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Abelian ideals in a Borel subalgebra of a complex simple Lie algebra

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Abstract. Let $\mathfrak{g}$ be a complex simple Lie algebra and $\mathfrak{b}$ a fixed Borel subalgebra of $\mathfrak{g}$. We describe the abelian ideals in $\mathfrak{b}$ in a uniform way, that is, independent of the classification of complex simple Lie algebras. As an application we derive a formula for the maximal dimension of a commutative Lie subalgebra of $\mathfrak{g}$.

1. Introduction

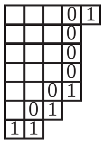
Let $\mathfrak{g}$ be a complex simple Lie algebra and $\mathfrak{b}$ a fixed Borel subalgebra of $\mathfrak{g}$.

This paper has three purposes. First, the maximal dimension among the commutative subalgebras of $\mathfrak{g}$ is determined purely in terms of certain invariants. These invariants involve the dual Coxeter number of $\mathfrak{g}$ and the numbers of positive roots of some associated root subsystems of $\mathfrak{g}$. Our formula gives a conceptual explanation of A. Malcev's classical result [Mal]. To assuage any possible curiosity we now list the maximal dimensions together with their computations for the five exceptional types. The whole picture will be revealed in the table on page 209.

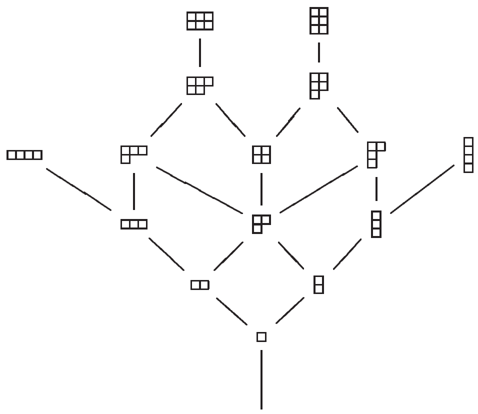
$$\begin{aligned}
g_{\mathsf{E}_6} - 1 + N_{\mathsf{A}_5} - N_{\mathsf{A}_4} &= 12 - 1 + 15 - 10 = 16\\
g_{\mathsf{E}_7} - 1 + N_{\mathsf{D}_6} - N_{\mathsf{D}_5} &= 18 - 1 + 30 - 20 = 27\\
g_{\mathsf{E}_8} - 1 + N_{\mathsf{A}_7} - N_{\mathsf{A}_6} &= 30 - 1 + 28 - 21 = 36\\
g_{\mathsf{F}_4} - 1 + N_{\mathsf{A}_1} - N_{\varnothing} &= 9 - 1 + 1 - 0 = 9\\
g_{\mathsf{G}_2} - 1 + N_{\varnothing} - N_{\varnothing} &= 4 - 1 + 0 - 0 = 3
\end{aligned}$$

Second, we answer a question of Panyushev and Röhrle [PR] who asked for a uniform explanation for the one-to-one correspondence between the maximal abelian ideals in $\mathfrak{b}$ and the long simple roots. More generally, in our approach all positive long roots will emerge in a natural way. We define a mapping from the set of nonzero abelian ideals in $\mathfrak{b}$ onto the set of positive long roots. This mapping was also discovered independently by Panyushev [Pan]. He proved that each fibre of this mapping is a poset having a unique maximal element and a unique minimal element and asked for further investigating the poset structure and, in particular, to find a general description of the maximal element of each fibre. The exact structure of the fibres was first announced in a preliminary version of this paper [Su3]. Some additional work by Cellini and Papi appeared recently [CP4].

Third, we keep the promise of giving a generalization and explanation of the symmetry property of a certain subposet of Young's lattice (the lattice of integer partitions) that was observed in [Su2] and which we now recall. For that consider the subposet $\mathbb{Y}_N$ of Young's lattice induced by the Young diagrams whose (largest) hook lengths are at most $N-1$. One sees easily that the poset $\mathbb{Y}_N$ has 2^{N-1} elements. This follows for instance by associating to each such diagram an integer between 0 and $2^{N-1}-1$ by the following procedure: in each column of the diagram write the figure 1 at the bottom and fill the rest by 0; then read the binary number along the rim.


$$\longmapsto 11010100001_2 = 1697$$

The main result of [Su2] states that the Hasse graph of $\mathbb{Y}_N$ (considered as an undirected graph) has the dihedral group Dih_N of order $2N$ as its automorphism group provided $N \geqslant 3$. The following figure exemplifies this fact for $N = 5$.



$\mathrm{Aut}(\mathbb{Y}_5) \cong \mathrm{Dih}_5$

The case dealt with in [Su2] is now seen as the A_{N-1} case, i. e., associated with the Lie algebra $\mathfrak{g} = \mathfrak{sl}_N(\mathbb{C})$. It is so to say the most spectacular case. The reason is that its affine Coxeter-Dynkin graph is a cycle of length N. Its dihedral symmetry induces a dihedral symmetry on a certain simplicial complex $\mathcal{C}$. The Hasse graph of $\mathbb{Y}_N$ can be geometrically realized as the 1-skeleton of the cell complex dual to $\mathcal{C}$.

