Two-Sided, Unbiased Version of Hall's Marriage Theorem

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Abstract. The standard conditions in Hall's perfect matching theorem for a bipartite graph G require that all subsets from one side of G are expanding. The unbiased extension identifies mixtures of subsets from both sides such that their expansions imply the standard conditions—hence a perfect matching.

The setting: $G = (V, W, N)$ is a bipartite graph, the two sides V, W are vertex sets of size n. The edges (v, w) between the sides induce the symmetric relation N; $N(S)$ is the set of neighbors of S on the other side.

A vertex set S is *expanding* if its size $|S|$ is at most $|N(S)|$.

A *perfect matching* [PM] in G is a set of n pairwise nonintersecting edges of G—hence a PM covers all the vertices of G.

Theorem (P. Hall "Marriage" Theorem [3]). *If all subsets S on one side of G are expanding, then G has a perfect matching.*

When (V, W) are sets of (men, women), while $(v, w) \in N$ signifies that this pair is a candidate for marriage, then a PM in G constitutes the matchmaker success of bringing n disjoint couples to matrimony. Also if W is a collection of subsets of V and $(v, w) \in N$ signifies that $v \in w$, then a PM gives a system of distinct representatives for these subsets [**3**].

Hall's theorem is a fundamental result in combinatorics [**1**]. It is applied in many proofs and constructions. The full symmetry between the sides in the setting suggests that some mixture of expansion conditions from both sides might also imply perfect matching in G. Strangely, this seemed unnoticed until 2015. Indeed,

Theorem. *The conditions* **U** *or* **D** *below imply the existence of a perfect matching between V and W in G.*

U: All sets of size $\leq p$ in V and all sets of size $\leq q$ in W are expanding, $p + q = n$.

D: All sets of size $> p$ in V and all sets of size $> q$ in W are expanding, $p + q = n$.

Notice that $(p, q) = (n, 0)$ is the standard Hall condition. Recently Ehrenborg [**2**] proved **U** by an inductive argument quite similar to the known proofs of Hall. We present a simple argument showing that:

Claim. **D** or **U** imply $(n, 0)$ (the standard Hall condition).

Proof. Observe first that

1. If $S \subseteq V$ is not expanding, then $T = W - N(S)$ is also not expanding—since $N(T)$ is fully inside $V - S$.

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For **D**, assume some $S \subseteq V$ of size $s \leq p$ is not expanding. Then $|N(S)| < p$ while T, being nonexpanding, satisfies $|T| \leq q$. Hence $|W| < p + q = n$, a contradiction. For proving **U**, assume $S \subseteq V$ is nonexpanding, of smallest size $s > p$. Clearly $|N(S)| = s - 1 \geq p$. While T, being nonexpanding, satisfies $|T| > q$. Hence $|W| > p + q = n$, a contradiction. ■

Remark 1. A sequence of expansion sizes implies perfect matching **only if** it includes **U** or includes **D**, as the following example (from [**2**]) shows:

$$G = K(p, p-1) \cup K(q-1, q).$$

Here $n = p + q - 1$ and G has no perfect matching. It has only one nonexpanding set of size p in V and only one of size q in W.

Remark 2. Condition **U** for perfect matching is already used in [**4**] to get an alternative, simpler proof of a theorem on completing partial Latin squares.

REFERENCES

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3. P. Hall, On representatives of subsets, *J. London Math. Soc.* **10** (1935) 26–30.
4. E. Shamir, Completing partial Latin squares—Alternative proof, `http://arxiv.org/abs1605.05931v1[math.CO]`.

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100 Years Ago This Month in *The American Mathematical Monthly*
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"The zero and principle of local value used by the Maya of Central America" is the subject of an interesting historical note by Professor FLORIAN CAJORI in *Science*, Nov. 17, 1916. Attention is called to the early use of a symbol for *zero* and the principle of local value of number symbols employed by the Maya probably dating back near the beginning of the Christian era. The Maya glyphs first deciphered by FÖRSTEMANN of Dresden, 1886, and independently by GOODMAN of California, relate for the most part to the calendar, to chronology, and to astronomy. The unit of this number system was 20, for which a special symbol, a half closed eye with a dot above, was used. Separate symbols of dots and bars represented the numbers from 1 to 19, each dot representing a unit, and each bar representing five units.

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