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ScienceDirect
Journal of Combinatorial Theory, Series A 113 (2006) 1536–1542
Journal of Combinatorial Theory Series A
www.elsevier.com/locate/jcta

Note

On a restricted cross-intersection problem ☆

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Received 20 June 2005
Available online 28 November 2005

Abstract

Suppose $\mathcal{A}$ and $\mathcal{B}$ are families of subsets of an n-element set and L is a set of s numbers. We say that the pair $(\mathcal{A}, \mathcal{B})$ is L-cross-intersecting if $|A \cap B| \in L$ for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Among such pairs $(\mathcal{A}, \mathcal{B})$ we write $P_L(n)$ for the maximum possible value of $|\mathcal{A}||\mathcal{B}|$. In this paper we find an exact bound for $P_L(n)$ when n is sufficiently large, improving earlier work of Sgall. We also determine $P_{\{2\}}(n)$ and $P_{\{1,2\}}(n)$ exactly, which respectively confirm special cases of a conjecture of Ahlswede, Cai and Zhang and a conjecture of Sgall.

Keywords: Extremal set theory; Restricted intersections; Cross-intersections

1. Introduction

We start by describing the fundamental results concerning extremal problems for set systems with restricted intersections. Throughout L will denote a set of s nonnegative integers. We say that a set system $\mathcal{A}$ is *L-intersecting* if for every $A, B \in \mathcal{A}$ we have $|A \cap B| \in L$. Ray-Chaudhuri and Wilson [14] and Frankl and Wilson [9] obtained tight bounds for L-intersecting set systems. They showed that an L-intersecting family on $[n] = \{1, \dots, n\}$ can have at most $\binom{n}{s}$ sets if it is uniform, and at most $\sum_{i=0}^{s} \binom{n}{i}$ if it is nonuniform. (A set system is *uniform* if all of its sets have the same size.) Frankl and Wilson also proved modular versions of these results. For p prime, they showed that the same bounds hold if the intersection sizes belong to L mod p and the sizes

☆ Research supported in part by NSF grant DMS-0355497, USA–Israeli BSF grant, and by an Alfred P. Sloan fellowship.

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0097-3165/$ – see front matter
doi:10.1016/j.jcta.2005.10.008

of the sets in $\mathcal{A}$ do not belong to L mod p. For an excellent introduction to this subject we refer the reader to [4].

A two-family variation on this problem has been considered in various contexts, most famously by Frankl and Rödl [8] to prove the following long-standing conjecture of Erdős: if $\mathcal{A}$ is a family on $[n]$ without $A, B \in \mathcal{A}$ such that $|A \cap B| = \lfloor n/4 \rfloor$ then $|\mathcal{A}| \leqslant 1.99^n$.

Suppose $\mathcal{A}$ and $\mathcal{B}$ are set systems on $[n]$ and L is a set of s numbers. We say that the pair $(\mathcal{A}, \mathcal{B})$ is *L-cross-intersecting* if $|A \cap B| \in L$ for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Among such pairs $(\mathcal{A}, \mathcal{B})$ we write $P_L(n)$ for the maximum possible value of $|\mathcal{A}||\mathcal{B}|$. Frankl and Rödl showed that for any $0 < \eta < 1/4$ there is $\epsilon > 0$ such that if $L = [n] \setminus \{\alpha n\}$ for some $\eta < \alpha < 1/2 - \eta$, i.e., $|A \cap B| \neq \alpha n$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$, then $P_L(n) \leqslant (4 - \epsilon)^n$.

This result has had a number of applications in combinatorics and computer science: see, e.g., [2,3,6,12]. The two-family problem also appears in coding theory [7] and communication complexity [1,15]. The authors of this paper have also previously considered a similar problem [11].