Here is a brief historical narrative around the topic of this paper. In his 1905 paper [Sch] in Crelle's journal I. Schur proved that the maximum number of linearly independent commuting $N \times N$ matrices is $\left\lfloor \frac{N^2}{4} \right\rfloor + 1$. In 1944 Jacobson [Jac] gave a simplified derivation of Schur's result. In the next year A. Malcev [Mal] determined the commutative subalgebras of maximum dimension of the semisimple complex Lie groups, or equivalently, their Lie algebras. The next entry in this short historical outline is Kostant's paper [Ko1] published in 1965. There he gave a connexion of Malcev's result with the maximal eigenvalue of the Laplacian acting on the exterior powers $\bigwedge^k \mathfrak{g}$ of the adjoint representation. Kostant [Ko2] again, in 1998, reconsidered the theme of abelian ideals in a Borel subalgebra of $\mathfrak{g}$ and reported inter alia about Peterson's proof that the number of abelian ideals in a fixed Borel subalgebra of $\mathfrak{g}$ is $2^{\operatorname{rank}\mathfrak{g}}$. This, quoting Kostant, utterly surprising and ingenious proof involves the affine Weyl group. It seems that Kostant's paper was the starting point of much recent activity. A natural generalization of Peterson's approach from abelian to ad-nilpotent ideals was developed recently by several authors [AKOP,CP1,CP2,CP3,KOP,Som], see also [Shi], and for Kostant's results [CMP].

The structure of the paper is as follows. In Sect. 2 we review the combinatorial setup for describing the abelian ideals in $\mathfrak{b}$. Such an ideal is associated with a subset Ψ of the set of positive roots. Kostant's theorem, which will be our main tool, characterizes the subsets Ψ that arise in this way. A central notion in our approach will be that of a ρ-point. The ρ-points are certain integral weights arising from the geometry of the affine Weyl group. The precise definition is given on page 183. We then reprove Peterson's theorem directly from Kostant's theorem (without using involutions in a maximal torus). In Sect. 3 we first observe that for each nonzero abelian ideal $\mathfrak{a} \trianglelefteq \mathfrak{b}$ its subspace spanned by the root spaces corresponding to those roots that are not perpendicular to the highest root is again an ideal, say $\mathfrak{a}^{\not\perp\theta}$. For each positive long root φ we then construct an abelian ideal $\mathfrak{a}^{\varphi,\min}$. (A word about the notation may be appropriate here. The $\mathfrak{a}$ in $\mathfrak{a}^{\varphi,\min}$ is purely notational whereas $\mathfrak{a}$ in $\mathfrak{a}^{\not\perp\theta}$ denotes a nonzero abelian ideal.) The image of the mapping $0 \neq \mathfrak{a} \mapsto \mathfrak{a}^{\not\perp\theta}$ is the set of abelian ideals of the form $\mathfrak{a}^{\varphi,\min}$. This will be proved in Theorem 17. The main theorem (Theorem 23) also describes the fibres of the mapping above. As a corollary we get the First Sum Formula (Theorem 24), which generalizes the fact that the sum of all entries in the first l rows of Pascal's triangle equals $2^l - 1$. In Sect. 4 we look at the maximal abelian ideals and in particular at those of maximal dimension. This links to A. Malcev's list for the maximal dimension of a commutative subalgebra in $\mathfrak{g}$. Corollary 25 gives a uniform formula for these dimensions. Next, the connexion between the maximal abelian ideals in $\mathfrak{b}$ and the set of long simple roots is explained. As a corollary we get the Second Sum Formula (Theorem 27) which generalizes the binomial expansion for $(1+1)^n = 2^n$. Section 5 deals with the symmetry of the Hasse graph of the poset of abelian ideals in $\mathfrak{b}$. As examples we display the Hasse graphs for the simple types of rank 4 and also the Hasse graph for E_6.

2. Notations and tools, review of some results

For basic facts about root systems, Weyl groups, and related topics see [Bou,Hum,Bro,Hil]. Let us fix a complex simple Lie algebra $\mathfrak{g}$ of rank l together with a Borel subalgebra $\mathfrak{b}$ and a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{b}$. Associated with these data there are quite a number of further objects whose notations are provided in the following table. Most of them are standard (but sometimes there are different conventions). We list the most important notations used here for the reader's convenience.

A	(closed) fundamental alcove (4),
C	(closed) dominant chamber (5),
F_i	$= H_i \cap A$ facets of type i of the fundamental alcove A,
g	dual Coxeter number,
$\mathfrak{g}_\varphi$	$= \{X \in \mathfrak{g} \mid [H, X] = \varphi(H)\, X \ \forall\, H \in \mathfrak{h}\}$ root space,
h	Coxeter number,
$\mathfrak{h}_{\mathbb{R}}^*$, $\mathfrak{h}_{\mathbb{R}}$	real vector space spanned by the roots, and its predual,
H_i	walls supporting the facets F_i ($i = 0, \ldots, l$) of A,
H_φ, $H_{k\delta-\varphi}$	walls $\mathrm{Fix}(s_\varphi)$, $\mathrm{Fix}(s_{k\delta-\varphi})$,
l	$= \operatorname{rank} \mathfrak{g}$,
ℓ	length function on $\widehat{W}$ or on W,
L	$L(\varphi) = \frac{2\,(\theta-\varphi\mid\rho)}{(\theta\mid\theta)}$,
m_i	exponents ($i = 1, \ldots, l$),
n_i	marks ($i = 1, \ldots, l$), $\theta = \sum_{i=1}^{l} n_i\, \alpha_i$, in addition, $n_0 = 1$,
N_{X}	number of positive roots for a root system of type X,
s_i	simple reflections ($i = 1, \ldots, l$), in addition, $s_0 : \mathfrak{h}_{\mathbb{R}}^* \to \mathfrak{h}_{\mathbb{R}}^*$, $s_0(\lambda) = \lambda - \langle \lambda, \theta^\vee \rangle\, \theta + g\theta$,
s_θ	reflection along the highest root, $s_\theta(\lambda) = s_0(\lambda) - s_0(0)$,
s_φ, $s_{k\delta-\varphi}$	reflections along the roots φ, $k\delta - \varphi$,
$w_\circ^\varphi$, $\widehat{w}_\circ^\varphi$	longest elements in $W_{\perp\varphi}$, $\widehat{W}_{\perp\varphi}$,
W, $\widehat{W}$	finite Weyl group, affine Weyl group,
$W_{\perp\varphi}$	$= \mathrm{gp}\big(s_i \mid \alpha_i \perp \varphi\ (i = 1, \ldots, l)\big)$,
$\widehat{W}_{\perp\varphi}$	$\widehat{W}_{\perp\varphi} = W_{\perp\varphi}$ if $\theta \not\perp \varphi$, $\widehat{W}_{\perp\varphi} = \mathrm{gp}\big(s_0, W_{\perp\varphi}\big)$ if $\theta \perp \varphi$,
α_i	simple roots ($i = 1, \ldots, l$), $\alpha_0 = \delta - \theta$,
θ	highest root,
ϖ_i	fundamental weights ($i = 1, \ldots, l$),
$\overset{\vee}{\varpi}_i$	$= \frac{1}{\lVert\alpha_i\rVert^2}\, \varpi_i$,
Π	$= \{\alpha_1, \ldots, \alpha_l\}$ set of simple roots,
Π^{long}	set of long simple roots,
ρ	$= \frac{1}{2} \langle \Phi_+ \rangle$ half the sum of positive roots,
$\varphi^\vee$	coroot corresponding to $\varphi \in \Phi$,
Φ, $\Phi_\pm$	root system, set of positive/negative roots,

