Turán's theorem: variations and generalizations

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PROBLEM:

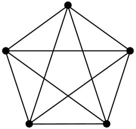
Determine or estimate the size of the largest configuration with a given property.

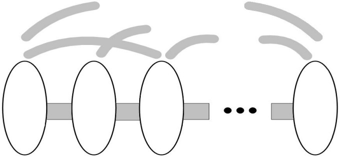
EXAMPLE: *Forbidden subgraph problem*

Given a fixed graph H, find

$$ex(n, H) = \max\left\{e(G) \mid H \not\subset G, |V(G)| = n\right\}$$

Which G are extremal, i.e., achieve maximum?

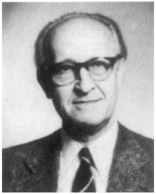




K_{r+1} = complete graph of order $r+1$

Turán graph $T_r(n)$: complete r-partite graph with equal parts.

$t_r(n) = e(T_r(n)) = \frac{r-1}{2r} n^2 + O(r)$



THEOREM: *(Turán 1941, Mantel 1907 for $r = 2$)*

For all $r \geq 2$, the unique largest K_{r+1}-free graph on n vertices is $T_r(n)$.

Definition:

Chromatic number of graph H

$$\chi(H) = \min\left\{k \mid V(H) = V_1 \cup \cdots \cup V_k,\ V_i = \text{independent set}\right\}$$

Theorem: (*Erdős-Stone 1946, Erdős-Simonovits 1966*)

Let H be a fixed graph with $\chi(H) = r + 1$. Then

$$ex(n, H) = t_r(n) + o(n^2) = (1 + o(1))\frac{r-1}{2r}n^2.$$

Remark:

This gives an asymptotic solution for non-bipartite H.

Local density

Problem: (*Erdős 1975*)

Suppose $0 \leq \alpha, \beta \leq 1$, $r \geq 2$, and G is a K_{r+1}-free graph on n vertices in which every αn vertices span at least βn^2 edges.

How large can β be as a function of α?

Example:

When $\alpha = 1$, Turán's theorem implies that $\beta = \frac{r-1}{2r}$.

Remark:

Szemerédi's regularity lemma implies that for fixed H with $\chi(H) = r + 1 \geq 3$, the bound on the local density for H-free graphs is the same as for K_{r+1}-free graphs.

CONJECTURE: (*Erdős, Faudree, Rousseau, Schelp*)

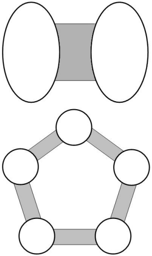
There exists a constant $c_r < 1$ such that for $c_r \leq \alpha \leq 1$, the Turán graph has the largest local density with respect to subsets of size αn.

THEOREM: (*Keevash and S., Erdős et al. for $r = 2$*)

There exists $\epsilon_r > 0$ such that if G is a K_{r+1}-free graph of order n and $1 - \epsilon_r \leq \alpha \leq 1$, then G contains a subset of size αn which spans at most

$$\frac{r-1}{2r}(2\alpha - 1)n^2$$

edges. Equality is attained only by the Turán graph $T_r(n)$.



CONJECTURE: (*Erdős, Faudree, Rousseau, Schelp*)

Any triangle-free graph G on n vertices should contain a set of αn vertices that spans at most

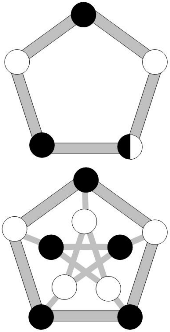
- $\frac{2\alpha-1}{4}n^2$ edges if $17/30 \leq \alpha \leq 1$.
- $\frac{5\alpha-2}{25}n^2$ edges if $1/2 \leq \alpha \leq 17/30$.

THEOREM: (*Krivelevich 1995*)

Conjecture holds for $0.6 \leq \alpha \leq 1$, i.e., the Turán graph $T_2(n)$ has the largest local density with respect to subsets in this range.

Conjecture: (*Erdős 1975*)

Any triangle-free graph G on n vertices should contain a set of $n/2$ vertices that span at most $n^2/50$ edges.



