Turán's Theorem: generalizations and applications

Benny Sudakov
UCLA

Extremal Graph Theory

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Determine or estimate the maximum or minimum possible size of a discrete structure (e.g., *graph or hypergraph*) satisfying certain restrictions.

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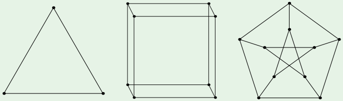
Examples and applications:

- Discrete geometry
- Additive number theory
- Probability
- Harmonic Analysis
- Computer Science
- Coding Theory

Problem:

Given a fixed graph H, e.g.,



determine $ex(n, H)$, the maximum number of edges in a graph on n vertices that does not contain a copy of H.

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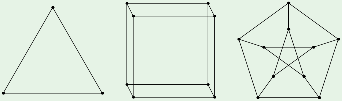

determine $ex(n, H)$, the maximum number of edges in a graph on n vertices that does not contain a copy of H.

MANTEL 1907: *Every triangle-free graph on n vertices has at most $\lfloor n^2/4 \rfloor$ edges.*

Probabilistic inequality

Theorem: (*Katona 1969*)

Let X_1, X_2 are i.i.d. random vectors in $\mathbb{R}^d$. Then

$$\mathbb{P}[\,|X_1 + X_2| \geq 1\,] \geq \frac{1}{2}\,\mathbb{P}^2[\,|X_1| \geq 1\,]$$

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Observation:

Let $v_1, \ldots, v_n$ be vectors in $\mathbb{R}^d$ with length at least 1. Then pairs (i,j) such that $|v_i + v_j| < 1$ can not form a triangle. Therefore there are at least $\frac{n(n-2)}{2}$ pairs $i \neq j$ with $|v_i + v_j| \geq 1$.

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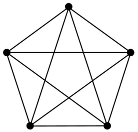
Proof. Let $a = \mathbb{P}[|X_1| \geq 1]$ and let $b = \mathbb{P}[|X_1 + X_2| \geq 1]$. Sample independently $X_1, \ldots, X_m$ from the distribution. Then there are $n \approx am$ vectors $|X_i| \geq 1$ and $\approx bm(m-1)$ pairs $|X_i + X_j| \geq 1$.

Thus $$bm(m-1) \geq \tfrac{n(n-2)}{2} \approx \tfrac{1}{2}a^2m^2.$$ □



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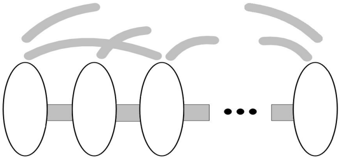
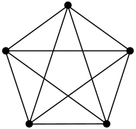


Turán graph $T_r(n)$: complete r-partite graph with equal parts.

$t_r(n) = e(T_r(n)) = \frac{r-1}{2r}n^2 + O(r)$

Turán's theorem



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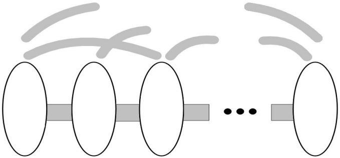

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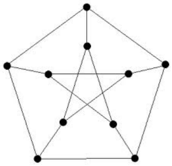
Theorem: *(Turán 1941, Mantel 1907 for $r = 2$)*

For all $r \geq 2$, the unique largest K_{r+1}-free graph on n vertices is $T_r(n)$.

Question:

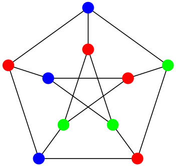
What is the Turán number $ex(n, H)$ for a general graph H?

E.g., $H =$



Definition:

The chromatic number $\chi(H)$ is the minimum number of colors needed to color $V(H)$, so that adjacent vertices have distinct colors.