Φ_+^{long}	set of positive long roots,
$\Phi(\mathfrak{a})$	set of weights of $\mathfrak{a}$, $\mathfrak{a} = \bigoplus_{\varphi\in\Phi(\mathfrak{a})} \mathfrak{g}_\varphi$,
Φ_w	$= \Phi_+ \cap w\Phi_-$,
$\widehat{\Phi}$, $\widehat{\Phi}_\pm$	affine root system, set of positive/negative affine roots,
$\widehat{\Phi}_{\widehat{w}}$	$= \widehat{\Phi}_+ \cap \widehat{w}\widehat{\Phi}_-$,
$(\ \mid\ )$	canonical bilinear form on $\mathfrak{h}^*_{\mathbb{R}}$, or on $\mathfrak{h}^*_{\mathbb{R}} \oplus \mathbb{R}\delta \oplus \mathbb{R}\Lambda_0$,
$\\|\ \\|$	norm from $(\ \mid\ )$,
$\langle\ \rangle$	$\langle\Psi\rangle = \sum_{\varphi\in\Psi} \varphi$ for $\Psi \subseteq \Phi_+$,
$\langle\ ,\ \rangle$	natural pairing,
$[\ ]$	$[n] = \frac{1-t^n}{1-t}$.

We denote by $\Phi_+ \subseteq \mathfrak{h}^*$ the set of positive roots. Here the convention is that the root spaces in $\mathfrak{b}$ belong to positive roots, i. e., $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\varphi\in\Phi_+} \mathfrak{g}_\varphi$ where $\mathfrak{g}_\varphi$ is the (1-dimensional) root space on which $\mathfrak{h}$ acts by the weight φ, that is, $\mathfrak{g}_\varphi = \left\{X \in \mathfrak{g} \;\middle|\; [H, X] = \varphi(H)\, X \;\; \forall\, H \in \mathfrak{h}\right\}$. As further pieces of notation we write $\Phi_- = -\Phi_+$ for the set of negative roots, $\Phi = \Phi_+ \dot{\cup} \Phi_-$ for the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$, and $\Pi \subseteq \Phi_+$ for the root basis. Recall that Π consists of the roots in Φ_+ that lie on the edges of the polyhedral (in fact, simplicial) cone spanned by the vectors in Φ_+. Each positive root is a linear combination of the vectors in Π with nonnegative integral coefficients. The Weyl group of Φ will be denoted by W. More about Weyl groups and some geometry associated with them will be recalled at the appropriate place below.

Now let $\mathfrak{a} \trianglelefteq \mathfrak{b}$ be an ideal. It is ad $\mathfrak{h}$-stable and hence compatible with the root space decomposition. If we further require that $\mathfrak{a}$ lies in the nilpotent radical $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$, we get that $\mathfrak{a}$ is of the form $\mathfrak{a} = \bigoplus_{\varphi\in\Psi} \mathfrak{g}_\varphi$ for some subset $\Psi \subseteq \Phi_+$ of positive roots. The ideal property of $\mathfrak{a}$ translates into the condition for Ψ that $\Psi \dot{+} \Phi_+ := (\Psi + \Phi_+) \cap \Phi_+ \subseteq \Psi$. If, in addition, $\mathfrak{a}$ is supposed to be abelian (so that $\mathfrak{a} \subseteq [\mathfrak{b}, \mathfrak{b}]$ holds automatically), we must have $\Psi \dot{+} \Psi := (\Psi + \Psi) \cap \Phi_+ = \varnothing$. It is clear that there is the following bijection.

$$\begin{aligned} \left\{\begin{matrix} \text{subsets } \Psi \subseteq \Phi_+ \text{ such that} \\ \Psi \dot{+} \Phi_+ \subseteq \Psi \text{ and } \Psi \dot{+} \Psi = \varnothing \end{matrix}\right\} &\xleftrightarrow{\;\cong\;} \left\{\text{abelian ideals } \mathfrak{a} \trianglelefteq \mathfrak{b}\right\} \\ \Psi &\longmapsto \mathfrak{a}_\Psi := \bigoplus_{\varphi\in\Psi} \mathfrak{g}_\varphi \end{aligned}$$

The inner product. Before we can go on and state Kostant's theorem, which will be an essential tool for our approach, we recall the canonical inner product on the real vector space $\mathfrak{h}^*_{\mathbb{R}}$ spanned by the (finite) irreducible (reduced) root system Φ. This inner product will be denoted by $(\ |\ )$ and the associated Euclidean norm by $\|\ \ \|$. It is characterized by being W-invariant and satisfying the normalization $\|\rho+\theta\|^2 - \|\rho\|^2 = 1$ where ρ is half the sum of positive roots and θ is the highest root.

