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THE ZERO FORCING NUMBER OF GRAPHS*

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Abstract. A subset S of initially infected vertices of a graph G is called zero forcing if we can infect the entire graph by iteratively applying the following process. At each step, any infected vertex which has a unique uninfected neighbor, infects this neighbor. The zero forcing number of G is the minimum cardinality of a zero forcing set in G. We study the zero forcing number of various classes of graphs, including graphs of large girth, H-free graphs for a fixed bipartite graph H, and random and pseudorandom graphs.

Key words. zero forcing sets, random graphs, graphs with large girth

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1. Introduction.

Let G be a simple, undirected graph on the vertex set V. The *zero forcing process* on G is defined as follows. Initially, there is a subset S of black vertices, while all other vertices are said to be white. At each time step, a black vertex with exactly one white neighbor will *force* its white neighbor to become black. The set S is said to be a *zero forcing set* if, by iteratively applying the forcing step, all of V becomes black. The *zero forcing number* of G is the minimum cardinality of a zero forcing set in G, denoted by $Z(G)$. Note that given an initial set of black vertices, the set of black vertices obtained by applying the forcing rule until no more changes are possible is unique. We will often use the adjective “forcing” instead of “zero forcing.”

The forcing process is an instance of a propagation process on graphs (in particular, it is a cellular automaton). Such processes are a common topic across mathematics and computer science (see, e.g., [15, 9, 31, 20]). In other fields (statistical mechanics [14], physics [7], social network analysis [26]), diverse graph processes are used to model technical or societal processes. For an overview of the different models and applications, refer to the book [10].

The zero forcing process was proposed in [12] and used in [11] as a criterion for quantum controllability of a system. Independently, the zero forcing number was introduced in [2] as a bound for the minimum rank or, equivalently, the maximum nullity of a graph G. Given an n-vertex graph G, let $M(G)$ denote the maximum nullity over all symmetric real-valued matrices A whose zero-nonzero pattern of the off-diagonal entries is described by the graph G. This means that for $i \neq j$, the entry A_{ij} is nonzero if and only if ij is an edge in G, whereas the diagonal entries are chosen freely. The minimum rank of G is $n - M(G)$. This parameter has been extensively studied in the last fifteen years, largely due to its connection to inverse eigenvalue problems for graphs, singular graphs, biclique partitions, and other problems. Among several tools introduced to study the minimum rank, the zero forcing number has the advantage that its definition is purely combinatorial. In [2], it was shown that

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$Z(G) \geqslant M(G)$ for all graphs G. To see this, suppose that A is a matrix whose off-diagonal zero pattern is described by G, S_0 is a zero forcing set in G, and S_{i+1} is the set obtained by applying the forcing rule to S_i for $i \geqslant 0$. If the cardinality $|S_0|$ is smaller than the nullity of A, we can construct a vector $x \neq 0$ such that $Ax = 0$ and $x|_{S_0} = 0$. But then looking at the entries of $Ax = 0$, we can iteratively deduce that $x|_{S_i} = 0$ for all i. As S_0 is zero forcing, we eventually get $x = 0$, contradicting the choice of x. The minimum rank and forcing number of some specific families of graphs have also been computed in [2]. As a simple example, the complete graph K_n on n vertices has $Z(K_n) = M(K_n) = n - 1$, whereas the n-vertex path P_n has $Z(P_n) = M(P_n) = 1$. More results on this topic can be found in [1] and [19].

Recently, there has been a lot of interest in studying the forcing number of graphs for its own sake, and its relation to other graph parameters, such as the path cover number [23], connected domination number [6], and the chromatic number [34]. Among others, [13] and [24] contain upper bounds on the zero forcing number of a graph in terms of its degrees. It is easy to see that a trivial lower bound on the zero forcing number of a graph is $Z(G) \geqslant \delta - 1$, where δ is the minimum degree of G. This bound cannot be improved without additional assumptions. For, a graph G_δ which consists of r cliques $K_1, K_2, \ldots, K_r$ of order δ and a matching between K_i and K_{i+1} for $i = 1, \ldots, r - 1$ has minimum degree δ and $Z(G_\delta) = \delta$. However, when G has girth $g \geqslant 3$ and minimum degree $\delta \geqslant 2$, Davila, Kalinowski, and Stephen [16] showed that $Z(G) \geqslant \delta + (\delta - 2)(g - 3)$, confirming the earlier conjecture from [17]. The *girth* is the length of the shortest cycle in a graph. Our first result substantially improves on this bound, with the exception of very small values of δ and $g = 3, 4$.

THEOREM 1.1. *Let G be a graph of girth g with minimum degree δ.*

(i) *If $g = 2k + 1$ for $k \in \mathbb{N}$, then* $Z(G) \geqslant e^{-1}\left(\dfrac{\delta^k}{k+1} - \delta^{k-1}\right)$.

(ii) *If $g = 2k + 2$ for $k \in \mathbb{N}$, then* $Z(G) \geqslant 2e^{-1}\left(\dfrac{\delta^k}{k+1} - \delta^{k-1}\right)$.

The crucial ingredient of the proof is an upper bound on the density of a graph which contains no cycles $C_3, C_4, \ldots, C_{g-1}$. This is an instance of the so-called Turán problem (see, e.g., [22]). Given a graph H, we define the Turán number $\mathrm{ex}(n, H)$ to be the maximum number of edges $e(G)$ over all the n-vertex graphs G not containing a subgraph isomorphic to H. In general, if a graph G does not contain H as a subgraph, we refer to it as *H-free*. The Turán numbers of graphs have been extensively studied, and the asymptotic value of $\mathrm{ex}(n, H)$ is known for all nonbipartite graphs H as a consequence of the Erdős–Stone–Simonovits theorem [18]. Denote the complete bipartite graph with vertex classes of order a and b by $K_{a,b}$. A celebrated theorem of Kövari, Sós, and Turán [32] says that for $a \leqslant b$, $\mathrm{ex}(n, K_{a,b}) = O(n^{2-1/a})$. This implies that for every bipartite graph H, there exists $c = c(H) < 1$ such that $\mathrm{ex}(n, H) = O(n^{1+c})$. Using our approach based on the Turán numbers, we can extend Theorem 1.1 to H-free graphs G, improving the trivial bound $Z(G) \geqslant \delta - 1$ by a power of δ whenever H is bipartite.

THEOREM 1.2. *Fix a graph H, and let $c = c(H)$, $\beta = \beta(H)$, and $\eta = \eta(H)$ be positive constants such that $\mathrm{ex}(n, H) \leqslant \beta n^{1+c}$ whenever $n \geqslant \eta$. If G is an H-free graph of minimum degree $\delta \geqslant 2\eta$, then*

$$Z(G) \geqslant 2^{-1-2/c}\left(\frac{\delta}{\beta}\right)^{1/c}.$$

We remark that in this theorem it is not assumed that H is bipartite, and since $\mathrm{ex}(n,H) \leqslant \binom{n}{2}$ for all n and H, it is always possible to choose constants c, β, and η with the required property. But only if H is bipartite can we take $c < 1$, and thus obtain a lower bound on $Z(G)$ which improves on the trivial bound.

The authors of [2] report that somewhat surprisingly, $M(G) = Z(G)$ for many graphs for which $M(G)$ was known. Our next theorem shows that for most graphs, $M(G)$ and $Z(G)$ are actually far apart. We consider the random graph model $G_{n,p}$. This is an n-vertex graph in which every pair of vertices is adjacent randomly and independently with probability p. With an abuse of notation, we write $G_{n,p}$ for the sampled graph, as well as the underlying probability space. The model $G_{n,1/2}$ is particularly interesting since it assigns the same probability to all the $2^{\binom{n}{2}}$ graphs, thus allowing us to make statements about a typical graph. We say that an event in $G_{n,p}$ holds *with high probability* if its probability tends to 1 as n tends to infinity. The standard O-notation is used for the asymptotic behavior of the relative order of magnitude of two sequences, depending on a parameter $n \to \infty$. The logarithm to base e is denoted by log. Hall et al. [27] have shown that with high probability, the maximum nullity of a random graph $G_{n,1/2}$ lies between $0.49n$ and $0.86n$. On the other hand, we will show that the zero forcing number of a typical graph is almost as high as n.

THEOREM 1.3. *Let $p = p(n)$ satisfy $(\log^2 n)/\sqrt{n} \leqslant p \leqslant 2/3$, then with high probability*

$$Z\left(G_{n,p}\right) = n - \left(2 + \sqrt{2} + o(1)\right) \cdot \frac{\log(np)}{-\log(1-p)}.$$

In particular, for $p = 1/2$ we have $Z(G_{n,1/2}) = n - (2 + \sqrt{2} + o(1)) \log_2 n$, whereas for $p = o(1)$ the formula simplifies to $Z(G_{n,p}) = n - (2 + \sqrt{2} + o(1))p^{-1} \log(np)$.

There is a natural trend in probabilistic combinatorics to explore the possible extensions of results about random graphs to the pseudorandom setting. A graph is pseudorandom if its edge distribution resembles the one of $G_{n,p}$. There are several formal approaches to pseudorandomness. Here, we will use the one based on the spectral properties of the graph. The *adjacency matrix* of a graph $G = (V, E)$ with vertex set $V = [n]$ is an $n \times n$ matrix whose entry a_{ij} is 1 if $\{i, j\} \in E$, and 0 otherwise. The *eigenvalues of a graph* G are the eigenvalues of its adjacency matrix. An (n, d, λ) *graph* is a d-regular n-vertex graph in which all eigenvalues but the largest one are at most λ in absolute value. If G is an (n, d, λ) graph, its largest eigenvalue is $\lambda_1 = d$, and the difference $d - \lambda$ is called the *spectral gap*. It is well known (see, e.g., [33]) that the larger this gap is, the more closely the edge distribution of a regular graph G approaches that of the random graph with the corresponding edge density. We prove a theorem which provides spectral bounds on the zero forcing number of a graph. The lower bound is given in terms of the smallest eigenvalue of G, akin to the celebrated result of Hoffman on the independence number [28]. Note that $\lambda_{\min}$ is negative, and not necessarily minimal in absolute value as its sign is taken into account. The previously defined parameter λ is used for the upper bound.