Examples:

- $C_5(n)$ = blow-up of 5-cycle.
  $e(C_5(n)) = \frac{1}{5}n^2$.
- $P(n)$ = blow-up of Petersen graph.
  $e(P(n)) = \frac{3}{20}n^2$.

Partial results

Theorem: (*Krivelevich 1995*)

Any triangle-free graph contains a set of size $n/2$ which spans at most $n^2/36$ edges.

Theorem: (*Keevash and S. 2005*)

- Let G be a triangle-free graph on n vertices with at least $n^2/5$ edges, such that every set of $\lfloor n/2 \rfloor$ vertices of G spans at least $n^2/50$ edges. Then $n = 10m$ for some integer m, and $G = C_5(n)$.
- Conjecture also holds for triangle-free graphs on n vertices with at most $n^2/12$ edges.

K_{r+1}-free graphs, $r \geq 3$

Conjecture: (*Chung and Graham 1990*)

Among K_{r+1}-free graphs of order n, the Turán graph $T_r(n)$ has the largest local density with respect to sets of size αn for all $\frac{1}{2} \leq \alpha \leq 1$ and $r \geq 3$.

In particular, every K_4-free graph on n vertices contains a set of size $n/2$ that spans at most $n^2/18$ edges.

Remark:

- For K_4-free graphs the result of Keevash and S. shows that the conjecture holds when $\alpha > 0.861$.
- It is easy to show that every K_4-free graph on n vertices contains a set of size $n/2$ that spans at most $n^2/16$ edges.

PROBLEM: (*Erdős*)

Let G be an H-free graph on n vertices. How many edges (as a function of n) does one need to delete from G to make it bipartite?

REMARK:

For every G it is enough to delete at most half of its edges to make it bipartite. Hence the extremal graph should be dense.

Max Cut versus local density

Observation: (*Krivelevich*)

Let G be a d-regular H-free graph on n vertices and S be a set of size $n/2$. Then

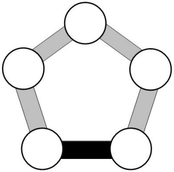
$$\begin{aligned} \frac{dn}{2} &= \sum_{s\in S} d(s) = 2e(S) + e(S,\bar{S}) \\ &= \sum_{s\in \bar{S}} d(s) = 2e(\bar{S}) + e(S,\bar{S}), \end{aligned}$$

i.e. $e(S) = e(\bar{S})$. Deleting the $2e(S)$ edges within S or $\bar{S}$ makes the graph bipartite, so if we could find S spanning at most βn^2 edges, we would delete at most $2\beta n^2$ edges and make G bipartite.

Triangle-free case

Conjecture: (*Erdős 1969*)

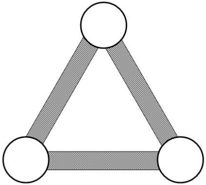
If G is a triangle-free graph of order n, then deleting at most $n^2/25$ edges is enough to make G bipartite.



Theorem: (*Erdős, Faudree, Pach, Spencer 1988*)

- If G has at least $n^2/5$ edges then the conjecture is true.
- Every triangle-free graph of order n can be made bipartite by deleting at most $(1/18 - \epsilon)n^2$ edges.

K_4-FREE GRAPHS



EXAMPLE:

The Turán graph $T_3(n)$ has $n^3/27$ triangles and every edge is in $\leq n/3$ of them. We need to delete $\geq \frac{n^3/27}{n/3} = n^2/9$ edges to make it bipartite.

CONJECTURE: (*Erdős*)

Every K_4-free graph with n vertices can be made bipartite by deleting at most $(1/9 + o(1))n^2$ edges.

Making K_4-free graph bipartite

Theorem: *(S. 2005)*

Every K_4-free graph G with n vertices can be made bipartite by deleting at most $n^2/9$ edges, and the only extremal graph which requires deletion of that many edges is the Turán graph $T_3(n)$.