Sgall [15] obtained a general upper bound of $P_L(n) \leqslant \binom{n}{s-1} 2^{n+s-1}$ (recall that $|L| = s$). In fact he proved this bound under the weaker assumption that every intersection size belongs to L mod m, for any $m \geqslant 2$ (not necessarily prime or prime-power). On the other hand, a lower bound of $\sum_{i=0}^{s-1} \binom{n}{i} 2^n$ can be obtained when $L = \{0, \ldots, s-1\}$ by taking $\mathcal{A} = \mathcal{P}[n]$ to consist of all subsets of $[n]$ and $\mathcal{B} = [n]^{(\leqslant s-1)}$ to consist of all subsets of $[n]$ of size at most $s-1$. Thus Sgall's general bound gives the correct order of magnitude for $P_L(n)$ among all sets L of size s. Also, when $L = \{0, \ldots, s-1\}$ he showed that the lower bound is tight, i.e., $P_L(n) = \sum_{i=0}^{s-1} \binom{n}{i} 2^n$. Indeed, in this case one can assume that both families are monotone decreasing and then apply Kleitman's correlation inequality [13] (see also [5, p. 149]). Our main theorem shows that when n is large the same bound holds for any L of size s.

Theorem 1.1. *If $|L| = s$ then $P_L(n) \leqslant \sum_{i=0}^{s-1} \binom{n}{i} 2^n$ for n sufficiently large, with equality only when $L = \{0, \ldots, s-1\}$.*

The problem has attracted particular attention in the case $|L| = 1$. Frankl and Rödl [8] showed that $P_\ell(n) \leqslant 2^{n-1}$ for $\ell \geqslant 1$ and $P_0(n) \leqslant 2^n$. (Note that here and in what follows we may find it more a pleasant notation to omit brace markers for sets when no ambiguity will arise.) For $\ell = 0$ and $\ell = 1$ these bounds are tight, but in general, the best known construction of an ℓ-cross-intersecting pair $(\mathcal{A}, \mathcal{B})$ is to take $\mathcal{A} = \{A\}$ to contain a single set of size $2\ell - 1$ and $\mathcal{B}$ to consist of all sets that meet A in ℓ points. In particular, this example shows that $P_\ell(n) \geqslant \binom{2\ell-1}{\ell} 2^{n-2\ell+1} = \binom{2\ell}{\ell} 2^{n-2\ell}$ for $n \geqslant 2\ell - 1$, and this is conjectured to be tight by Ahlswede et al. [1]. We will prove this when $\ell = 2$.

Theorem 1.2. $P_2(n) = 3 \cdot 2^{n-3}$ *for $n \geqslant 3$.*

The next natural case to attempt to find tight bounds is when $|L| = 2$. Sgall observed that in the case when $L = \{\ell, \ell + t\}$ with $t > 1$ we have L mod $t = \{\ell\}$, and his result gives $P_L(n) \leqslant 2^n$. So we only need to consider the case $L = \{\ell, \ell + 1\}$. Our Theorem 1.1 gives an upper bound of $(n + 1)2^n$ for large n, which is tight for $\ell = 0$, but for larger ℓ one might expect better bounds. The best known construction of an $\{\ell, \ell + 1\}$-cross-intersecting pair, due to Sgall, is to take $\mathcal{A}$ to contain a fixed set A and all of its one-element extensions $\{A \cup \{x\}: x \in [n] \setminus A\}$, and $\mathcal{B}$ to consist of all sets that meet A in ℓ points: taking $|A| = 2\ell - 1$ shows that $P_{\{\ell,\ell+1\}}(n) = \Omega(n 2^n / \sqrt{\ell})$. When $\ell = 1$ we can take $|A| = 1$ to get $P_{\{1,2\}}(n) \geqslant n \cdot 2^{n-1}$. We show that this bound is tight.

Theorem 1.3. *$P_{\{1,2\}}(n) = n \cdot 2^{n-1}$ for $n \geqslant 1$.*

2. Key lemmas

In this section we assemble some lemmas that will be used for the proofs in the following section. Given a set system $\mathcal{F}$ on $[n]$ and $i \in [n]$ we define two systems on $[n] \setminus \{i\}$ by $\mathcal{F}_i = \{F \setminus \{i\}\colon i \in F \in \mathcal{F}\}$ and $\mathcal{F}_{\bar{i}} = \{F\colon i \notin F \in \mathcal{F}\}$. Note that $|\mathcal{F}| = |\mathcal{F}_i| + |\mathcal{F}_{\bar{i}}|$. Call $\mathcal{F}$ reducible if there is some i for which $|\mathcal{F}_{\bar{i}}| \geqslant |\mathcal{F}_i|$. Say that the pair $(\mathcal{A}, \mathcal{B})$ is reducible if one of $\mathcal{A}$ or $\mathcal{B}$ is reducible.