THEOREM 1.4. *Let G be an (n, d, λ)-graph with smallest eigenvalue $\lambda_{\min}$. Then*

(i) $Z(G) \geqslant n\left(1 + \dfrac{2\lambda_{\min}}{d - \lambda_{\min}}\right)$, *and*

(ii) $Z(G) \leqslant n\left(1 - \dfrac{1}{2(d-\lambda)} \log\left(\dfrac{d-\lambda}{2\lambda+1}\right)\right)$.

The bound on $n-Z(G)$ implied by inequality (i) is tight, whereas the one implied by (ii) is tight up to a constant factor.

The rest of this paper is organized as follows. Section 2 contains results on H-free graphs with a forbidden bipartite graph H. In section 3, we asymptotically determine the zero forcing number of $G_{n,p}$. Section 4 contains bounds based on spectral properties of a graph.

2. Graphs with forbidden subgraphs. In this section, we bound the forcing number of graphs with a forbidden bipartite subgraph. We start by restating and proving Theorem 1.1. Namely, if G is a graph with girth g and minimum degree δ, then

(i) $Z(G) \geqslant e^{-1}\left(\dfrac{\delta^k}{k+1} - \delta^{k-1}\right)$ for $g = 2k+1, k \in \mathbb{N}$, and

(ii) $Z(G) \geqslant 2e^{-1}\left(\dfrac{\delta^k}{k+1} - \delta^{k-1}\right)$ for $g = 2k+2, k \in \mathbb{N}$.

Proof of Theorem 1.1. We will use a slightly weakened version of a result due to Alon, Hoory, and Linial [3]—a graph G_1 of girth g and average degree d satisfies

$$|V(G_1)| \geqslant \begin{cases} (d-1)^k, & g = 2k+1, \\ 2(d-1)^k, & g = 2k+2. \end{cases} \tag{1}$$

Let G be an n-vertex graph with girth $g = 2k+1$ and minimum degree δ. The proof for the case of even girth is the same. Let S be a zero forcing set in G of order s. It suffices to show that

$$(k+1)s \geqslant \left(\frac{k\delta}{k+1} - 1\right)^k. \tag{2}$$

Indeed, using the inequalities $(k/(k+1))^k \geqslant e^{-1}$ and $(1-\alpha)^k \geqslant 1 - k\alpha$ for $\alpha < 1, k \in \mathbb{N}$, the bound (2) implies that

$$\begin{aligned}(k+1)s &\geqslant \left(\frac{k\delta}{k+1} - 1\right)^k = \delta^k \left(\frac{k}{k+1}\right)^k \left(1 - \frac{k+1}{k\delta}\right)^k \\ &\geqslant \delta^k e^{-1}\left(1 - \frac{k(k+1)}{k\delta}\right) = e^{-1}\left(\delta^k - (k+1)\delta^{k-1}\right),\end{aligned}$$

which is the required result.

The proof of (2) splits into two cases. First, if $s > n/(k+1)$, we apply (1) to the entire graph G and obtain $(k+1)s > n \geqslant (\delta-1)^k > \left(k\delta/(k+1) - 1\right)^k$, so we are done.

Hence we assume that $s \leqslant n/(k+1)$. Starting from a set S of black vertices, we run an implementation of the zero forcing process, forcing the vertices of $V(G) \setminus S$ one by one to become black in an arbitrary order. To be more precise, in each step, we choose one specific black vertex u which has a unique white neighbor v. We say that u forces v and as a result, v becomes black. The process is stopped once we have reached a set of black vertices T with $|T| = (k+1)s$. Let U be the set containing all vertices u which forced some vertex of v during this implementation of the process. Then, since each vertex can force only one of its neighbors, $|U| \geqslant ks$. Moreover, by the forcing rule, all the edges with an endpoint in U lie inside T. Denoting the number of edges with both endpoints in T by $e(T)$, we have $e(T) \geqslant |U|\delta/2 \geqslant k|S|\delta/2$.

In other words, recalling that $|T| = (k+1)s$, the graph $G[T]$ has average degree at least $k\delta/(k+1)$. Applying (1) to the graph $G[T]$ gives precisely inequality (2), which completes the proof. ∎

It is worth mentioning that already for rather small values of δ, our lower bound exceeds the value $\delta + (\delta - 2)(g-3)$ conjectured in [17]. Even for girth five, by taking $k = 2$, we obtain $Z(G) \geqslant 1/3\,(2\delta/3 - 1)^2$, which implies the conjecture of Davila and Kenter for $\delta \geqslant 22$.

The previous approach will now be used to establish a bound which applies to H-free graphs for any bipartite graph H. We do not try to optimize the constants in this proof.

Proof of Theorem 1.2. It will be useful to rearrange our hypothesis on the Turán number of H, which is that for $n \geqslant \eta$, $\mathrm{ex}(n, H) \leqslant \beta n^{1+c}$, where η, β, and c are positive constants depending on H. Suppose that G_1 is an H-free graph with average degree $d \geqslant \eta$. In particular, G_1 has at least η vertices. Denoting $n_1 = |V(G_1)|$, the hypothesis gives $n_1 d/2 = e(G_1) \leqslant \beta n_1^{1+c}$, hence, $n_1 \geqslant (d/(2\beta))^{1/c}$. The proof reduces to the following claim, which we state formally because we will use it to prove Corollary 2.2.

Claim 2.1. *Suppose that any H-free graph G_1 with average degree $d \geqslant \eta$ satisfies*

$$|V(G_1)| \geqslant \left(\frac{d}{2\beta}\right)^{1/c} \tag{3}$$

with $c, \beta > 0$. Let G be an H-free graph with minimum degree $\delta \geqslant 2\eta$. Then

$$Z(G) \geqslant \frac{1}{2}\left(\frac{\delta}{4\beta}\right)^{1/c}.$$

To see this, let S be a zero forcing set in G of order s. Assume that $s < n/2$, since otherwise we can apply (3) to the entire graph G to get

$$s \geqslant \frac{n}{2} \geqslant \frac{1}{2}\left(\frac{\delta}{2\beta}\right)^{1/c} \geqslant \frac{1}{2}\left(\frac{\delta}{4\beta}\right)^{1/c}.$$

As in the previous proof, starting from S, we run the zero forcing process vertex by vertex until we have reached a set of black vertices T with $|T| = 2s$. Let U be the set containing all vertices u which forced some vertex of $T \setminus S$ during our process. Since each vertex can force only one of its neighbors, $|U| = s$. Moreover, all the edges with an endpoint in U lie inside T. Hence $e(T) \geqslant |U|\delta/2 = s\delta/2$, and we conclude that the average degree in $G[T]$ is at least $2(s\delta/2)/|T| = \delta/2$. Now we can apply (3) to $G[T]$, and obtain

$$|T| = 2s \geqslant \left(\frac{\delta}{4\beta}\right)^{1/c},$$

as required. ∎

Recall that $K_{a,b}$ denotes the complete bipartite graph with parts of order a and b. Using the well-known result of [32], we give explicit bounds for $K_{a,b}$-free graphs.

Corollary 2.2. *Let G be a $K_{a,b}$-free graph with minimum degree $\delta \geqslant 4a - 4$. Then*

$$Z(G) \geqslant \frac{1}{2}\left(\frac{\delta}{4(b-1)^{1/a}}\right)^{a/(a-1)}.$$

Proof. We use a statement of the aforementioned Kövari–Sós–Turán theorem proved in [22, Theorem 2.22]. The original proof in [32] was for the case $a = b$, but it immediately implies that for an n-vertex $K_{a,b}$-free graph with average degree d,

$$d \leqslant (b-1)^{1/a} n^{1-1/a} + (a-1). \tag{4}$$

It follows that $H = K_{a,b}$ satisfies the hypothesis of Claim 2.1 with $c = (a-1)/a$, $\beta = (b-1)^{1/a}$, and $\eta = 2a-2$. Let G_1 be an n-vertex $K_{a,b}$-free graph with average degree $d \geqslant 2a-2$. This implies that the first summand in the right-hand side of (4) is at least $a-1$ and, therefore,

$$d \leqslant (b-1)^{1/a} n^{1-1/a} + (a-1) \leqslant 2(b-1)^{1/a} n^{1-1/a}.$$

Rearranging, we get

$$n \geqslant \left(\frac{d}{2(b-1)^{1/a}} \right)^{a/(a-1)}.$$

Our bound on the zero forcing number of $K_{a,b}$-free graphs follows from Claim 2.1. ∎

Since for $b > (a-1)!$ and a constant c_a, there are constructions of $K_{a,b}$-free graphs on only $c_a \delta^{a/(a-1)}$ vertices with minimum degree δ (see, e.g., [4]), the result of Corollary 2.2 is tight up to a constant factor.

3. The random graph. Here we prove that for $(\log^2 n)/\sqrt{n} \leqslant p = o(1)$, with high probability

$$Z\left(G_{n,p}\right) = n - \left(2 + \sqrt{2} + o(1)\right) \cdot \frac{\log(np)}{p}.$$

For $p = o(1)$, this establishes the bound of Theorem 1.3 as $-\log(1-p) = (1+o(1))p$. Restricting to $p = o(1)$ in this section keeps the calculations clearer. The case of constant p is easier (as we explain below) and can be proved similarly. In what follows, we mostly omit floor and ceiling signs for the sake of clarity of presentation. The logarithms are in base e unless stated otherwise. All the inequalities will hold for large enough n. In a graph G, we denote the set of edges between vertex sets S and T by $E(S,T)$. We write $u \sim v$ if the vertices u and v are adjacent in G, and $u \nsim v$ otherwise.