General graphs

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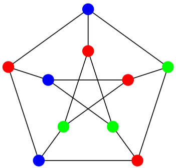

Theorem: (*Erdős-Stone 1946, Erdős-Simonovits 1966*)

Let H be a fixed graph with $\chi(H) = r + 1$. Then

$$ex(n, H) = t_r(n) + o(n^2) = (1 + o(1))\frac{r-1}{2r}n^2.$$

Remark: Determines the asymptotics of Turán numbers $ex(n, H)$ for all graphs with chromatic number at least 3.

Corollary:

For any constant $\varepsilon > 0$ and large n, every n-vertex graph with at least εn^2 edges contains all fixed bipartite graphs.

Complete bipartite graphs

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Application: Let $S \subset \mathbb{Z}^2$ and define *density* of S

$$d(S) = \limsup_{k\to\infty} d_k(S), \quad \text{where} \quad d_k(S) = \max_{\substack{A,B\subset\mathbb{Z} \\ |A|=|B|=k}} \frac{S \cap A \times B}{|A||B|}.$$

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Theorem: (*Kővári, Sós and Turán 1954*)

Let $K_{r,s}$ be a complete bipartite graph with parts of size r and s. Then for all $r \leq s$ there is a constant $c = c(r,s)$ such that

$$ex(n, K_{r,s}) \leq c\, n^{2-1/r}.$$

Unit distances

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What is the maximum possible number of unit distances among n points in the plane?

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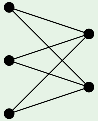
Observation:

Connect two points by an edge if the distance between them is 1. Note that this graph can not have $K_{2,3}$. Thus, from estimate on $ex(n, K_{2,3})$ the number of unit distances is at most $O(n^{3/2})$.



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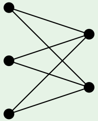

Remarks:

- The best current upper bound for this problem is $O(n^{4/3})$.
- It is conjectured that the number of unit distance is $\leq n^{1+o(1)}$.

Representing squares economically

Problem: (*Wooley, Erdős-Newman*)

Let $A \subset \mathbb{Z}$ such that $A + A = \{a + a' | a, a' \in A\}$ contains $1^2, 2^2, \ldots n^2$. How small can set A be?

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Sketch: For every $1 \leq x \leq n$ connect some pair (a, a') such that $a + a' = x^2$ by an edge. If $|A| = m = n^{2/3-\epsilon}$ then this graph has m vertices, $n \geq m^{3/2+\epsilon}$ edges and by estimate on $ex(m, K_{2,s})$ contains a pair a_1, a_2 with at least $s = n^\delta$ common neighbors. Then $a_1 - a_2$ can be written as a difference of two squares in n^δ ways and hence has too many devisor. □

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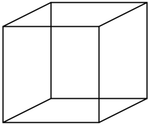
Open:

- Complete bipartite graph with equal parts of size 4.
- Cycle of length 8.
- The 3-cube.



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CONJECTURE: (*Erdős 1966*)

Every r-degenerate bipartite H satisfies $ex(n, H) \leq O(n^{2-1/r})$.

Remark: For all r this estimate is best possible.

Degenerate bipartite graphs

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Theorem: (*Alon-Krivelevich-S. 2003*)

Conjecture holds for every H in which vertices of one part have degrees at most r. For general r-degenerate bipartite H

$$ex(n, H) \leq O(n^{2-\frac{1}{4r}}).$$

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- $h_H(G)$ = the number of homomorphisms from H to G.
- $t_H(G) = \frac{h_H(G)}{|G|^{|H|}}$ = fraction of mappings from H to G which are homomorphisms.

Sidorenko's conjecture

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Conjecture: (*Erdős-Simonovits 84, Sidorenko 93*)

For every bipartite H and every n-vertex G with $pn^2/2$ edges,

$$t_H(G) \geq p^{e(H)}.$$

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Analytical form

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- Known for *trees, even cycles, complete bipartite graphs, cubes.*
- Has connections to matrix theory [BR], Markov chains [BP], graph limits [L], and quasi-randomness.