Lemma 2.1. *Suppose that pair $(\mathcal{A}, \mathcal{B})$ is L-cross-intersecting on $[n]$ and $\ell = \max L$. If $(\mathcal{A}, \mathcal{B})$ is not reducible then every set in $\mathcal{A}$ and $\mathcal{B}$ has size at most $2\ell - 1$, and $n \leqslant 4\ell - 3$.*

Proof. Suppose neither $\mathcal{A}$ nor $\mathcal{B}$ is reducible, i.e., for every i we have $|\mathcal{A}_i| > |\mathcal{A}|/2$ and $|\mathcal{B}_i| > |\mathcal{B}|/2$. Consider $A \in \mathcal{A}$. Since $|A \cap B| \leqslant \ell$ for every $B \in \mathcal{B}$ we have

$$\ell|\mathcal{B}| \geqslant \sum_{B \in \mathcal{B}} |A \cap B| = \sum_{i \in A} |\mathcal{B}_i| > |A||\mathcal{B}|/2.$$

This implies that $|A| < 2\ell$ for every $A \in \mathcal{A}$ and by symmetry the same bound holds for $\mathcal{B}$. Now

$$(2\ell - 1)|\mathcal{A}| \geqslant \sum_{A \in \mathcal{A}} |A| = \sum_{i=1}^{n} |\mathcal{A}_i| > n|\mathcal{A}|/2,$$

so $n < 4\ell - 2$. This proves the lemma. □

Next we need a lemma of Sgall [15, Lemma 3.1]. Since the proof is very short we reproduce it for the convenience of the reader.

Lemma 2.2. *If nonnegative numbers x, x', y, y', m, M satisfy $xy \leqslant m$, $x'y \leqslant M$, $xy' \leqslant M$, $x'y' \leqslant M$ then $(x + x')(y + y') \leqslant 2(M + m)$.*

Proof. The statement is trivial if either of x or y is zero. Otherwise let $X = x'/x$, $Y = y'/y$ and $Z = M/(xy)$. By our assumptions X, Y and XY are all nonnegative and at most Z. Also, we have $X + Y \leqslant 1 + Z$. To see this, note that it is immediate if $X \leqslant 1$, as we have $Y \leqslant Z$. Now if $X \geqslant 1$, since $X + Y \leqslant X + Z/X$, which is a concave function of X, the maximum occurs at an extreme value $X = 1$ or $X = Z$, so we see that $X + Y \leqslant 1 + Z$. Therefore

$$\begin{aligned}(x + x')(y + y') &= (1 + X)(1 + Y)xy = \bigl(1 + (X + Y) + XY\bigr)xy \\ &\leqslant \bigl(1 + (1 + Z) + Z\bigr)xy \leqslant 2(M + m). \qquad \square\end{aligned}$$

Our final lemma provides the induction step for all our proofs. We use a similar approach to Sgall [15], but obtain a stronger conclusion by assuming reducibility.

Lemma 2.3. *Let $(\mathcal{A}, \mathcal{B})$ be an L-cross-intersecting reducible pair on $[n]$ and let L' be the proper subset of L such that $\ell \in L'$ iff both $\ell, \ell + 1$ are in L. Then*

$$|\mathcal{A}||\mathcal{B}| \leqslant 2\bigl(P_L(n-1) + P_{L'}(n-1)\bigr).$$

Proof. Since the pair $(\mathcal{A}, \mathcal{B})$ is reducible we suppose without loss of generality that $|\mathcal{A}_{\bar{n}}| \geqslant |\mathcal{A}_n|$. Write

$$|\mathcal{A}||\mathcal{B}| = \bigl(|\mathcal{A}_n| + |\mathcal{A}_{\bar{n}}|\bigr)\bigl(|\mathcal{B}_n \cup \mathcal{B}_{\bar{n}}| + |\mathcal{B}_n \cap \mathcal{B}_{\bar{n}}|\bigr).$$