Our approach combines the first and the second moment method, and is somewhat similar to the argument used to determine the independence number of $G_{n,p}$ (see, e.g., [29, Chapter 7, section 1]). However, the proof contains some delicate points, which we try to explain before giving the formal argument. We start this discussion by considering the special case, $G_{n,1/2}$, in which the constant edge density allows for a simple proof.

The zero forcing number of the random graph is governed by the occurrence of a specific substructure called a witness. In a graph G on the vertex set V, a k-witness (or a witness of order k) is a pair of ordered vertex k-tuples $\big((s_i)_{i\in[k]}, (t_i)_{i\in[k]}\big)$ such that $s_i, t_i \in V$, $s_i \sim t_i$ for each i, and $s_i \nsim t_j$ for $i < j$. The definition requires $s_i \neq s_j$ and $t_i \neq t_j$ for $i \neq j$, but it might happen that $s_i = t_j$ for some $i > j$. The adjacency matrix of a k-witness, where the rows and columns are indexed by (s_i) and (t_i), respectively, can be found in Figure 1(a). For any pair of k-tuples $\big((s_i)_{i\in[k]}, (t_i)_{i\in[k]}\big)$, we define the set of *superdiagonal* pairs to be $\{(s_i, t_j) : i, j \in [k], i < j\}$, and the set of

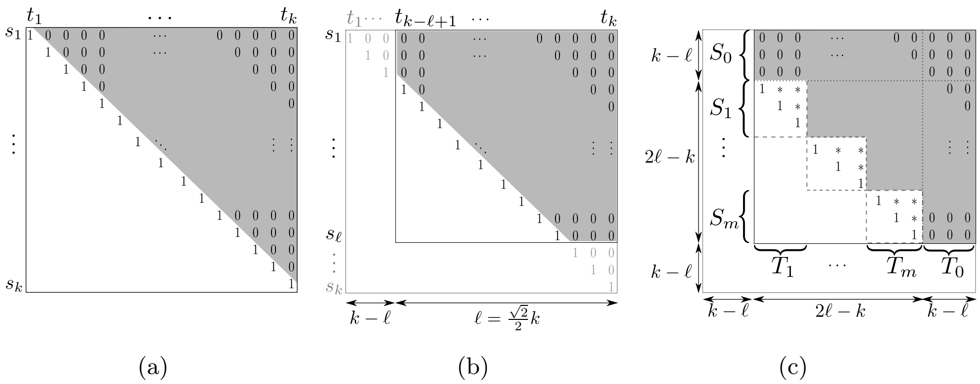

FIG. 1. *Adjacency matrices of a k-witness, k-subwitness, and a loose k-subwitness, respectively. The regions which are required to contain only zeros are shaded. In* (c)*, the stars mark the entries which are superdiagonal in this particular ordering of the vertices, but not required to contain zeros.*

diagonal pairs to be $\{(s_i, t_i) : i \in [k]\}$. It is easy to see that if G has a forcing set S of order at most $n-k$, then it contains a k-witness (Lemma 3.1). Therefore, our aim is to find the order of the largest witness in $G_{n,1/2}$. The first subtlety arises in trying to use the first moment computation to guess the forcing number of $G_{n,1/2}$. Let R_k be the random variable counting the k-witnesses in $G_{n,1/2}$. For fixed k-tuples (s_i) and (t_i), the probability that they form a witness is $2^{-\binom{k+1}{2}}$ since they determine $\binom{k+1}{2}$ superdiagonal and diagonal pairs. Using linearity of expectation, we can multiply this probability with the number of choices for (s_i) and (t_i) to obtain

$$\mathbb{E}\left[R_k\right] \leqslant \left(\frac{n!}{(n-k)!}\right)^2 \cdot 2^{-\binom{k+1}{2}} \leqslant n^{2k} 2^{-k^2/2} = \left(n^2 2^{-k/2}\right)^k .$$

For $k \geqslant (4+\epsilon)\log_2 n$, it holds that $\mathbb{E}\left[R_k\right] = o(1)$, so Markov's inequality implies that with high probability $G_{n,1/2}$ contains no such k-witness. On the other hand, if $k \leqslant (4-\epsilon)\log_2 n$ the expected number of k-witnesses tends to infinity with n, so it is reasonable to take $(4+o(1))\log_2 n$ as the first guess for the order of the largest witness. For example, in finding the independence number of $G_{n,p}$, the first moment computation gives the correct value throughout the range of p. However, in our case, the actual asymptotic order of the largest witness, $k_c := \left(2+\sqrt{2}\right)\log_2 n$, is smaller than $4\log_2 n$.

The reason for this is that a substructure of a k-witness $\left((s_i)_{i\in[k]}, (t_i)_{i\in[k]}\right)$ obtained by discarding a final segment of (s_i) and an initial segment of (t_i) has a lower expected number of copies inside $G_{n,1/2}$ than the witness itself and, therefore, gives a stronger bound on k. The discarded vertices contribute a factor of order n each to the number of choices for a potential k-witness, but pose few restrictions on the adjacencies. The correct asymptotic value k_c is obtained by taking $\ell = \ell(k) = \sqrt{2}k/2$ and counting substructures called k-subwitnesses. The adjacency matrix of a subwitness is depicted in Figure 1(b). For integers $a \leqslant b$, define the interval $[a \mathinner{..} b] = \{a, a+1, \ldots, b\}$. A *$k$-subwitness* in G is a pair of ℓ-tuples $\left((s'_i)_{i\in[\ell]}, (t'_i)_{i\in[k-\ell+1\mathinner{..}k]}\right)$ such that $s'_i, t'_j \in V$, $s'_i \sim t'_i$ for each $i \in [k-\ell+1 \mathinner{..} \ell]$, and $s'_i \nsim t'_j$ for $1 \leqslant i < j \leqslant k$. Clearly, if $\left((s_i)_{i\in[k]}, (t_i)_{i\in[k]}\right)$ is a witness in G, then the restrictions $(s_1, \ldots, s_\ell)$ and $(t_{k-\ell+1}, \ldots, t_k)$ form a subwitness.

Another piece of intuition on why we can discard those segments of (s_i) and (t_i) is that if $k = (1-\epsilon)k_c$, the number of discarded columns $t_1, \dots, t_{k-\ell}$ is $q = (1-\epsilon)\log_2 n$. Therefore, given a $(1-\epsilon)k_c$-subwitness in $G_{n,1/2}$, we can extend it to a $(1-\epsilon)k_c$-witness with high probability. Indeed, to find the remaining columns $t_1, \dots, t_q$, we only need to consider their adjacencies with $s_1, \dots, s_q$. But a short computation shows that for $q = (1-\epsilon)\log_2 n$, $G_{n,1/2}$ has the following extension property with high probability. For any $s_1, s_2, \dots, s_q$, there will be at least $n^{\epsilon/2}$ vertices in $G_{n,1/2}$ satisfying any given adjacency restriction with $s_1, \dots, s_q$. This extension property of $G_{n,1/2}$ allows us to find the missing columns $t_1, \dots, t_q$, as well as the rows $s_{\ell+1}, \dots, s_{\ell+q}$.

The argument of Lemma 3.2 for $p = 1/2$ would amount to counting k-subwitnesses for $k = (1+\epsilon)k_c = (1+\epsilon)\left(2+\sqrt{2}\right)\log_2 n$, and showing that with high probability, $G_{n,1/2}$ contains no such k-subwitness. This intuition gives the correct answer for $G_{n,1/2}$, but when $p = o(1)$ the computation of the expected number of k-subwitnesses contains another caveat. In a subwitness, $(s'_i)_{i\in[\ell]}$ and $(t'_i)_{i\in[k-\ell+1..k]}$ are orderings of the corresponding vertex sets. In the first moment computation, the ordering contributes a factor of $(\ell!)^2 = k^{2\ell+o(k)}$. This factor was negligible when $p = 1/2$ and $k_c = \Theta(\log n)$, but for $p = o(1)$, $G_{n,p}$ contains witnesses whose order is polynomial in n, so we need to be more careful. Next, we explain how to shave off a factor of $k^{k+o(k)}$. A subwitness is modified so that the adjacency matrix is invariant under reordering large subsets of the vertices, at the price of discarding a small number of its zero entries. Figure 1(c) illustrates this trade-off.

In a graph G, we define a *loose k-subwitness* (or just *loose subwitness*) to be a substructure labeled by sets $S_0, S_1, \dots S_m, T_0, T_1, \dots, T_m \subseteq V$ and bijections $f_i : S_i \to T_i$ for $i = 1, \dots, m$, where $m = (2\ell - k)p = \left(\sqrt{2}-1\right)kp$ and the following conditions are satisfied.

(i) The sets S_i are pairwise disjoint for $i = 0, \dots, m$, as well as the sets T_i. Denoting $S = \bigcup_{i=0}^m S_i$ and $T = \bigcup_{i=0}^m T_i$, we have $|S| = |T| = \ell$ and $|S_0| = |T_0| = k - \ell = \left(1 - \sqrt{2}/2\right)k$. The sets $S \backslash S_0$ and $T \backslash T_0$ are partitioned *equitably* into S_i, and T_i, that is, $0 \leqslant |S_i| - |S_j| \leqslant 1$ for $1 \leqslant i < j \leqslant m$. Therefore, $1/p - 1 \leqslant |S_i| = |T_i| \leqslant 1/p + 1$ and the orders of S_i are nonincreasing for $i \in [m]$.

(ii) For the edges between S and T, we require $E(T_0, S) = \emptyset$, and $E(S_i, T_j) = \emptyset$ whenever $0 \leqslant i < j \leqslant m$. For every $i \in [m]$ and $v \in S_i$, there is an edge in G between v and $f_i(v)$. In other words, for every $i \in [m]$, the bijection f_i determines a matching between S_i and T_i.