Remark The canonical inner product is the restriction to $\mathfrak{h}^*_{\mathbb{R}}$ of the symmetric bilinear form dual to the Killing form of $\mathfrak{g}$. There are several alternative descriptions of the same normalization. Here is a short list.

$\|\rho+\theta\|^2 - \|\rho\|^2 = 1$, i. e., the eigenvalue of the Casimir operator associated to the Killing form is 1 for the adjoint representation;
$\|\theta\|^{-2} = g$, the dual Coxeter number;
$\|\rho\|^2 = \frac{1}{24}\dim\mathfrak{g}$, the "strange formula" of Freudenthal and de Vries;
$\sum\limits_{\varphi\in\Phi} \|\varphi\|^2 = \operatorname{rank}\mathfrak{g}$, a formula due to G. Brown;
$\|\theta\|^2 + \sum\limits_{i=1}^{l} n_i\,\|\alpha_i\|^2 = 1$, where $n_1,\dots,n_l$ are the marks and $\alpha_1,\dots,\alpha_l$ the simple roots. (The formula looks funnier if one substitutes $\sum\limits_{i=1}^{l} n_i\,\alpha_i$ for θ.) One can show the formula by writing $\|\theta\|^{-2} = g$ and using the connexion between the Coxeter number and the dual Coxeter number. Another derivation will be given in the remark beginning on page 211.

Definition For $\Psi \subseteq \Phi_+$ we define $\langle\Psi\rangle := \sum\limits_{\varphi\in\Psi}\varphi$.

Lemma 1 (Kostant) *Let $\Psi_i \subseteq \Phi_+$ with $\Psi_i \dot{+} \Phi_+ \subseteq \Psi_i$ $(i = 1, 2)$ (ideals) such that $\langle\Psi_1\rangle = \langle\Psi_2\rangle$. Then $\Psi_1 = \Psi_2$.*

Proof Let $\Psi := \Psi_1 \cap \Psi_2$. Assume to the contrary that $\Psi_1 \neq \Psi_2$. Then since $\langle\Psi_1\rangle = \langle\Psi_2\rangle$ both $\Psi_1 - \Psi$ and $\Psi_2 - \Psi$ are nonempty. Let $\varphi_i \in \Psi_i - \Psi$ $(i = 1, 2)$. We must have $(\varphi_1|\varphi_2) \leqslant 0$. Otherwise $\varphi_1 - \varphi_2$ would be a root which can be assumed positive by possibly interchanging the indices 1 and 2. By the ideal property $\Psi_i \dot{+} \Phi_+ \subseteq \Psi_i$ we then have $\varphi_1 = \varphi_2 + (\varphi_1 - \varphi_2) \in \Psi_2$, a contradiction. Thus $(\varphi_1|\varphi_2) \leqslant 0$. Hence since $\langle\Psi_1 - \Psi\rangle = \langle\Psi_2 - \Psi\rangle$ we obtain
$$0 \leqslant \big\|\langle\Psi_i - \Psi\rangle\big\|^2 = \big(\langle\Psi_1 - \Psi\rangle \bigm| \langle\Psi_2 - \Psi\rangle\big) \leqslant 0$$
and so $\Psi = \Psi_1 = \Psi_2$. □

Like the previous lemma the following theorem is due to Kostant and was published in 1965.

Theorem 2 (Kostant) *Let $\Psi \subseteq \Phi_+$ be a set of positive roots. Further let $\mathfrak{a}_\Psi := \bigoplus\limits_{\varphi\in\Psi}\mathfrak{g}_\varphi \subseteq \mathfrak{b}$ be the corresponding subspace. Then one always has the inequality*
$$\Big\|\rho + \sum_{\varphi\in\Psi}\varphi\Big\|^2 - \|\rho\|^2 \leqslant |\Psi|$$

with equality if and only if $\mathfrak{a}_\Psi$ is an abelian ideal in $\mathfrak{b}$ (and every abelian ideal in $\mathfrak{b}$ is of this form).

In particular, one recovers the normalization $\|\rho+\theta\|^2 - \|\rho\|^2 = 1$ because the root space $\mathfrak{g}_\theta \trianglelefteq \mathfrak{b}$ is an abelian ideal in $\mathfrak{b}$ (the unique 1-dimensional one).

Reflections and Weyl groups. Each sum $\rho + \sum_{\varphi\in\Psi} \varphi = \rho + \langle\Psi\rangle$ that occurs in Kostant's theorem (Theorem 2) and such that $\mathfrak{a}_\Psi$ is an abelian ideal in $\mathfrak{b}$ will be shown to be of the form $\rho + \langle\Psi\rangle = \widehat{w}\rho$ for some element $\widehat{w}$ in the affine Weyl group $\widehat{W}$. Here, the affine Weyl group is the group of affine isometries of $\mathfrak{h}_{\mathbb{R}}^*$ generated by the finite Weyl group W—which is itself generated by the simple reflections $s_1, \ldots, s_l$ along the simple roots $\alpha_1, \ldots, \alpha_l$, that is,

$$s_i : \lambda \longmapsto \lambda - \frac{2\,(\lambda|\alpha_i)}{(\alpha_i|\alpha_i)}\,\alpha_i = \lambda - \langle\lambda, \alpha_i^\vee\rangle\,\alpha_i$$