Problem:

Prove that deleting at most $\frac{r-2}{4r}n^2$ edges for even $r \geq 4$ and $\frac{(r-1)^2}{4r^2}n^2$ edges for odd $r \geq 5$ will be enough to make every K_{r+1}-free graph of order n bipartite.

Turán's theorem revisited

Problem: (*Erdős 1983*)

Find conditions on a graph G which imply that the largest K_{r+1}-free subgraph and the largest r-partite subgraph of G have the same number of edges.

Theorem: (*Babai, Simonovits and Spencer 1990*)

Almost all graphs have this property, i.e., the largest K_{r+1}-free subgraph and the largest r-partite subgraph of the random graph $G(n, 1/2)$ almost surely have the same size.

Large minimum degree is enough

Theorem: (*Alon, Shapira, S. 2005*)

Let H be a fixed graph of chromatic number $r+1 \geq 3$ which contains an edge whose removal reduces its chromatic number, e.g., H is the clique K_{r+1}. Then there is a constant $\mu = \mu(H) > 0$ such that if G is a graph on n vertices with minimum degree at least $(1-\mu)n$ and Γ is the largest H-free subgraph of G, then

Γ is r-partite.

Remark:

- In the special case when H is a triangle, this was proved by Bondy, Shen, Thomassé, Thomassen and in a stronger form by Balogh, Keevash, S.
- In this theorem μ is of order r^{-2}.

When is the max. $\triangle$-free subgraph bipartite?

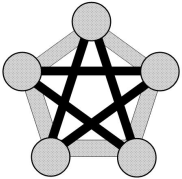
Conjecture: (*Balogh, Keevash, S.*)

Let G be a graph of order n with min. degree $\delta(G) \geq \left(\frac{3}{4} + o(1)\right)n$. Then the largest triangle-free subgraph of G is bipartite.

Example:

Substitute $\forall$ vertex of a 5-cycle by a clique of size $n/5$, $\forall$ edge by a complete bipartite graph, add remaining edges with probability $\theta < 3/8$. The min. degree can be as close to $3n/4$ as needed.

$$\text{Max Cut} = \left(\frac{17}{100} + \frac{2}{25}\theta\right)n^2 < n^2/5.$$



Theorem: (*Balogh, Keevash, S., extending Bondy et al.*)

If the minimum degree $\delta(G) \geq 0.791n$, then the largest triangle-free subgraph of G is bipartite.

Large minimum degree and H-free subgraphs

Theorem: *(Alon, Shapira, S.)*

Let H be a fixed graph with chromatic number $r+1>3$. There exist constants $\gamma=\gamma(H)>0$ and $\mu=\mu(H)>0$ such that if G is a graph on n vertices with minimum degree at least $(1-\mu)n$ and Γ is the largest H-free subgraph of G, then Γ can be made r-partite by deleting $O\left(n^{2-\gamma}\right)$ edges.

Remarks:

- When G is a complete graph K_n, this gives the Erdős-Stone-Simonovits theorem.
- The error term $n^{2-\gamma}$ cannot be avoided.

Edge-deletion problems

Definition:

A graph property $\mathcal{P}$ is *monotone* if it is closed under deleting edges and vertices. It is *dense* if there are n-vertex graphs with $\Omega(n^2)$ edges satisfying it.

Examples:

- $\mathcal{P} = \{G \text{ is 5-colorable}\}$.
- $\mathcal{P} = \{G \text{ is triangle-free}\}$.
- $\mathcal{P} = \{G \text{ has a 2-edge coloring with no monochromatic } K_6\}$

Definition:

Given a graph G and a monotone property $\mathcal{P}$, denote by

$$E_{\mathcal{P}}(G) = \text{smallest number of edge deletions needed to turn } G \text{ into a graph satisfying } \mathcal{P}.$$

Approximation and hardness

Theorem: *(Alon, Shapira, S. 2005)*

- For every monotone $\mathcal{P}$ and $\epsilon > 0$, there exists a linear time, deterministic algorithm that given graph G on n vertices computes number X such that $\left|X - E_{\mathcal{P}}(G)\right| \leq \epsilon n^2$.
- For every monotone dense $\mathcal{P}$ and $\delta > 0$ it is *NP*-hard to approximate $E_{\mathcal{P}}(G)$ for graph of order n up to an additive error of $n^{2-\delta}$.