The key point is that any graph that contains a k-witness, also contains a loose k-subwitness, which we use to get an improved bound. Since the sets S_i and T_i are not ordered, but only paired by the bijections f_i for $i \in [m]$, we gain a factor of $k^{-k+o(k)}$ in the first moment computation. On the other hand, for $i \in [m]$, we do not care about the adjacency relation between S_i and T_i apart from the diagonal vertex pairs, but that costs us a much less significant factor $(1-p)^{|S_i|^2 m} = (1-p)^{(2\ell-k)/p}$. In words, the definition of a loose subwitness allows a large number of vertex permutations. However, since those permutations preserve most of the superdiagonal pairs, the adjacency matrix of a loose subwitness still contains most of the zeros which were previously required.

Estimating the expected number of loose subwitnesses in $G_{n,p}$, we can match the bound obtained from the second moment method (Lemma 3.3). This computation requires some additional understanding of how two witnesses can interact, but the ideas are explicit in the argument. We are now ready to present the formal proof. We first establish the relationship between the zero forcing number of a graph and the occurrence of a witness. We will be using the shortened notation $\mathbf{s} = (s_i)_{i\in[k]}$ and denote the image of this k-tuple by $\mathbf{s}[k] = \{s_i : i \in [k]\}$.

LEMMA 3.1. *Let G be an n-vertex graph and $k \in \mathbb{N}$. If $Z(G) \leqslant n - k$, then G contains a k-witness. Moreover, if G contains a k-witness $(\mathbf{s}, \mathbf{t})$ with $\mathbf{s}[k] \cap \mathbf{t}[k] = \emptyset$, then $Z(G) \leqslant n - k$.*

Proof. Assume that $Z(G) \leqslant n-k$, that is, G has a forcing set S with $n-|S| \geqslant k$. Index the vertices of $V(G) \setminus S$ according to the order in which they were forced, so t_1 is the first forced vertex, t_2 the second, and so on, up to t_k. For $1 \leqslant i \leqslant k$, let s_i be a vertex which forced t_i. Then by the definition of a zero forcing set, $\big((s_i)_{i\in[k]}, (t_i)_{i\in[k]}\big)$ is a witness.

Conversely, let $(\mathbf{s}, \mathbf{t})$ be a k-witness with $\mathbf{s}[k] \cap \mathbf{t}[k] = \emptyset$. Then $V(G) \setminus \mathbf{t}[k]$ is a forcing set in G, since the vertices $t_1, \dots, t_k$ can be forced by the vertices $s_1, \dots, s_k$, respectively. ∎

We now formalize the ideas outlined in the previous discussion.

LEMMA 3.2. *Let $C/n < p(n) < 1$ for a large constant C, and define*

$$k = \left(2+\sqrt{2}\right) p^{-1} \left(\log(np) + \log\log(np)\right).$$

With high probability, $G_{n,p}$ contains no k-witness, and therefore $Z\left(G_{n,p}\right) \geqslant n - k$.

Proof. For a graph G on the vertex set V, a k-witness and a loose k-subwitness have been defined in the previous section. We recall that

$$k = \left(2+\sqrt{2}\right) p^{-1} \left(\log(np) + \log\log(np)\right),\ \ \ell = \sqrt{2}k/2,$$

and set

$$r = 2\ell - k = \frac{\sqrt{2}}{p}\left(\log(np) + \log\log(np)\right), \quad m = pr = \sqrt{2}\left(\log(np) + \log\log(np)\right).$$

The crucial fact is that if G contains a witness $\big((s_i)_{i\in[k]}, (t_i)_{i\in[k]}\big)$, then a loose subwitness can be found as follows. S_0 consists of the first $k-\ell$ rows of the witness, $S_0 := \{s_1, \dots, s_{k-\ell}\}$, and T_0 of the last $k-\ell$ columns, $T_0 := \{t_{\ell+1}, \dots, t_k\}$. The sets $S_1, \dots, S_m$ are constructed by ordering the vertices $s_{k-\ell+1}, s_{k-\ell+2}, \dots, s_\ell$ and partitioning them into m equitable intervals. Naturally, $T_1, \dots, T_m$ are the corresponding columns, $T_j = \{t_i : s_i \in S_j\}$, and the bijections f_j map s_i to t_i for $i \in [k-\ell+1 \mathinner{..} \ell]$.

Let Y_k denote the number of loose k-subwitnesses in $G_{n,p}$. Our aim is to show $\mathbb{E}\left[Y_k\right] \to 0$. Fix the sets S_j, T_j and bijections f_j which satisfy (i). In particular, $|S_1 \cup \dots \cup S_m| = |T_1 \cup \dots \cup T_m| = 2\ell - k = r$ and $|S_i| = |T_i| \in \left[\frac{1}{p} - 1, \frac{1}{p} + 1\right]$ for $i \in [m]$. For S_j, T_j, and f_j to span a loose subwitness, i.e., for (ii) to be satisfied, we require $(\ell^2 - r^2) + (\binom{r}{2} - m \cdot \binom{r/m}{2})$ pairs to be nonedges in $G_{n,p}$. The first summand, $\ell^2 - r^2 = 2(k-\ell)\ell - (k-\ell)^2$, comes from $E(S_0, T) = E(T_0, S) = \emptyset$, and the second from $E(S_i, T_j) = \emptyset$ for $i < j$. The $m \cdot \binom{r/m}{2}$ pairs have been subtracted since we do not impose any restrictions on $E(S_i, T_i)$, and they will turn out to be negligible. Moreover, (ii) requires r diagonal edges to be present in $G_{n,p}$, so the probability that our fixed S_j, T_j and bijections f_j satisfy (ii) is at most

$$p^r (1-p)^{\ell^2 - r^2/2 - r^2/(2m)}.$$

Now we will take the union bound over all the potential loose subwitnesses. There are at most $\binom{n}{\ell-r}^2$ choices for S_0, T_0, and $\binom{n}{r/m}^m$ choices for the sets S_j, $j \in [m]$. Each vertex $v \in S_j$ gets assigned a vertex $f_j(v)$, which can be done in at most n^r ways. This assignment also determines the sets $T_j = f_j(S_j)$ for $j \in [m]$, and we get

$$\begin{aligned}\mathbb{E}[Y_k] &\leqslant \binom{n}{\ell-r}^2 \binom{n}{r/m}^m n^r p^r (1-p)^{\ell^2 - r^2/2 - r^2/(2m)} \\ &\leqslant \left(\frac{en}{\ell-r}\right)^{2\ell-2r} \left(\frac{enm}{r}\right)^r n^r p^r e^{-p\left(\ell^2 - r^2/2 - r^2/(2m)\right)}.\end{aligned}$$

We use the inequalities $\ell - r \geqslant k/4$ and $r \geqslant k/4$ to obtain

$$\begin{aligned}\mathbb{E}[Y_k] &\leqslant \left(\frac{4en}{k}\right)^{2\ell-2r} \left(\frac{4en}{k}\right)^r m^r n^r p^r e^{-p\left(\ell^2 - r^2/2 - r^2/(2m)\right)} \\ &\leqslant C_1^k (np)^{2\ell} (kp)^{-2\ell+r} m^r e^{-p\left(\ell^2 - r^2/2\right)},\end{aligned}$$

where $C_1 = 4e^2$. For the second inequality, we used $m = pr$, so that $e^{(pr^2)/(2m)} = e^{r/2} \leqslant e^k$. Finally, substituting $\ell = \sqrt{2}k/2$, $\ell^2 - r^2/2 = (\sqrt{2}-1)k^2$, and noting that $(kp)^{-2\ell+r} m^r \leqslant (kp)^{-2\ell+2r} < 1$, we obtain

$$\mathbb{E}[Y_k] < C_1^k \,(np)^{\sqrt{2}k}\, e^{-pk^2\left(\sqrt{2}-1\right)}.$$

Taking np sufficiently large and recalling that $\left(\sqrt{2}-1\right) pk = \sqrt{2}\,(\log(np) + \log\log(np))$, we get

$$\mathbb{E}[Y_k] \leqslant \left(C_1 (np)^{\sqrt{2}} e^{-\sqrt{2}(\log(np) + \log\log(np))}\right)^k < 2^{-k}.$$

Markov's inequality implies that with high probability, $G_{n,p}$ contains no loose k-subwitnesses, and therefore no k-witnesses. By Lemma 3.1, $Z(G) \geqslant n - k$ with high probability. ∎

For the so-called 1-statement of Theorem 1.3, we set $k = (1-\epsilon)k_c$ and show that with high probability, $G_{n,p}$ contains such a k-witness $(\mathbf{s}, \mathbf{t})$ with $\mathbf{s}[k] \cap \mathbf{t}[k] = \emptyset$ using a well-known consequence of Chebyshev's inequality. Let X_n be a sequence of nonnegative random variables indexed by some parameter n going to infinity. If $\mathbb{E}[X_n] \to \infty$ and $\mathrm{Var}[X_n] / (\mathbb{E}[X_n])^2 \to 0$, then with high probability $X_n > 0$. The proof can be found for example in [5, Corollary 4.3.2].