—and, in addition, the affine reflection

$$s_0 : \lambda \mapsto \lambda - \left(\frac{2\,(\lambda|\theta)}{(\theta|\theta)} - g\right)\theta = \lambda - \big(\langle\lambda, \theta^\vee\rangle - g\big)\theta = s_\theta\lambda + g\theta. \quad (1)$$

Here, $\langle\ ,\ \rangle : \mathfrak{h}_{\mathbb{R}}^* \times \mathfrak{h}_{\mathbb{R}} \to \mathbb{R}$ stands for the natural pairing; $\alpha_1^\vee, \ldots, \alpha_l^\vee, \theta^\vee$ are the coroots corresponding to $\alpha_1, \ldots, \alpha_l, \theta$. More generally, for any root $\varphi \in \Phi$ the corresponding coroot $\varphi^\vee \in \mathfrak{h}_{\mathbb{R}}$ is defined by

$$\langle\lambda, \varphi^\vee\rangle = \frac{2\,(\lambda|\varphi)}{(\varphi|\varphi)} \quad \forall\,\lambda \in \mathfrak{h}_{\mathbb{R}}^*.$$

The affine Weyl group $\widehat{W}$ is a Coxeter group with Coxeter generators $s_0, \ldots, s_l$. Let $\ell : \widehat{W} \to \mathbb{Z}_{\geqslant 0}$ be the usual length function, that is, $\ell(\widehat{w}) = r$ if $\widehat{w} = s_{i_1} \ldots s_{i_r}$ with $i_1, \ldots, i_r \in \{0, \ldots, l\}$ and r minimal. Similarly, denoting again by $\ell : W \to \mathbb{Z}_{\geqslant 0}$ the length function of the parabolic subgroup $W \subseteq \widehat{W}$, one knows that it coincides with the restriction of the length function of $\widehat{W}$.

Remark The definition of the affine Weyl group is not exactly the standard but a scaled one and has the effect that $s_0\rho = \rho+\theta$. One has the well-known decomposition $\widehat{W} \cong gM \rtimes W$ of $\widehat{W}$ as a semidirect product of W acting on the normal subgroup gM, the lattice spanned by the long roots and dilated by the factor g, in the obvious way. Each element $\mu \in gM$ acts as the translation $\lambda \mapsto \lambda + \mu$.

There is of course also the linear version of the affine Weyl group acting on $\mathfrak{h}_{\mathbb{R}}^* \oplus \mathbb{R}\delta \oplus \mathbb{R}\Lambda_0$ as in Kac's book [Kac]. One extends the inner product in $\mathfrak{h}_{\mathbb{R}}^*$ to a nondegenerate symmetric bilinear form, again denoted $(\ |\ )$, by declaring that δ and Λ_0 are isotropic vectors perpendicular to $\mathfrak{h}_{\mathbb{R}}^*$ and such that $(\delta|\Lambda_0) = 1$. By some slight abuse of notation we define the reflections $s_0, \ldots, s_l \in \mathrm{O}\big(\mathfrak{h}_{\mathbb{R}}^* \oplus \mathbb{R}\delta \oplus \mathbb{R}\Lambda_0, (\ |\ )\big)$ by the formula

$$s_i : \lambda \longmapsto \lambda - \frac{2\,(\lambda|\alpha_i)}{(\alpha_i|\alpha_i)}\,\alpha_i,$$

where $\alpha_1, \ldots, \alpha_l$ are the simple roots as usual and $\alpha_0 = \delta - \theta$. The group generated by $s_0, \ldots, s_l$ will again by abuse of notation be denoted by $\widehat{W}$. Each affine hyperplane $\mathfrak{h}_{\mathbb{R}}^* \oplus \mathbb{R}\delta + c\Lambda_0$ is mapped onto itself by the reflections $s_0, \ldots, s_l$. The action of $\widehat{W}$ on $\mathfrak{h}_{\mathbb{R}}^*$ defined previously comes from the action of $\widehat{W}$ on the subquotient $\mathfrak{h}_{\mathbb{R}}^* \oplus \mathbb{R}\delta + \frac{1}{2}\Lambda_0 \pmod{\mathbb{R}\delta}$ if one identifies this subquotient with $\mathfrak{h}_{\mathbb{R}}^*$ in the evident way. In fact, for $\varphi \in \Phi_+$ and $\lambda \in \mathfrak{h}_{\mathbb{R}}^* \oplus \mathbb{R}\delta$ we compute

$$\begin{aligned} s_{\delta-\varphi}\big(\lambda + \tfrac{1}{2}\Lambda_0\big) &= \lambda + \tfrac{1}{2}\Lambda_0 - \frac{2\left(\lambda + \frac{1}{2}\Lambda_0|\delta-\varphi\right)}{(\delta-\varphi|\delta-\varphi)}\,(\delta-\varphi) \\ &\in \lambda + \tfrac{1}{2}\Lambda_0 - \frac{2\,(\lambda|\varphi)}{(\varphi|\varphi)}\,\varphi + \frac{1}{(\varphi|\varphi)}\,\varphi + \mathbb{R}\delta \\ &= s_\varphi\lambda + \|\varphi\|^{-2}\,\varphi + \mathbb{R}\delta + \tfrac{1}{2}\Lambda_0, \end{aligned}$$

which for $\varphi = \theta$ reduces to the formula (1). Let us also pin down the expression for the affine action of $s_{\delta-\varphi}$ on $\mathfrak{h}_{\mathbb{R}}^*$, namely,

$$s_{\delta-\varphi}\lambda = s_\varphi\lambda + \|\varphi\|^{-2}\,\varphi. \tag{2}$$

Lemma 3 *Let $\varphi \in \Phi_+$ be a positive root. Then $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$ satisfies*

$$\|\lambda + \varphi\|^2 - \|\rho\|^2 = \|\lambda\|^2 - \|\rho\|^2 + 1 \tag{3}$$

if and only if $\lambda + \varphi = s_{\delta-\varphi}\lambda$.

