\big)\\ &= \max\big((1+o(1))np + \Delta_p^{1/3}, (1+o(1))\sqrt{\Delta_p} + \Delta_p^{1/3}\big)\\ &= (1+o(1))\max\big(\sqrt{\Delta_p}, np\big) = (1+o(1))\max\big(\sqrt{\Delta(G)}, np\big),\end{aligned}$$

and completes the proof of the theorem. □

4. Concluding remarks

In this paper we have found the asymptotic value of the largest eigenvalue of the random graph $G(n,p)$, or the spectral radius of the corresponding random real symmetric matrix.

It would be quite interesting to obtain more accurate estimates on the error term in the asymptotic estimate for $\lambda_1(G(n,p))$, given by Theorem 1.1. Notice that, owing to the recent concentration result of Alon, Krivelevich and Vu [2], the standard deviation of $\lambda_1(G(n,p))$ can be asymptotically bounded by an absolute constant, and this random variable is sharply concentrated. Our proof methods do not allow us to locate the expectation of λ_1 with such a degree of precision. Neither are we able to obtain a limit distribution of λ_1, as has been done by Füredi and Komlós [7] for the case of a constant edge probability p. This is another attractive open question.

We can also try to determine when the largest eigenvalue of a random graph has multiplicity one, and then to understand the typical structure of the first eigenvector of $G(n,p)$. While, for the case $p \gg \log n/n$, where the graph $G(n,p)$ becomes a.s. almost regular, the first eigenvector will be a.s. almost collinear to the all-1 vector, the picture becomes more complicated for smaller values of $p(n)$. Notice that for $p(n) \ll \log n/n$ the graph $G(n,p)$ is a.s. disconnected, and therefore the support of the first eigenvector will be at most as large as the size of its largest connected component.

Consider the case $p = c/n$, for a constant $c > 0$. Performing direct calculations similar to those of Section 2 of the present paper, we can show that in this case $G(n,p)$ contains almost surely an unbounded collection of vertices of degree $\Delta(G)(1-o(1))$ at distance at least three from each other. Considering then the subgraph of G spanned by those vertices and their neighbours shows that a.s. $G(n,p)$ has an unbounded number of eigenvalues $\lambda_i = (1-o(1))\lambda_1$.

Another observation for the case $p = c/n$ is that, according to Corollary 1.2, the first eigenvalue of $G(n,c/n)$ remains asymptotically the same for all values of the constant $c > 0$, and thus appears to be quite insensitive to the growth of $c > 0$. This is in sharp contrast to many other properties of random graphs, such as the appearance of the giant component (all components of $G(n,c/n)$ are a.s. at most logarithmic in size for $c < 1$, while for $c > 1$ $G(n,p)$ contains a.s. one component of linear size, and the rest are $O(\log n)$) or planarity ($G(n,c/n)$ is a.s. planar for $c < 1$ and a.s. non-planar for $c > 1$).

Another related problem is to investigate the spectrum of the Laplacian of a random graph $G(n,p)$. For a graph G, the Laplacian $L = L(G)$ is defined as $L = D - A$, where A is the adjacency matrix of G and D is the diagonal matrix whose diagonal entries are degrees of corresponding vertices. For any graph G, the Laplacian $L(G)$ is easily seen to be a real symmetric matrix with nonnegative eigenvalues, the smallest of them being zero. One may study the so-called spectral gap (the smallest positive eigenvalue of the Laplacian) of random graphs $G(n,p)$ for various values of $p(n)$.

The methods of this paper can possibly be applied to the study of the spectrum of dilute random matrices. A dilute random matrix A is defined by

$$A_{i,j} = a_{i,j}b_{i,j}, \quad 1 \leqslant i \leqslant j \leqslant n$$
$$A_{j,i} = A_{i,j}, \quad 1 \leqslant i < j \leqslant n.$$

where $a_{i,j}$ are jointly independent and not necessarily identically distributed random variables with zero mean and variance 1, and $b_{i,j}$ are also jointly independent and independent from $\{a_{i,j}\}$, where $b_{i,j} = 1$ with probability $p = p(n)$ and $b_{i,j} = 0$ with probability $1 - p$. In other words, the dilute random matrix is obtained by replacing each entry of a matrix from the so-called Wigner ensemble by zero independently with probability $q = 1 - p$. As such, it unifies the notions of the Wigner random matrices and random graphs. Khorunzhy proved in [9], under additional (and rather standard) assumptions on the moments of variables a_{ij} (see his paper for details), that the spectral norm of the dilute random matrix is asymptotically equal to $2\sqrt{np}$ in the case $p(n) \gg \log n/n$ and is asymptotically much larger than $\sqrt{np}$ for $p(n) \ll \log n$. It would be quite interesting to determine the asymptotic behaviour of the spectral radius of the dilute random matrix for the case of small values of $p(n)$.

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