Clearly $(\mathcal{A}_{\bar{n}}, \mathcal{B}_n \cup \mathcal{B}_{\bar{n}})$ and $(\mathcal{A}_{\bar{n}}, \mathcal{B}_n \cap \mathcal{B}_{\bar{n}})$ are L-cross-intersecting on $[n-1]$, so $|\mathcal{A}_{\bar{n}}||\mathcal{B}_n \cup \mathcal{B}_{\bar{n}}| \leqslant P_L(n-1)$ and $|\mathcal{A}_{\bar{n}}||\mathcal{B}_n \cap \mathcal{B}_{\bar{n}}| \leqslant P_L(n-1)$. By reducibility we also have that

$$|\mathcal{A}_n||\mathcal{B}_n \cup \mathcal{B}_{\bar{n}}| \leqslant |\mathcal{A}_{\bar{n}}||\mathcal{B}_n \cup \mathcal{B}_{\bar{n}}| \leqslant P_L(n-1).$$

Finally, note that $(\mathcal{A}_n, \mathcal{B}_n \cap \mathcal{B}_{\bar{n}})$ is L'-cross-intersecting. Indeed, given $A \in \mathcal{A}_n$ and $B \in \mathcal{B}_n \cap \mathcal{B}_{\bar{n}}$ we have, by definition, $A \cup \{n\} \in \mathcal{A}$ and $B, B \cup \{n\} \in \mathcal{B}$. Therefore

$$\bigl|\bigl(A \cup \{n\}\bigr) \cap B\bigr| = |A \cap B| \quad \text{and} \quad \bigl|\bigl(A \cup \{n\}\bigr) \cap \bigl(B \cup \{n\}\bigr)\bigr| = |A \cap B| + 1$$

both belong to L. This implies $|\mathcal{A}_n||\mathcal{B}_n \cap \mathcal{B}_{\bar{n}}| \leqslant P_{L'}(n-1)$, and we can apply Lemma 2.2 with $x = |\mathcal{A}_n|$, $y = |\mathcal{B}_n \cap \mathcal{B}_{\bar{n}}|$, $x' = |\mathcal{A}_{\bar{n}}|$, $y' = |\mathcal{B}_n \cup \mathcal{B}_{\bar{n}}|$, $m = P_{L'}(n-1)$ and $M = P_L(n-1)$ to complete the proof. □

3. Proofs of theorems

We start by proving Theorem 1.1, which states that if $|L| = s$ and n is sufficiently large as a function of L then $P_L(n) \leqslant \sum_{i=0}^{s-1} \binom{n}{i} 2^n$, with equality only when $L = \{0, \ldots, s-1\}$. In the case when $L = \{0, \ldots, s-1\}$ this was proved by Sgall, and the other cases are covered by the following theorem.

Theorem 3.1. *If L is a fixed set of s integers such that $L \neq \{0, \ldots, s-1\}$ then*

$$P_L(n) \leqslant \bigl(1 + o(1)\bigr)\binom{n}{s-1} 2^{n-1}.$$

Proof. We argue by induction on s, the case $s = 1$ holding by a result of Frankl and Rödl mentioned in the introduction. Suppose $\epsilon > 0$ is given. Let L' be the subset of L such that $\ell \in L'$ iff both $\ell, \ell + 1$ are in L. Clearly $|L'| \leqslant |L| - 1 = s - 1$ and since $L \neq \{0, \ldots, s-1\}$ we have that $L' \neq \{0, \ldots, s-2\}$. Thus, by induction hypothesis, we can choose n_0 so that $P_{L'}(n) \leqslant (1 + \epsilon/2)\binom{n}{s-2} 2^{n-1}$ for $n \geqslant n_0$. We also choose $n_0 \geqslant 4(\max L) - 3$, so that by Lemmas 2.1 and 2.3 we have $P_L(n) \leqslant 2(P_L(n-1) + P_{L'}(n-1))$ for $n > n_0$. Repeatedly applying this inequality gives

$$P_L(n) \leqslant 2^{n-n_0} P_L(n_0) + \sum_{i=n_0}^{n-1} 2^{n-i} P_{L'}(i). \tag{1}$$

By choice of n_0 we have $P_{L'}(i) \leqslant (1 + \epsilon/2)\binom{i}{s-2} 2^{i-1}$ for all $i \geqslant n_0$ and we can use the trivial estimate $P_L(n_0) \leqslant 2^{2n_0}$ to conclude that

$$\begin{aligned}
P_L(n) &\leqslant 2^{n-n_0} P_L(n_0) + \sum_{i=n_0}^{n-1} 2^{n-i} P_{L'}(i) \leqslant 2^{n+n_0} + (1 + \epsilon/2) 2^{n-1} \sum_{i=n_0}^{n-1} \binom{i}{s-2} \\
&= 2^{n+n_0} + (1 + \epsilon/2) 2^{n-1} \left(\binom{n}{s-1} - \binom{n_0}{s-1}\right) \\
&\leqslant 2^{n-1} \left(2^{n_0+1} + (1 + \epsilon/2)\binom{n}{s-1}\right).
\end{aligned}$$