Lemma 3.3. *Let $p = p(n)$ satisfy $(\log^2 n)/\sqrt{n} < p = o(1)$ and $\epsilon > 0$. With high probability*

$$Z(G_{n,p}) \leqslant n - (1-\epsilon) \cdot \frac{\left(2+\sqrt{2}\right)\log(np)}{p}.$$

Proof. Partition the vertex set V of $G_{n,p}$ into V_1 and V_2 with $|V_1| = \lfloor n/2 \rfloor$ and $|V_2| = \lceil n/2 \rceil$. Fix $k = (1-\epsilon)\, p^{-1} \left(2+\sqrt{2}\right) \log(np)$. We say that a pair $(\mathbf{s}, \mathbf{t})$ (or the corresponding k-witness) is *divided* if $s_i \in V_1$ and $t_i \in V_2$ for $i \in [k]$. The set of all divided pairs is denoted by $\mathcal{D}$. We will show that with high probability, $G_{n,p}$ contains a divided k-witness. Let X_k denote the number of such k-witnesses in $G_{n,p}$. Furthermore, for a pair of k-tuples $(\mathbf{s}, \mathbf{t})$, we denote the event that $(\mathbf{s}, \mathbf{t})$ is a k-witness by $W_{\mathbf{s},\mathbf{t}}$. Let us first estimate the expectation of X_k. We will denote $n' = n/2$, and define the falling factorial power $(n')_k = n'!/(n'-k)!$. We crudely bound $(n')_k \geqslant (n'/2)^k$ for $k \leqslant n'/2$. The inequality $1-p \geqslant e^{-(p+p^2)} \geqslant e^{-1.1p}$ for $0 \leqslant p \leqslant 0.1$ is also used. We get

$$\mathbb{E}[X_k] = \sum_{(\mathbf{s},\mathbf{t})\in\mathcal{D}} \mathbb{P}[W_{\mathbf{s},\mathbf{t}}] = ((n')_k)^2 p^k (1-p)^{\binom{k}{2}} \geqslant \left(\frac{n}{4}\right)^{2k} p^k e^{-1.1pk^2/2}$$
$$\geqslant \left(\frac{1}{16} n^2 p e^{-1.1\left(1+\sqrt{2}/2\right)\log(np)}\right)^k > \left(\frac{1}{16} n^2 p \,(np)^{-1.9}\right)^k,$$

where in the second line, we used $pk \leqslant \left(2+\sqrt{2}\right)\log(np)$. It follows that $\mathbb{E}[X_k] \geqslant \left(n^{0.1}\right)^k \longrightarrow \infty$.

To use Chebyshev's inequality, we will need second moment estimates. Fix a specific divided pair $(\mathbf{s},\mathbf{t})$. The events $W_{\mathbf{s}',\mathbf{t}'}$ are symmetric over all the divided pairs $(\mathbf{s}',\mathbf{t}')$, so a standard computation (see, e.g., [5, section 4.3]) gives

$$\operatorname{Var}[X_k] \leqslant \mathbb{E}[X_k] \sum_{\substack{(\mathbf{s}',\mathbf{t}')\in\mathcal{D} \\ \mathbf{s}[k]\cap\mathbf{s}'[k]\neq\emptyset \\ \mathbf{t}[k]\cap\mathbf{t}'[k]\neq\emptyset}} \mathbb{P}[W_{\mathbf{s}',\mathbf{t}'} \mid W_{\mathbf{s},\mathbf{t}}].$$

We remark that as soon as $\mathbf{s}'[k]\cap\mathbf{s}[k]=\emptyset$ or $\mathbf{t}'[k]\cap\mathbf{t}[k]=\emptyset$, the events $W_{\mathbf{s},\mathbf{t}}$ and $W_{\mathbf{s}',\mathbf{t}'}$ are independent, so such pairs $(\mathbf{s}',\mathbf{t}')$ do not contribute to the second moment. Moreover, the sum includes the case $(\mathbf{s}',\mathbf{t}')=(\mathbf{s},\mathbf{t})$, so we do not need the additional summand $\mathbb{E}[X_k]$ which often appears in the formula. For any pair of k-tuples $(\mathbf{s}',\mathbf{t}')$, we have defined the set of superdiagonal pairs to be $\left\{\left(s_i',t_j'\right) : i,j\in[k], i<j\right\}$, and the set of diagonal pairs to be $\{(s_i',t_i') : i\in[k]\}$. We now partition the pairs $(\mathbf{s}',\mathbf{t}')$. Let $P_{a,b,d}$ denote the set of divided pairs $(\mathbf{s}',\mathbf{t}')$ such that

- $|\mathbf{s}[k]\cap\mathbf{s}'[k]|=a$, $|\mathbf{t}[k]\cap\mathbf{t}'[k]|=b$, and
- the number of vertex pairs which are diagonal in both $(\mathbf{s},\mathbf{t})$ and $(\mathbf{s}',\mathbf{t}')$ is d.

Moreover, we define the term $T_{a,b,d} = \mathbb{E}[X_k]^{-1}\sum_{(\mathbf{s}',\mathbf{t}')\in P_{a,b,d}} \mathbb{P}[W_{\mathbf{s}',\mathbf{t}'} \mid W_{\mathbf{s},\mathbf{t}}]$. If $d>a$ or $d>b$, $T_{a,b,d}=0$, so we let d run up to a for simplicity. Using this partition, our sum can be written as

$$\frac{\operatorname{Var}[X_k]}{(\mathbb{E}[X_k])^2} \leqslant \sum_{a,b=1}^{k}\sum_{d=0}^{a} T_{a,b,d}.$$

We now analyze the term $T_{a,b,d}$. We start by counting the divided pairs in $P_{a,b,d}$. There are at most $(n'-k)_{k-a}(n'-k)_{k-b}$ ways of selecting and indexing the vertices of $\mathbf{s}'[k]\setminus\mathbf{s}[k]$ and $\mathbf{t}'[k]\setminus\mathbf{t}[k]$. Then we select d distinct pairs (s_i,t_i) which are diagonal in $(\mathbf{s},\mathbf{t})$ and will also be diagonal in $(\mathbf{s}',\mathbf{t}')$, which can be done in at most $\binom{k}{d}$ ways. The number of ways to place those pairs into $(\mathbf{s}',\mathbf{t}')$, i.e., to choose the index j such that $s_j'=s_i$ and $t_j'=t_i$ is at most $(k)_d$. Similarly, we choose $a-d$ of the remaining vertices in $\mathbf{s}[k]$, and assign them any preimage under $\mathbf{s}'$, which gives an additional factor of $\binom{k-d}{a-d}(k-d)_{a-d}$. Finally, we do the same for $\mathbf{t}$ and $\mathbf{t}'$. Altogether,

(5)
$$|P_{a,b,d}| \leqslant (n'-k)_{k-a}(n'-k)_{k-b}\binom{k}{d}(k)_d\binom{k-d}{a-d}(k-d)_{a-d}\binom{k-d}{b-d}(k-d)_{b-d}$$
$$\leqslant (n'-k)_{k-a}(n'-k)_{k-b}\,k^{a+b-d}\binom{k}{d}\binom{k-d}{a-d}\binom{k-d}{b-d}.$$

To bound the probability of $(\mathbf{s}',\mathbf{t}')\in P_{a,b,d}$ being a k-witness, we first bound the number of pairs which are superdiagonal for both $(\mathbf{s},\mathbf{t})$ and $(\mathbf{s}',\mathbf{t}')$. We can even forget about how the overlap vertices are placed in $(\mathbf{s}',\mathbf{t}')$, that is, the number of common superdiagonal pairs is bounded above by $\Phi(a,b)$, where

$$\Phi(a,b) = \max_{A,B\subseteq[k],\,|A|=a,\,|B|=b} \left|\{(i,j)\in A\times B : i<j\}\right|.$$

Note that trivially $\Phi(a,b) \leqslant ab$. By definition of Φ, $\mathbb{P}\left[W_{\mathbf{s}',\mathbf{t}'} \mid W_{\mathbf{s},\mathbf{t}}\right] \leqslant p^{k-d}(1-p)^{\binom{k}{2}-\Phi(a,b)}$. Dividing by $\mathbb{E}\left[X_k\right] = \left((n')_k\right)^2 p^k(1-p)^{\binom{k}{2}}$, summing over $(\mathbf{s}',\mathbf{t}') \in P_{a,b,d}$, and using (5), we get

$$\begin{aligned} T_{a,b,d} &\leqslant \mathbb{E}\left[X_k\right]^{-1} |P_{a,b,d}| p^{k-d}(1-p)^{\binom{k}{2}-\Phi(a,b)} \\ &\leqslant \frac{(n'-k)_{k-a}}{(n')_k}\frac{(n'-k)_{k-b}}{(n')_k} k^{a+b-d}\binom{k}{d}\binom{k-d}{a-d}\binom{k-d}{b-d} p^{-d}(1-p)^{-\Phi(a,b)} \qquad (6)\\ &\leqslant \left(\frac{4}{n}\right)^{a+b} k^{a+b-d}\binom{k}{d}\binom{k}{a-d}\binom{k}{b-d} p^{-d}(1-p)^{-\Phi(a,b)}, \end{aligned}$$

where the last inequality is a consequence of

$$\frac{(n'-k)_{k-a}}{(n')_k} \leqslant \frac{(n'-a)_{k-a}}{(n')_k} = \frac{(n'-a)!(n'-k)!}{(n'-k)!n'!} = \frac{1}{(n')_a} \leqslant \left(\frac{4}{n}\right)^a,$$

(and similarly with b instead of a), which in turn is valid since $a,b \ll n$.

The following lemma is essential to our argument.

LEMMA 3.4. *For any $k\in\mathbb{N}$ and $a,b\in[k]$,* $\dfrac{\Phi(a,b)}{k(a+b)} \leqslant 1-\dfrac{\sqrt{2}}{2}$.

From its proof, which we defer to the end of this section, it will be clear that the maximum value of $T_{a,b,d}$ is achieved at $a=b=\sqrt{2}k/2$.