Proof The equation (3) is equivalent to $2\,(\lambda|\varphi) + \|\varphi\|^2 = 1$. Now by (2)

$$s_{\delta-\varphi}\lambda = s_\varphi\lambda + \|\varphi\|^{-2}\,\varphi = \lambda - \frac{2\,(\lambda|\varphi) - 1}{\|\varphi\|^2}\,\varphi.$$

Hence $s_{\delta-\varphi}\lambda = \lambda + \varphi$ is equivalent to $2\,(\lambda|\varphi) + \|\varphi\|^2 = 1$. □

The fundamental weights $\varpi_1, \ldots, \varpi_l \in \mathfrak{h}_{\mathbb{R}}^*$ are the basis dual to the basis $\alpha_1^\vee, \ldots, \alpha_l^\vee$ of $\mathfrak{h}_{\mathbb{R}}$. Also recall that $\rho = \sum_{i=1}^{l} \varpi_i$. The next definition is slightly non-standard: define $\overset{\vee}{\varpi}_1, \ldots, \overset{\vee}{\varpi}_l \in \mathfrak{h}_{\mathbb{R}}^*$ by $\big(\overset{\vee}{\varpi}_i\big|\alpha_j\big) = \delta_{ij}\,\frac{1}{2}$, that is, $\overset{\vee}{\varpi}_i = \frac{1}{\|\alpha_i\|^2}\,\varpi_i$.

The mapping $\widehat{W} \to \mathfrak{h}_{\mathbb{R}}^*$, $\widehat{w} \mapsto \widehat{w}\rho$ is injective. Its image will be termed the set of **ρ-points**. Let

$$\begin{aligned} A &= \left\{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \;\middle|\; (\lambda|\alpha) \geqslant 0 \text{ for all } \alpha \in \Pi \text{ and } \langle\lambda, \theta^\vee\rangle \leqslant g\right\} \\ &= \left\{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \;\middle|\; 0 \leqslant (\lambda|\varphi) \leqslant \tfrac{1}{2} \text{ for all } \varphi \in \Phi_+\right\} \end{aligned} \tag{4}$$

be the (closed) fundamental alcove, which is a fundamental domain for $\widehat{W}$ acting on $\mathfrak{h}_{\mathbb{R}}^*$. The fundamental alcove A is the simplex whose vertices are $0, \frac{\overset{\vee}{\varpi}_1}{n_1}, \dots, \frac{\overset{\vee}{\varpi}_l}{n_l}$ where $n_1, \dots, n_l$ are the marks, i. e., the (positive integer) coefficients in $\theta = \sum_{i=1}^{l} n_i \alpha_i$.

The cone with apex 0 spanned by A is the dominant chamber

$$C = \left\{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \;\middle|\; (\lambda|\alpha) \geqslant 0 \text{ for all } \alpha \in \Pi\right\}. \tag{5}$$

It is a fundamental domain for the finite Weyl group W.

The $\widehat{W}$-translates of the fundamental alcove are called alcoves. For each $i = 0, \dots, l$ one has the (affine if $i = 0$) hyperplane

$$H_i := \mathrm{Fix}(s_i) = \left\{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \;\middle|\; s_i\lambda = \lambda\right\},$$

and their $\widehat{W}$-translates are termed walls. They are the fixed point sets of a reflection in $\widehat{W}$, that is, of the form

$$H_{k\delta-\varphi} = \mathrm{Fix}(s_{k\delta-\varphi}) = \left\{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \;\middle|\; (\lambda|\varphi) = \tfrac{k}{2}\right\}$$

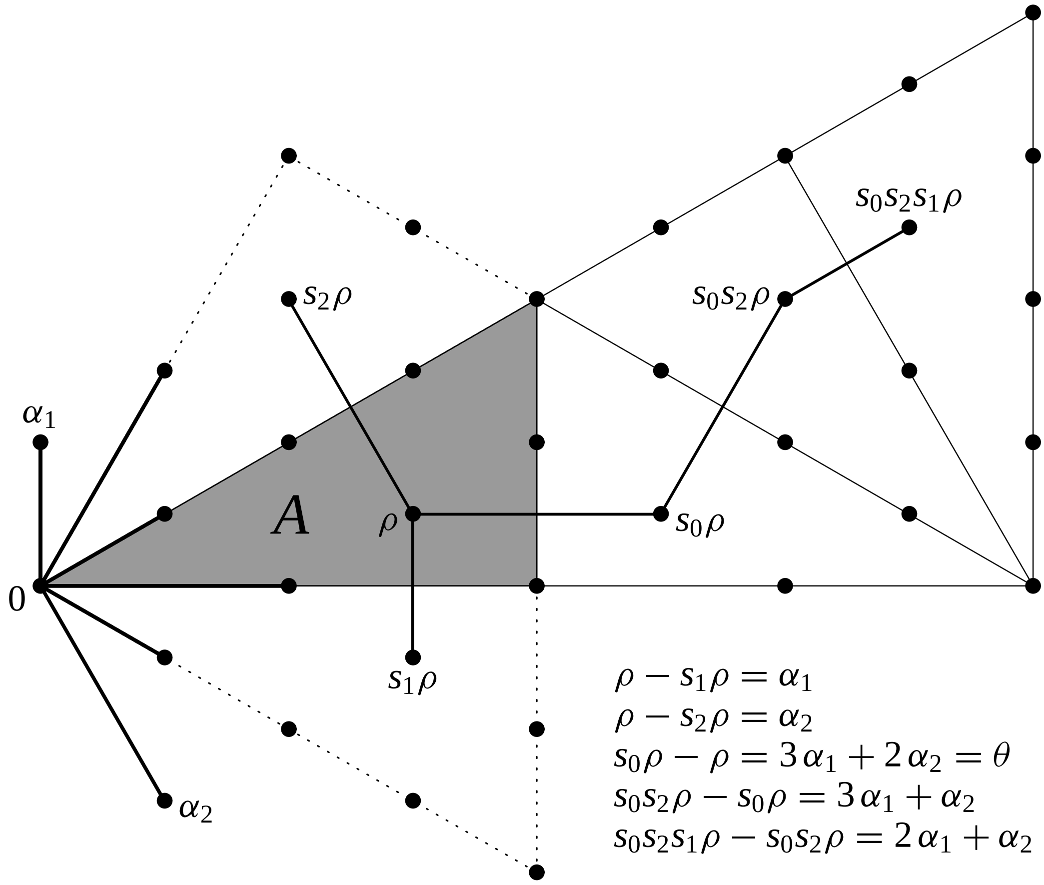
for some $k \in \mathbb{Z}$ and $\varphi \in \Phi_+$ (note that $H_{k\delta-\varphi} = H_{-(k\delta-\varphi)}$).