By choosing n_1 large enough so $\epsilon\binom{n_1}{s-1} \geqslant 2^{n_0+2}$ we obtain that $P_L(n) \leqslant (1+\epsilon)\binom{n}{s-1}2^{n-1}$ for $n \geqslant n_1$. Since ϵ can be made arbitrarily small we have $P_L(n) \leqslant (1+o(1))\binom{n}{s-1}2^{n-1}$, as required. □

Remark. Note that the value of n for which Theorem 1.1 holds can be made very reasonable by feeding in Sgall's bounds to the above proof, rather than the trivial bound $P_L(n_0) \leqslant 2^{2n_0}$. Consider for example the case $|L| = 2$. Sgall [15] showed that $P_L(n) \leqslant (2n-1)2^n$, and in fact $P_L(n) \leqslant 2^n$ unless $L = \{\ell, \ell+1\}$ for some ℓ, where we can suppose $\ell > 0$. Then $L' = \{\ell\}$, $P_\ell(i) \leqslant 2^{i-1}$ and by taking $n_0 = 4\ell + 1$ in (1) we obtain

$$P_{\{\ell,\ell+1\}}(n) \leqslant 2^{n-n_0} P_{\{\ell,\ell+1\}}(n_0) + \sum_{i=n_0}^{n-1} 2^{n-i} P_\ell(i) \leqslant 2^{n-n_0}(2n_0-1)2^{n_0} + (n-n_0)2^{n-1}$$
$$= (n+3n_0-2)2^{n-1} = (n+12\ell+1)2^{n-1}.$$

This is smaller than $(n+1)2^n$ for $n \geqslant 12\ell$.

Next we prove Theorem 1.2, which states that $P_2(n) = 3 \cdot 2^{n-3}$ for $n \geqslant 3$.

Proof of Theorem 1.2. Suppose $(\mathcal{A}, \mathcal{B})$ is a 2-cross-intersecting pair on $[n]$ with $n \geqslant 3$. We will show that $|\mathcal{A}||\mathcal{B}| \leqslant 3 \cdot 2^{n-3}$ by induction on n. First suppose that $n = 3$. Then at most one of $\mathcal{A}$ and $\mathcal{B}$ can contain the set 123. If $\mathcal{A}$ and $\mathcal{B}$ both contain a set of size 2 then they must contain the same one, and no others. So it is clear that the maximum value of $|\mathcal{A}||\mathcal{B}| = 3$ is achieved when, say, $\mathcal{A} = \{123\}$ and $\mathcal{B} = \{12, 13, 23\}$.

For general n, if $(\mathcal{A}, \mathcal{B})$ is reducible we have $|\mathcal{A}||\mathcal{B}| \leqslant 2P_2(n-1) \leqslant 3 \cdot 2^{n-3}$ by Lemma 2.3 and induction hypothesis. So we can assume that $(\mathcal{A}, \mathcal{B})$ is not reducible. By Lemma 2.1 we have $n = 4$ or $n = 5$ and every set in $\mathcal{A} \cup \mathcal{B}$ has size 2 or 3.

Consider any $A \in \mathcal{A}$ and let $\bar{A} = [n] \setminus A$. For every $B \in \mathcal{B}$ we have $|B \cap \bar{A}| = |B| - |A \cap B| = |B| - 2 \leqslant 1$. Now $\mathcal{B}$ is not reducible, i.e., $|\mathcal{B}_i| < |\mathcal{B}|/2$ for every i, so

$$|\mathcal{B}| \geqslant \sum_{B \in \mathcal{B}} |B \cap \bar{A}| = \sum_{i \in \bar{A}} |\mathcal{B}_i| > |\bar{A}||\mathcal{B}|/2.$$

Therefore $|\bar{A}| < 2$ and similarly $|\bar{B}| < 2$ for all $B \in \mathcal{B}$. It follows that we cannot have $n = 5$, so $n = 4$ and every set in $\mathcal{A} \cup \mathcal{B}$ has size 3. Clearly $\mathcal{A}$ and $\mathcal{B}$ cannot have the same set of size 3, so $|\mathcal{A}||\mathcal{B}| \leqslant 2 \cdot 2 = 4 < 6$, and the bound holds for $n = 4$. This proves the theorem. □

Finally we indicate a proof of Theorem 1.3, which states that $P_{\{1,2\}}(n) = n \cdot 2^{n-1}$ for $n \geqslant 1$. The argument is very similar to that in the previous theorem, although the analysis of the base cases requires rather more work, which we will not give in full detail.