We now split into two cases according to the value of $a+b$. First, assume that $a+b \geqslant 24/(\epsilon p)$. In this case, we trivially bound the product of the three remaining binomial coefficients in (6) by 2^{3k}. The whole purpose of analyzing the term $T_{a,b,d}$ was to gain an extra factor of k^{-d} compared to the trivial bound on the number of vertex orderings in $(\mathbf{s},\mathbf{t})$. This gives

$$T_{a,b,d} \leqslant 4^{a+b}2^{3k}(np)^{-(a+b)}(kp)^{a+b-d}e^{\left(p+p^2\right)\Phi(a,b)},$$

and summing over d and using $\sum_{d=0}^{a}(kp)^{-d} \leqslant \sum_{d=0}^{a}\left(2\log(np)\right)^{-d} < 4$, we obtain

$$\sum_{d=0}^{a} T_{a,b,d} \leqslant 4^{a+b+1}\cdot 2^{3k}(np)^{-(a+b)}(kp)^{a+b}e^{\left(p+p^2\right)\Phi(a,b)}.$$

With Lemma 3.4 and $k/(a+b) \leqslant \epsilon\log(np)/6$, we get

$$\sum_{d=0}^{a} T_{a,b,d} \leqslant \left(8kp\cdot 2^{\epsilon\log(np)/2}(np)^{-1}e^{\left(1-\sqrt{2}/2\right)k\left(p+p^2\right)}\right)^{a+b}.$$

Let n be large enough so that $p \leqslant 0.1\epsilon$. Now $kp = (1-\epsilon)\left(2+\sqrt{2}\right)\log(np)$ gives $e^{\left(1-\sqrt{2}/2\right)k\left(p+p^2\right)} \leqslant e^{(1-\epsilon)(1+0.1\epsilon)\log(np)} \leqslant (np)^{(1-0.9\epsilon)}$ and, hence,

$$\sum_{d=0}^{a} T_{a,b,d} \leqslant \left(8(1-\epsilon)(2+\sqrt{2})\log(np)(np)^{((\log 2)/2-0.9)\epsilon}\right)^{a+b} \leqslant (np)^{-\epsilon(a+b)/4}$$

for large enough n. Summing over a, b, we get

$$\sum_{\substack{a,b\in[k]\\ a+b\geqslant\epsilon/(24p)}} \sum_{d=0}^{a} T_{a,b,d} \leqslant \sum_{a=1}^{k}\sum_{b=1}^{k}(np)^{-\epsilon(a+b)/4} = \left(\sum_{a=1}^{k}(np)^{-\epsilon a/4}\right)^2 \longrightarrow 0. \tag{7}$$

In the second case, $a+b < 24/(\epsilon p)$, we bound the binomial coefficients by $\binom{k}{i} \leqslant k^i$, $i \in \{d,\, a-d,\, b-d\}$. Then

$$T_{a,b,d} \leqslant 4^{a+b} n^{-(a+b)} k^{2a+2b-2d} p^{-d}(1-p)^{-ab}.$$

Given that $p < 1/2$ for large n, we use the inequality $(1-p) \geqslant e^{-p-p^2} \geqslant e^{-2p}$, which implies $(1-p)^{-ab} \leqslant e^{2pab} \leqslant e^{p(a+b)^2/2}$. Furthermore, $kp > 1$, so

$$T_{a,b,d} \leqslant \left(4n^{-1}k^2 e^{p(a+b)/2}\right)^{a+b} k^{-d}(kp)^{-d} \leqslant \left(4n^{-1}k^2 e^{p(a+b)/2}\right)^{a+b}.$$

Using the assumption that $p \geqslant (\log^2 n)/\sqrt{n}$, it follows that

$$n^{-1}k^2 \leqslant n^{-1}p^{-2}(2+\sqrt{2})^2 \log^2(np) \leqslant n^{-1}\cdot\frac{\left(2+\sqrt{2}\right)^2}{\log^4 n}\cdot n\log^2(np) \leqslant \frac{16}{\log^2 n}.$$

Moreover, $e^{p(a+b)/2} \leqslant e^{12/\epsilon}$, so altogether, $T_{a,b,d} \leqslant (1/\log n)^{a+b}$ for large enough n. Summing up,

$$\begin{aligned}\sum_{\substack{a,b\in[k]\\ a+b<\epsilon/(24p)}} \sum_{d=0}^{a} T_{a,b,d} &\leqslant \sum_{a=1}^{k}\sum_{b=1}^{k}(a+1)(\log n)^{-(a+b)}\\ &= \left(\sum_{a=1}^{k}(a+1)(\log n)^{-a}\right)\left(\sum_{b=1}^{k}(\log n)^{-b}\right) \longrightarrow 0.\end{aligned}$$

We conclude that $\mathrm{Var}\left[X_k\right]/\left(\mathbb{E}\left[X_k\right]\right)^2 = o(1)$, and hence $X_k > 0$ with high probability. Upon this event, Lemma 3.1 implies $Z(G) \leqslant n-k$. ∎

Now we prove Lemma 3.4, which is essentially finding the induced subgraph of a k-witness with a minimum expected number of copies in $G_{n,p}$.

Proof of Lemma 3.4. Let k be fixed. Recall that we need to prove an upper bound on $\Phi(a,b)/(k(a+b))$ for $(a,b) \in [k]^2$, where

$$\Phi(a,b) = \max_{A,B\subseteq[k],\,|A|=a,\,|B|=b} \left|\{(i,j)\in A\times B : i<j\}\right|.$$

For $a+b \leqslant k$, the claimed bound follows immediately from the trivial bound $\Phi(a,b) \leqslant ab$ and the arithmetic mean–geometric mean (AM-GM) inequality. So we may assume $a+b > k$. We first fix $|A| = a$, $|B| = b$, and $|A\cap B| = g$. If A and B are a selection of rows and columns of a $k\times k$ matrix, then g denotes the number of diagonal entries at the intersection of selected rows and columns. Define $\phi(A,B) = |\{(i,j)\in A\times B : i<j\}|$. We claim that $\phi(A,B) \leqslant ab-(g+1)g/2$. To see this, note that each element (c_1,c_2) with $c_1,c_2 \in A\cap B$ and $c_1 \geqslant c_2$ is contained in $A\times B$, but not counted by ϕ. There are $(g+1)g/2$ such pairs (in fact, such (c_1,c_2) are indices of the subdiagonal matrix entries selected by A and B).

Now we minimize g for fixed a and b. From the identity $|A\cup B|+|A\cap B| = |A|+|B|$ and $|A\cup B| \subseteq [k]$, we get $g \geqslant a+b-k$, so $\phi(A,B) \leqslant ab-(a+b-k+1)(a+b-k)/2$. Taking the maximum over A and B and using the AM-GM inequality, we get

$$\frac{\Phi(a,b)}{k(a+b)} \leqslant \frac{1}{k(a+b)}\left(ab - \frac{(a+b-k+1)(a+b-k)}{2}\right)$$
$$\leqslant \frac{1}{k(a+b)}\left(\left(\frac{a+b}{2}\right)^2 - \frac{(a+b-k)^2}{2}\right).$$

We substitute $a+b=2\alpha k$, so that the problem reduces to maximizing the function $f(\alpha) = \left(2\alpha^2-(2\alpha-1)^2\right)/(4\alpha)$ for $1/2 < \alpha \leqslant 1$. This is a simple calculus exercise, but we provide details for the sake of transparency. $f'(\alpha) = \left(-2\alpha^2+1\right)/(4\alpha^2)$, so f attains its local maximum at $\alpha_0 = \sqrt{2}/2$. Evaluating f at α_0, we get

$$\Phi(a,b) \leqslant \max_{\alpha} f(\alpha) = f(\alpha_0) = 1-\sqrt{2}/2$$

for all $a,b \subseteq [k]$. Note that equality is asymptotically attained when $A=[\ell]$, $B=[k-\ell+1,k]$ with $\ell = \lfloor \sqrt{2}k/2 \rfloor$, which we have used in Lemma 3.2 (the 0-statement of Theorem 1.3). ∎

4. Spectral bounds. In this section we discuss the bounds on the zero forcing number in terms of the graph eigenvalues. The study of spectral properties and their relation to other graph parameters is an established area of research with many diverse techniques and applications, surveyed, for example, in the monograph of Godsil and Royle [25]. One of the earliest results of this type is Hoffman's bound on the independence number of a graph [28]. Namely, let G be an n-vertex d-regular graph, and let $\lambda_{\min}$ denote its smallest eigenvalue. Hoffman proved that then G contains no independent set of order larger than $-\lambda_{\min} n/(d-\lambda_{\min})$. Note that since the trace of the adjacency matrix of a graph is zero, $\lambda_{\min}$ is negative. There are many examples showing the bound to be tight.

We establish an analogue of Hoffman's bound for the zero forcing number by showing that $Z(G) \geqslant n\left(1+2\lambda_{\min}/(d-\lambda_{\min})\right)$. To prove this result we use the following well-known estimate on the edge distribution of a graph in terms of its eigenvalues. Part (ii) of Theorem 4.1 is provided in, e.g., [33], whereas the variant (i) follows from the same proof. For a graph $G=(V,E)$ and two sets $U,W \subseteq V$, denote the number of edges with one endpoint in U and the other one in W by $e(U,W)$. Any edge with both endpoints in $U \cap W$ is counted twice. Recall that an (n,d,λ) graph is a d-regular n-vertex graph in which all eigenvalues but the largest one are at most λ in absolute value.

THEOREM 4.1. *Let G be an (n,d,λ)-graph, and denote its smallest eigenvalue by $\lambda_{\min}$. Then for any two vertex subsets U,W of G,*

(i) $$\frac{d|U||W|}{n} - e(U,W) \leqslant -\lambda_{\min}\sqrt{|U||W|\left(1-\frac{|U|}{n}\right)\left(1-\frac{|W|}{n}\right)},$$

(ii) $$\left|\frac{d|U||W|}{n} - e(U,W)\right| \leqslant \lambda\sqrt{|U||W|\left(1-\frac{|U|}{n}\right)\left(1-\frac{|W|}{n}\right)}.$$

Proof of Theorem 1.4(i). Let $(\mathbf{s},\mathbf{t})$ be a k-witness in G, and $k=2\mu n$. By definition of a witness, there are no edges between the sets $U=\{s_1,s_2,\ldots,s_{\mu n}\}$ and

$W = \{t_{\mu n+1}, \dots, t_k\}$. Hence, using Theorem 4.1(i),

$$0 = e(U, W) \geqslant d\mu^2 n + \lambda_{\min}\mu n(1-\mu) = \mu n\left(d\mu + \lambda_{\min}(1-\mu)\right) \\ = \mu n\left(\mu(d-\lambda_{\min}) + \lambda_{\min}\right),$$

which implies $\mu \leqslant -\lambda_{\min}/(d-\lambda_{\min})$. Hence the largest witness in G has order at most $-2\lambda_{\min}n/(d-\lambda_{\min})$, which by Lemma 3.1 implies

$$Z(G) \geqslant n\left(1 + \frac{2\lambda_{\min}}{d-\lambda_{\min}}\right).$$

∎

Surprisingly, the additional factor of two in the abovementioned bound that looks like an artefact of the proof, turns out to be necessary, and the result of Theorem 1.4(i) is shown to be tight by the following example.