The ρ-points are precisely the integral weights in the interior of an alcove. So there are the natural bijections

$$\begin{array}{ccccc} \widehat{W} & \xleftrightarrow{\cong} & \{\text{alcoves}\} & \xleftrightarrow{\cong} & \{\rho\text{-points}\} \\ \widehat{w} & \longleftrightarrow & \widehat{w}A & \longleftrightarrow & \widehat{w}\rho. \end{array}$$

We have already mentioned above that $\rho + \theta$ is the ρ-point of the alcove s_0A. The ρ-points of the other neighbours $s_1A, \dots, s_lA$ of the fundamental alcove are $\rho - \alpha_1, \dots, \rho - \alpha_l$. This follows because s_i (for $i = 1, \dots, l$) permutes all positive roots other than α_i and $s_i(\alpha_i) = -\alpha_i$.

The following picture shows a small part of the tessellation of the plane by alcoves for type G_2. The shaded region marks the fundamental alcove A. The boundaries of the four alcoves in $2A = \{2\lambda \mid \lambda \in A\}$ are drawn in solid lines.



Let us recall that there is a close connexion between reduced expressions for elements $\widehat{w} \in \widehat{W}$ and minimal galleries going from A to $\widehat{w}A$. In fact, in general, one has the bijection

$$\begin{aligned} \{\text{words in } s_0, \ldots, s_l\} &\overset{\cong}{\longleftrightarrow} \{\text{(non-stuttering) galleries beginning at } A\} \\ s_{i_1}s_{i_2}\ldots s_{i_k} &\longleftrightarrow A,\ s_{i_1}A,\ s_{i_1}s_{i_2}A,\ \ldots,\ s_{i_1}s_{i_2}\ldots s_{i_k}A, \end{aligned}$$

and reduced words correspond to minimal galleries. Note that $\widehat{w}A$ and $\widehat{w}s_iA$ are adjacent alcoves with a common facet[1] of type i. Instead of keeping track of the types of the facets where adjacent alcoves meet, one can also specify the list of separating walls in a gallery. The two alcoves $\widehat{w}A$ and $\widehat{w}s_iA = s_{\widehat{w}\alpha_i}\widehat{w}A$ are separated by the wall $H_{\widehat{w}\alpha_i} = \mathrm{Fix}(s_{\widehat{w}\alpha_i})$. (Surely, we must consider the linear action of the affine Weyl group when we write $\widehat{w}\alpha_i$. So $\widehat{w}\alpha_i \in \widehat{\Phi} = \Phi + \mathbb{Z}\delta$, the set of so-called real affine roots.) It is well-known that the length $\ell(\widehat{w})$ is the number of walls which separate A from $\widehat{w}A$.

Lemma 4 *Let $\mathfrak{a}_0 \subseteq \mathfrak{a}$ be abelian ideals in $\mathfrak{b}$. Then there is a flag*

$$\mathfrak{a}_0 \subseteq \mathfrak{a}_1 \subseteq \cdots \subseteq \mathfrak{a}_m = \mathfrak{a}$$

[1] We follow the traditional terminology which speaks of “facets” for “faces of codimension one”. The terminology in French is “face” for “facette de codimension une”.

of abelian ideals in $\mathfrak{b}$ *such that*

$$\dim \mathfrak{a}_k = \dim \mathfrak{a}_0 + k \qquad (k = 0, \ldots, m).$$

Proof There is nothing to prove if $m = 0$. Let us denote by $\Phi(\mathfrak{a}_0) \subseteq \Phi(\mathfrak{a})$ the sets of weights of $\mathfrak{a}_0$ and $\mathfrak{a}$. Suppose now that $m > 0$ and take a root $\varphi \in \Phi(\mathfrak{a}) - \Phi(\mathfrak{a}_0)$ of minimal height. (Recall that the height $\mathrm{ht}(\varphi)$ of a root $\varphi = \sum_{i=1}^{l} c_i\, \alpha_i$ is defined to be $\mathrm{ht}(\varphi) = \sum_{i=1}^{l} c_i$, its coefficient sum with respect to the basis of simple roots. The simple roots are those having height 1, and the highest root θ is the root whose height is maximal, namely, $\mathrm{ht}(\theta) = h - 1$, one less than the Coxeter number of $\mathfrak{g}$.) Now let $\mathfrak{a}_{m-1}$ be the sum of root spaces such that $\mathfrak{a}_m = \mathfrak{a}_{m-1} \oplus \mathfrak{g}_\varphi$. The choice of φ guarantees that $\mathfrak{a}_{m-1}$ is again an ideal (of course an abelian one) in $\mathfrak{b}$. This completes the proof by induction. □

Proposition 5 *Let* $\mathfrak{a} \trianglelefteq \mathfrak{b}$ *be an abelian ideal and* $\Phi(\mathfrak{a})$ *its set of weights. Then* $\rho + \langle \Phi(\mathfrak{a}) \rangle$ *is a* ρ*-point,* $\rho + \langle \Phi(\mathfrak{a}) \rangle = \widehat{w}\rho$ *with* $\ell(\widehat{w}) = \dim \mathfrak{a}$*. Moreover,* $\rho + \langle \Phi(\mathfrak{a}) \rangle \in 2A$.