Sketch proof of Theorem 1.3. Suppose $(\mathcal{A}, \mathcal{B})$ is a $\{1, 2\}$-cross-intersecting pair on $[n]$ with $n \geqslant 1$. Assume that we know the result for $n - 1$. Then if $(\mathcal{A}, \mathcal{B})$ is reducible, using Lemma 2.3, the induction hypothesis and the Frankl–Rödl bound $P_1(n-1) \leqslant 2^{n-2}$, we have

$$|\mathcal{A}||\mathcal{B}| \leqslant 2\big(P_{\{1,2\}}(n-1) + P_1(n-1)\big) \leqslant 2\big((n-1)2^{n-2} + 2^{n-2}\big) = n \cdot 2^{n-1},$$

as required. Thus it suffices to consider the nonreducible cases, when $n \leqslant 5$.

We will just give the proof that when $n = 5$ and $(\mathcal{A}, \mathcal{B})$ is not reducible then $|\mathcal{A}||\mathcal{B}| \leqslant 5 \cdot 2^4 = 80$. The remaining (easier) cases $1 \leqslant n \leqslant 4$ are left to the reader. First note that every set in

$\mathcal{A} \cup \mathcal{B}$ has size 2 or 3: no larger by Lemma 2.1, and no smaller since if $\{x\} \in \mathcal{A}$ then $x \in B$ for every $B \in \mathcal{B}$ and each such B contains at most 2 of the remaining 4 points, and the usual double counting argument shows that $\mathcal{B}$ is reducible. Also, each of $\mathcal{A}$ and $\mathcal{B}$ contains more sets of size 3 than it does of size 2, otherwise they would be reducible.

Now no triple can appear in both $\mathcal{A}$ and $\mathcal{B}$. Further, we can assume that all 10 triples appear in $\mathcal{A} \cup \mathcal{B}$. Otherwise, between them they contain at most 9 triples and at most 7 pairs (since the number of triples is larger than the number of pairs in each), so that even if they contain 8 sets each we have $|\mathcal{A}||\mathcal{B}| \leqslant 8 \cdot 8 < 80$. A closer inspection of this argument reveals that the only potential way to get $|\mathcal{A}||\mathcal{B}| > 80$ is when each of $\mathcal{A}$ and $\mathcal{B}$ contain 5 triples and 4 pairs. To dispose of this case, note that since $\mathcal{A}$ contains 4 pairs on 5 points there must be two with a common point y. Now it is easy to see that at least 3 pairs in $\mathcal{B}$ must contain y. Then the pairs in $\mathcal{A}$ are exactly those containing y, and so the same is true of the pairs in $\mathcal{B}$. Finally one of $\mathcal{A}$ or $\mathcal{B}$ has a triple T that does not contain y, and then the other contains the pair that is the complement of T, contradiction. □

4. Concluding remarks

It is likely that our method can be used to verify the conjectures of Ahlswede, Cai and Zhang and of Sgall for other small values of ℓ, although the necessary case analysis for small n will quickly become prohibitive without further ideas.

The modular version of the problem merits further study. Sgall was able to obtain a bound of the right order of magnitude under the weaker assumption that every cross-intersection size belongs to L mod m, for any $m \geqslant 2$. It is particularly interesting that m need not necessarily be a prime or prime-power, as for the basic restricted intersection problem, the bounds go from polynomial when the modulus is a prime or prime-power case, to super-polynomial in the general case (see [10]). In fact, Sgall makes a general conjecture that, if true, would imply that the bound we obtain in Theorem 1.1 is also valid mod m, for any $m \geqslant 2$. This, in turn, would give a simple proof for the famous result of Frankl and Rödl mentioned earlier. Unfortunately, our methods do not seem to be applicable to the modular version.

Acknowledgments

We thank the anonymous referees for some helpful comments.

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