Remark:

Prior to this result, it was not even known that computing $E_{\mathcal{P}}(G)$ *precisely* for dense $\mathcal{P}$ is *NP*-hard. We thus answer (in a stronger form) a question of Yannakakis from 1981.

Hardness: example

Setting:

$\mathcal{P}$ = property of being H-free, $\chi(H) = r + 1$.

$E_{r\text{-}col}(F)$ = number of edge-deletions needed to make graph F r-colorable. Computing $E_{r\text{-}col}(F)$ is NP-hard.

Reduction:

- Given F, let F' = blow-up of F : vertex $\leftarrow$ large independent set, edge $\leftarrow$ complete bipartite graph. Add edges to F' in a pseudo-random way to get a graph G with large minimum degree.
- $E_{r\text{-}col}(F)$ changes in a controlled way, i.e., knowledge of an accurate estimate for $E_{r\text{-}col}(G)$ tells us the value of $E_{r\text{-}col}(F)$.
- Since G has large minimum degree,

$$\left|E_{r\text{-}col}(G) - E_{\mathcal{P}}(G)\right| \le n^{2-\gamma}.$$

- Thus, approximating $E_{\mathcal{P}}(G)$ up to an additive error of $n^{2-\delta}$ is as hard as computing $E_{r\text{-}col}(F)$.

Another extension

Claim: (*Folklore*)

Every graph G contains a K_{r+1}-free subgraph with at least $\frac{r-1}{r}e(G)$ edges.

Question:

For which G is the size of the largest K_{r+1}-free subgraph

$$\frac{r-1}{r}e(G) + o\big(e(G)\big)?$$

Examples:

- Holds for the complete graph K_n by Turán's theorem.
- Hold almost surely for the random graph $G(n,p)$ of appropriate density.

Notation:

The adjacency matrix A_G of a graph G has $a_{ij} = 1$ if $(i,j) \in E(G)$ and 0 otherwise. It is a symmetric matrix with real eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. If G is d-regular, then $\lambda_1 = d$.

Definition:

G is an (n,d,λ)-graph if it is d-regular, has n vertices, and

$$\max_{i \geq 2} |\lambda_i| \leq \lambda.$$

Remark:

A large *spectral gap*, i.e., when $\lambda \ll d$, implies that the edges of G are distributed as in the random graph $G\left(n, \frac{d}{n}\right)$.

Properties of (n, d, λ)-graphs

Proposition: *(Alon)*

Let G be an (n, d, λ)-graph and $B, C \subseteq V(G)$. Then

$$\left| e(B, C) - \frac{d}{n}|B||C| \right| \leq \lambda\sqrt{|B||C|}.$$

Facts:

- Let $B = C$ be the set of neighbors of a vertex v in G. Then $|B| = |C| = d$ and the above inequality gives that if
$$d^2 \gg \lambda n,$$
then there is an edge in the neighborhood of v, i.e., G contains a triangle.
- Using induction one can show that if $d^r \gg \lambda n^{r-1}$ then every (n, d, λ)-graph contains cliques of size $r + 1$.

Spectral Turán's theorem

Theorem: *(S., Szabó, Vu 2005)*

Let $r \geq 2$, and let G be an (n, d, λ)-graph with $d^r \gg \lambda n^{r-1}$. Then the size of the largest K_{r+1}-free subgraph of G is

$$\frac{r-1}{r} e(G) + o\big(e(G)\big).$$

Remarks:

- The complete graph K_n has $d = n - 1$ and $\lambda = 1$. Thus we have an asymptotic extension of Turán's theorem.
- The theorem is tight for $r = 2$. By the result of Alon, there are (n, d, λ)-graphs with $d^2 = \Theta(\lambda n)$ which contain no triangles.

Problem:

Find constructions of K_{r+1}-free (n, d, λ)-graphs with $d^r \approx \lambda n^{r-1}$.