PROPOSITION 4.2. *For any even $D \geqslant 2$, and for infinitely many values of N, there exists an N-vertex D-regular graph G^* whose smallest eigenvalue is $\lambda_{\min} = -2$, and which satisfies $N - Z(G^*) \geqslant 4N/(D+2) - 2$.*

Proof. Let $d = D/2 + 1$, and let G be an n-vertex d-regular graph which contains a Hamilton cycle consisting of edges $e_1, e_2, \dots, e_n$ in this order. Clearly, such graphs do exist for all d and infinitely many n. Let G^* be the line graph of G, that is, G^* has the vertex set $E(G)$ with two vertices adjacent if the corresponding edges in G share a vertex. Then G^* has $N = nd/2$ vertices and is D-regular with $D = 2d-2$. Moreover, Hoffman [28] has observed that the smallest eigenvalue of G^* is -2.

Note that the vertices in G^* corresponding to $e_1, e_2, \dots, e_n$ form an induced cycle. This implies that $V(G^*)\backslash\{e_3, e_4, \dots, e_n\}$ is a zero forcing set in G^*. Namely, the vertex e_i forces e_{i+1} for $i = 2, 3, \dots, n-1$. This zero forcing set has order $N-(n-2)$. Finally, notice that in G^*, we have

$$-\frac{2\lambda_{\min}N}{D-\lambda_{\min}} = \frac{4\cdot(nd/2)}{2d-2+2} = n.$$

∎

Next, we turn our attention to the second part of Theorem 1.4, which says that any (n, d, λ)-graph G satisfies

$$Z(G) \leqslant n\left(1 - \frac{1}{2(d-\lambda)}\log\left(\frac{d-\lambda}{2\lambda+1}\right)\right).$$

In particular, if $\lambda = d^{1-\epsilon}$ for some $\epsilon > 0$, then $n - Z(G) = \Omega\left((n\log d)/d\right)$.

Proof of Theorem 1.4(ii). We greedily construct a witness. In each step i, we will select vertices $s_i, t_i \in U_{i-1}$ and a set $U_i \subseteq U_{i-1}$. Start with $U_0 = V$, the vertex set of G. Assuming that the steps $1, \dots, i-1$ were executed, let s_i be any vertex in U_{i-1} satisfying $1 \leqslant \deg_{G[U_{i-1}]}(s_i) \leqslant (d-\lambda)|U_{i-1}|/n + \lambda$. As usual, $N_G(s_i)$ denotes the neighborhood of s_i in G. We fix any $t_i \in N_G(s_i) \cap U_{i-1}$, and set $U_i = U_{i-1} \setminus (N_G(s_i) \cup \{s_i\})$. The algorithm continues as long as $|U_i| > \lambda n/(d+\lambda)$. Denote the total number of steps by k.

By construction, the pair $(\mathbf{s}, \mathbf{t})$ is a witness. We will show that there is a choice for s_i throughout the algorithm, and that

$$k \geqslant \frac{n}{2(d-\lambda)}\log\left(\frac{d-\lambda}{2\lambda+1}\right).$$

Claim 4.3. *If $|U_i| > \lambda n/(d+\lambda)$, then the induced subgraph $G[U_i]$ contains a vertex u satisfying $1 \leqslant \deg_{G[U_i]}(u) \leqslant (d-\lambda)|U_i|/n + \lambda$.*

Proof. Suppose that some set U_i does not satisfy the claim. By Theorem 4.1(ii), U_i is not an independent set as $|U_i| > \lambda n/(d+\lambda)$. Therefore, removing the isolated vertices in $G[U_i]$, we get a nonempty set $W \subseteq U_i$ in which every vertex u satisfies $\deg_{G[W]}(u) > (d-\lambda)|U_i|/n + \lambda$. In particular,

$$e\,(W,W) = \sum_{u\in W} \deg_{G[W]}(u) > |W| \left((d-\lambda)\frac{|U_i|}{n} + \lambda \right),$$

recalling that each edge in $E(W,W)$ is counted twice in $e(W,W)$. On the other hand, Theorem 4.1(ii) implies

$$e(W,W) \leqslant |W| \left(\frac{d|W|}{n} + \lambda \left(1 - \frac{|W|}{n}\right) \right) \leqslant |W| \left((d-\lambda)\frac{|U_i|}{n} + \lambda \right).$$

We reached a contradiction, which completes the proof of the claim. ∎

Now denote $a_i = |U_i|/n$. By construction, $a_0 = 1$ and, for $i \geqslant 1$,

$$\begin{aligned} a_i &= \frac{|U_{i-1}| - \deg_{G[U_{i-1}]}(s_i) - 1}{n} \\ &\geqslant \frac{1}{n}\left(|U_{i-1}| - \frac{(d-\lambda)|U_{i-1}|}{n} - \lambda - 1 \right) = \left(1 - \frac{d-\lambda}{n}\right) a_{i-1} - \frac{\lambda+1}{n}. \end{aligned}$$

By induction on i, this implies that for all i,

$$a_i \geqslant \left(1 + \frac{\lambda+1}{d-\lambda}\right)\left(1 - \frac{d-\lambda}{n}\right)^i - \frac{\lambda+1}{d-\lambda}. \tag{8}$$

Claim 4.4. *For $i \leqslant \frac{n}{2(d-\lambda)} \log \frac{d-\lambda}{2\lambda+1}$, $a_i \geqslant \lambda/(d+\lambda)$.*

Proof. We use (8) to estimate a_i for $i \leqslant \frac{n}{2(d-\lambda)} \log \frac{d-\lambda}{2\lambda+1}$, ignoring the constant $(1 + (\lambda+1)/(d-\lambda))$ and using the inequality $1 - (d-\lambda)/n \geqslant e^{-2(d-\lambda)/n}$ for $(d-\lambda)/n < 1/2$. This gives

$$a_i \geqslant \exp\left(-\frac{2(d-\lambda)}{n} \cdot \frac{n}{2(d-\lambda)} \log\left(\frac{d-\lambda}{2\lambda+1}\right)\right) - \frac{\lambda+1}{d-\lambda} = \frac{2\lambda+1}{d-\lambda} - \frac{\lambda+1}{d-\lambda} = \frac{\lambda}{d+\lambda},$$

as required. We conclude that the algorithm continues for at least $k = \frac{n}{2(d-\lambda)} \log \frac{d-\lambda}{2\lambda+1}$ steps, so

$$n - Z(G) \geqslant \frac{n}{2(d-\lambda)} \log \frac{d-\lambda}{2\lambda+1}.$$ ∎

This concludes the proof of Theorem 1.4(ii). ∎

To show that for $\lambda \leqslant d^{1-\Omega(1)}$, the bound on $n - Z(G)$ is tight up to a constant factor, we exhibit a sequence of (n,d,λ)-graphs G_m with $\lambda = O(\sqrt{d})$ satisfying $n - Z(G_m) \leqslant n\left((\log_2 d)/(2d) + o(1)\right)$. We use the following construction from [33, section 3]. For an odd integer m, the vertices of G_m are all binary vectors of length m with an odd number of ones except for the all-one vector. Two distinct vertices are adjacent iff the inner product of the corresponding vectors is 1 modulo 2. This graph has $n_m = 2^{m-1} - 1$ vertices, degree $d_m = (n_m - 3)/2$, and second largest eigenvalue $\lambda(G_m) = 1 + 2^{(m-3)/2} = O\left(\sqrt{d_m}\right)$. It is easy to check that if $(\mathbf{s}, \mathbf{t})$ is a k-witness in G_m, then the vectors corresponding to $t_1, t_2, \ldots, t_k$ are linearly independent, and therefore $k \leqslant m = (1 + o(1)) \log_2 n_m = (1 + o(1)) n_m/(2d_m) \log_2 d_m$. This gives the required bound on $n - Z(G_m)$.

5. Concluding remarks.

- Theorem 1.3 can be extended to $p \gg n^{-1}$. The proof combines our second moment estimates with Talagrand's inequality, along the lines of [29, Theorem 7.4], which finds independent sets of order $p^{-1}\log(np)$ in $G_{n,p}$. Since this proof gives no additional combinatorial insight, we put it into the appendix.
- We were wondering whether a linear lower bound on the minimum rank of a random graph from [27], $mr\left(G_{n,1/2}\right) \geqslant 0.14n$, can be extended to (n, d, λ)-graphs G with d linear in n and $\lambda < d^{1-\Omega(1)}$. A negative answer was reached in conversation with Babai [8]. We considered the graph H_t whose vertex set corresponds to t-element subsets of $\left[t^2\right]$, and two vertices are adjacent if and only if the corresponding sets intersect (the complement of a Kneser graph, described, for instance, in [25]). The minimum rank of H_t is $mr(H_t) = o(\log^2(n_t))$, where $n_t = \binom{t^2}{t}$ is the corresponding number of vertices. It would be interesting to see if there are such (n, d, λ)-graphs with minimum rank as low as $O(\log n)$, which is the lower bound implied by Theorem 1.4(ii).
- What is the asymptotic value of $n - Z(G_{n,d})$, where d is a large constant and $G_{n,d}$ is a graph chosen uniformly at random from all n-vertex d-regular graphs? A greedy argument (see, e.g., [6]) shows that $Z(G_{n,d}) \leqslant n(1 - 1/(d-1))$ deterministically, whereas Theorem 1.4(ii) implies that for large d, with high probability, $Z\left(G_{n,d}\right) \leqslant n\left(1 - \log d/(4d)\right)$. This follows from the fact that with high probability, $G_{n,d}$ is an (n, d, λ)-graph with $\lambda \leqslant 3\sqrt{d}$ (see, e.g., [21]). The lower bound, $Z\left(G_{n,d}\right) \geqslant n\left(1 - 40\log d/d\right)$, is an immediate consequence of the fact that with high probability, $G_{n,d}$ contains edges between any two sets S, T with $|S|, |T| \geqslant 20n\log d/d$ (see, e.g., [30, Lemma 3.6]).