Conjecture:

μ is the Lebesgue measure on $[0,1]$, $h(x,y) \geq 0$ is bounded, measurable function on $[0,1]^2$, $H = (U, V, E)$ is bipartite graph with $U = \{u_1, \ldots, u_t\}$, $V = \{v_1, \ldots, v_s\}$ and $|E(H)| = q$. Then

$$\int \prod_{(u_i,v_j)\in E} h(x_i, y_j) d\mu^{s+t} \geq \left(\int h d\mu^2 \right)^q .$$

Theorem: (*Conlon-Fox-S. 2010*)

Sidorenko's conjecture holds for every bipartite $H = (U, W)$ which has a vertex $u^* \in U$ adjacent to all vertices in the part W. This also gives an asymptotic version of the conjecture for all graphs.

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Key idea:

- Let G be an n-vertex graph with $pn^2/2$ edges and let v be a random vertex of G. Then almost all small subsets $S \subset N(v)$ have at least $c_H p^{|S|} n$ common neighbors, which, apart from the constant factor c_H, is the expected size of the common neighborhood of a subset of size $|S|$ in the random graph $G_{n,p}$.

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- Use this observation to show that there exist a constant c_H such that a probability $t_H(G)$ that a random mapping from H to G is a homomorphism is at least $c_H p^{e(H)}$.

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- Use this observation to show that there exist a constant c_H such that a probability $t_H(G)$ that a random mapping from H to G is a homomorphism is at least $c_H p^{e(H)}$.
- Use a tensor power trick to show that $c_H = 1$. $\square$

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Turán's theorem: *Equality if G is a complete graph.*

Problem: (*Erdős 1983*)

Find conditions on a graph G which imply that the largest K_{r+1}-free subgraph and the largest r-partite subgraph of G have the same number of edges.

Large minimum degree is enough

Theorem: *(Alon, Shapira, S. 2009)*

Let H be a fixed graph with chromatic number $r+1>3$. There exist constants $\gamma=\gamma(H)>0$ and $\mu=\mu(H)>0$ such that if G is a graph on n vertices with minimum degree at least $(1-\mu)n$ and Γ is the largest H-free subgraph of G, then

- Γ can be made r-partite by deleting $O\left(n^{2-\gamma}\right)$ edges.
- If H is a is a clique K_{r+1}, then Γ is r-partite.

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Remark: Since a complete graph has minimum degree $n-1$, this extends Turán's and Erdős-Stone-Simonovits theorems to all graphs with large minimum degree.

Edge-deletion problems

Definition:

A graph property $\mathcal{P}$ is *monotone* if it is closed under deleting edges and vertices. It is *dense* if there are n-vertex graphs with $\Omega(n^2)$ edges satisfying it.

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- $\mathcal{P} = \{G \text{ is 5-colorable}\}$.
- $\mathcal{P} = \{G \text{ is triangle-free}\}$.
- $\mathcal{P} = \{G \text{ has a 2-edge coloring with no monochromatic } K_6\}$.

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Definition:

Given a graph G and a monotone property $\mathcal{P}$, let

$$E_{\mathcal{P}}(G) = \text{smallest number of edge deletions needed to turn } G \text{ into a graph satisfying } \mathcal{P}.$$

Approximation and hardness

Theorem: (*Alon, Shapira, S. 2009*)

- For every monotone $\mathcal{P}$ and $\epsilon > 0$, there exists a linear-time deterministic algorithm that, given a graph G on n vertices, computes a number X such that $\left|X - E_{\mathcal{P}}(G)\right| \leq \epsilon n^2$.
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- First result uses a strengthening of Szemerédi regularity lemma to approximate G by a fixed size weighted graph W.
- Second result uses generalizations of Turán and Erdős-Stone-Simonovits theorems together with spectral techniques.

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These connections with other mathematical disciplines and the fundamental nature of the area will ensure that in the future Extremal Graph Theory will continue to play an essential role in the development of mathematics.