Appendix A. The random graph with small p. To extend Lemma 3.3 to small p we use Talagrand's inequality (see, e.g., [29, Theorem 2.29]).

LEMMA A.1. *Assume that $1 \ll np < \sqrt{n}\log^2 n$. Let the vertex set V of $G_{n,p}$ be partitioned into V_1 and V_2 with $|V_1| = \lfloor n/2 \rfloor$ and $|V_2| = \lceil n/2 \rceil$, and let $k_{-\epsilon} = (1-\epsilon)(2+\sqrt{2})p^{-1}\log(np)$ with $0 < \epsilon < 1/2$. With high probability (whp), $G_{n,p}$ contains a divided $k_{-\epsilon}$-witness.*

Proof. Denote by $w(G)$ the order of the largest divided witness. We will actually show that $w\left(G_{n,p}\right) \geqslant k_{-2\epsilon} = (1-2\epsilon)(2+\sqrt{2})p^{-1}\log(np)$ whp, which is sufficient as ϵ is arbitrary. The first step is a second moment lower bound on the probability $\mathbb{P}\left[w\left(G_{n,p}\right) \geqslant k_{-\epsilon}\right]$. For now, we write $k = k_{-\epsilon}$. The proof is identical to the proof of Lemma 3.3 down to inequality (7). In particular, the case $a + b \geqslant 24/(\epsilon p)$ remains unchanged (and that is the case which determines k). Hence we assume $a + b < 24/(\epsilon p)$. Inequality (6) from the proof of Lemma 3.3 implies, using $kp > 1$ and

$1-p \geqslant e^{-2p}$ for $p \geqslant 0$,

$$T_{a,b,d} \leqslant \left(\frac{4}{n}\right)^{a+b} k^{a+b-d} p^{-d} \binom{k}{d} \binom{k}{a-d} \binom{k}{b-d} (1-p)^{-ab}$$
$$\leqslant \left(\frac{4k}{n}\right)^{a+b} \binom{k}{d} \binom{k}{a} \binom{k}{b} e^{2pab}.$$

Denote $a+b=2u$, and using the inequality $\binom{k}{d} \leqslant \binom{k}{u}$ which follows from $d \leqslant a \ll k$, as well as $\binom{k}{a}\binom{k}{b} \leqslant \binom{k}{u}^2$ and $ab \leqslant u^2$, we have

$$T_{a,b,d} \leqslant \left(\frac{4k}{n}\right)^{2u} \binom{k}{u}^3 e^{2pu^2} \leqslant \left(\frac{16k^2}{n^2} \cdot \frac{e^3k^3}{u^3}\right)^u e^{2pu^2}.$$

Define $\xi = 16e^3k^5/n^2$ and $g(u) = u \log\left(\xi u^{-3}\right) + 2pu^2$, so that $\log T_{a,b,d} \leqslant g(u)$ for all a, b, d. We will now use basic calculus to bound the function g on $[1, 12/(\epsilon p)]$, so a first-time reader may find it helpful to skip over the proof of the claim. The main point is that the function g has a global maximum $u_0 \ll k$.

Claim A.2. *For* $u \in [1, 12/(\epsilon p)]$, $g(u) = u\log\xi - 3u\log u + 2pu^2 \leqslant k(np)^{-1/2}$.

Proof. Let $u_2 = 12/(\epsilon p)$ and $u_1 = 4ek\,(k/n)^{2/3}$ or, equivalently, $(ek/u_1)^3 = (n/k)^2/64$. We compute $g'(u) = \log\xi - 3\log u - 3 + 4pu = \log\left(\xi u^{-3}\right) - 3 + 4pu$ and claim that

$$g'(1) > 0, \quad g'(u_1) < 0, \quad \text{and} \quad g'(u_2) < 0.$$

To see the first inequality, recall that $k > 1/p > \sqrt{n}/\log^2 n$ and therefore $\xi \gg 1$. It follows that $g'(1) = \log\xi - 3 + 4p > 0$. Evaluating g' at u_1 and using

$$pu_1 = 4pek^{5/3}n^{-2/3} \leqslant 100\,(np)^{-2/3}\log^{5/3}(np),$$

we get

$$g'(u_1) = \log(\xi u_1^{-3}) - 3 + 4pu_1 = \log(1/4) - 3 + o(1) < 0.$$

Finally, $\xi u_2^{-3} = O\left(\xi p^3\right) = O\left(\frac{\log^5(np)}{(np)^2}\right) = o(1)$, so $g'(u_2) = \log\left(\xi u_2^{-3}\right) - 3 + 48\epsilon^{-1} < 0$.

From $g'(1) > 0$ and $g'(u_1) < 0$ it follows that there is a $u_0 \in [1, u_1]$ with $g'(u_0) = 0$. We will show that $g(u_0) \leqslant 3u_0 \leqslant k(np)^{-1/2}$ and, moreover, that u_0 is the global maximum of g on $[1, u_2]$.

To see the first claim, note that $g'(u_0) = \log\left(\xi u_0^{-3}\right) - 3 + 4pu_0 = 0$ and hence

$$g(u_0) = u_0(3 - 4pu_0) + 2pu_0^2 \leqslant 3u_0 \leqslant 3u_1 = 12ek\,(k/n)^{2/3} < k\,(np)^{-1/2}.$$

It remains to show that g has no other extrema. This follows from the fact that $g''(u) = -3/u + 4p$ is increasing in u. Therefore g' has exactly one local minimum at $u = 3/(4p) > u_1$. If g' vanishes at more than one point in $(1, u_2)$, then either g' has multiple local extrema, or $g'(1) > 0$ and $g'(u_2) > 0$, both of which lead to a contradiction. Hence u_0 is the unique local extremum of g, and it is indeed the global maximum since the sign of g' changes from positive to negative at u_0. ∎

We conclude that for all a, b, d such that $a + b < 24/(\epsilon p)$,

$$T_{a,b,d} \leqslant e^{g((a+b)/2)} \leqslant e^{k(np)^{-1/2}}.$$

Using $p < \log^2 n/\sqrt{n}$ we bound the exponent by

$$k(np)^{-1/2} = \frac{(1-\epsilon)(2+\sqrt{2})p^{-3/2}\log(np)}{\sqrt{n}} \geqslant \frac{n^{3/4}}{\sqrt{n}\log^3 n} \geqslant n^{1/8}.$$

Summing the bound for $T_{a,b,d}$ over all a, b, d and using (7) we get

$$\frac{\mathrm{Var}\left[X_k\right]}{\left(\mathbb{E}\left[X_k\right]\right)^2} \leqslant \sum_{\substack{a,b\in[k]\\ a+b\geqslant 24/(\epsilon p)}} \sum_{d=0}^{a} T_{a,b,d} + \sum_{\substack{a,b\in[k]\\ a+b< 24/(\epsilon p)}} \sum_{d=0}^{a} T_{a,b,d} \leqslant o(1) + k^3 e^{k(np)^{-1/2}} \leqslant e^{2k(np)^{-1/2}}.$$

We will apply a stronger form of Chebyshev's inequality, which reads $\mathbb{P}\left[X_k > 0\right] \geqslant \left(\mathbb{E}\left[X_k\right]\right)^2 / \mathbb{E}\left[X_k^2\right]$ (see [29, Remark 3.1] for details). Using

$$\frac{\mathbb{E}\left[X_k^2\right]}{\left(\mathbb{E}\left[X\right]_k\right)^2} = \frac{\mathrm{Var}\left[X_k\right]}{\left(\mathbb{E}\left[X_k\right]\right)^2} + 1 \leqslant e^{4k(np)^{-1/2}},$$

we obtain

$$\mathbb{P}\left[X_k > 0\right] \geqslant e^{-4k(np)^{-1/2}}. \tag{9}$$

To show a concentration of the order of the largest divided witness $w\left(G_{n,p}\right)$, we apply Talagrand's inequality. The random graph $G_{n,p}$ is modeled using vertex exposure. Formally, we fix an ordering of the vertices $v_1, v_2, \ldots, v_n$, and define mutually independent random variables $\left(Z_i\right)_{i\in[n]}$, where Z_i exposes the backward edges from the vertex v_i. Then $w\left(G_{n,p}\right)$ is a function of $Z_1, \ldots, Z_n$. This function is 1-Lipschitz, that is, if graphs G and G' differ only at the vertex v_i, then $|w(G) - w(G')| \leqslant 1$. Moreover, whenever $w(G_{n,p}) \geqslant k$, there exist $2k$ certificate vertices, namely, the vertices of a divided k-witness, which are responsible for the fact that $w(G_{n,p}) \geqslant k$. Hence we may apply [29, Theorem 2.29] with $\sigma(k) = 2k$ in their notation. Recalling that $k_{-\epsilon} = (1-\epsilon)(2+\sqrt{2})p^{-1}\log(np)$, we have

$$\mathbb{P}\left[w(G) \leqslant k_{-2\epsilon}\right] \mathbb{P}\left[w(G) \geqslant k_{-\epsilon}\right] \leqslant e^{-(k_{-\epsilon}-k_{-2\epsilon})^2/(8k_{-\epsilon})} \leqslant e^{-\epsilon^2 k_{-\epsilon}/8}.$$

Inequality (9) says $\mathbb{P}\left[w(G) \geqslant k_{-\epsilon}\right] \geqslant e^{-4k(np)^{-1/2}}$ and taking $np > (10/\epsilon)^4$, we conclude

$$\mathbb{P}\left[w(G) \leqslant k_{-2\epsilon}\right] \leqslant e^{-\epsilon^2 k_{-\epsilon}/8 + 4k_{-\epsilon}(np)^{-1/2}} \leqslant e^{-\epsilon^2 k_{-\epsilon}/16} \longrightarrow 0,$$

as required. ∎

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