Proof Let $d = \dim \mathfrak{a}$ and $\Phi(\mathfrak{a}) = \{\varphi_1, \ldots, \varphi_d\}$. According to Lemma 4 we can assume that the enumeration of the roots is such that $\mathfrak{g}_{\varphi_1} \oplus \cdots \oplus \mathfrak{g}_{\varphi_k}$ is an abelian ideal for each $k = 0, \ldots, d$, and hence

$$\|\rho + \varphi_1 + \cdots + \varphi_k\|^2 - \|\rho\|^2 = k$$

by Kostant's theorem (Theorem 2). Applying Lemma 3 d times, we conclude that

$$\rho + \langle \Phi(\mathfrak{a}) \rangle = s_{\delta-\varphi_d} \ldots s_{\delta-\varphi_1} \rho \tag{6}$$

is a ρ-point. The walls that separate A from $s_{\delta-\varphi_d} \ldots s_{\delta-\varphi_1} A$ are exactly $H_{\delta-\varphi_1}, \ldots, H_{\delta-\varphi_d}$. Hence $\ell(s_{\delta-\varphi_d} \ldots s_{\delta-\varphi_1}) = d = \dim \mathfrak{a}$.

It is clear from formula (6) that $\rho + \langle \Phi(\mathfrak{a}) \rangle \in 2A$. In fact,

$$A,\ s_{\delta-\varphi_1} A,\ s_{\delta-\varphi_2} s_{\delta-\varphi_1} A, \ldots,\ s_{\delta-\varphi_d} \ldots s_{\delta-\varphi_1} A$$

is a gallery starting with A, and the common walls are all different from the walls $2H_0, H_1, \ldots, H_l$ which bound $2A$.

We shall give a different proof that $\rho + \langle \Phi(\mathfrak{a}) \rangle \in 2A$. Let us first show that $\rho + \langle \Phi(\mathfrak{a}) \rangle$ lies in the dominant chamber. Consider a minimal gallery from the fundamental alcove A to the alcove containing $\rho + \langle \Phi(\mathfrak{a}) \rangle$. If the latter would lie outside C, then the gallery would contain two adjacent alcoves A' (inside C) and A'' (outside C). But then the ρ-points ρ' of A' and ρ'' of A'' would satisfy $\|\rho'\| = \|\rho''\|$ which contradicts Kostant's theorem (Theorem 2).

Finally we show that the ρ-point $\rho + \langle \Phi(\mathfrak{a}) \rangle$ and the origin lie on the same side of the wall $2H_0 = H_{2\delta-\theta} = \left\{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \;\middle|\; (\lambda|\theta) = 1\right\}$. We give more than one argument. Consider a minimal gallery from the fundamental alcove to the alcove that contains $\rho + \langle \Phi(\mathfrak{a}) \rangle$. If the gallery would cross the wall $2H_0$, say ρ' (with $(\rho'|\theta) < 1$) and ρ'' (with $(\rho''|\theta) > 1$) are the ρ-points of two adjacent alcoves with a common facet in $2H_0$, then $\rho' + \theta = \rho''$. But the root θ already occurred right at the beginning as $\rho + \theta = s_0\rho$, and surely $\rho' \neq \rho$. Alternatively one can make a contradiction with Kostant's theorem (Theorem 2). Still another way to show that $\big(\rho + \langle \Phi(\mathfrak{a}) \rangle \bigm| \theta\big) < 1$ is to invoke Proposition 11. □

Lemma 6 *Let H be a wall which cuts $2A$ into two connected components. Then there is a positive root $\varphi \in \Phi_+$ such that*

$$H = H_{\delta-\varphi} = \mathrm{Fix}(s_{\delta-\varphi}) = \left\{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \;\middle|\; (\lambda|\varphi) = \tfrac{1}{2}\right\}.$$

Proof Recall that the fundamental alcove A (see (4)) can be written as the intersection

$$A = \bigcap_{\varphi\in\Phi_+} \left\{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \;\middle|\; 0 \leqslant (\lambda|\varphi) \leqslant \tfrac{1}{2}\right\}$$

of the strips bounded by the walls H_φ and $H_{\delta-\varphi}$. Hence $2A$ is the intersection of the strips bounded by the walls H_φ and $2H_{\delta-\varphi} = H_{2\delta-\varphi}$ where φ runs through the positive roots. The lemma follows as any wall is the fixed point set $H_{k\delta-\varphi}$ of some reflection $s_{k\delta-\varphi}$ with $k \in \mathbb{Z}$ and $\varphi \in \Phi_+$. □

Theorem 7 *The mapping*

$$\begin{aligned} \{\textit{abelian ideals in } \mathfrak{b}\} &\longrightarrow \{\rho\textit{-points in } 2A\} \\ \mathfrak{a} &\longmapsto \rho + \langle \Phi(\mathfrak{a}) \rangle \end{aligned}$$

($\langle \Phi(\mathfrak{a}) \rangle$ is the sum of the weights of $\mathfrak{a}$) is a bijection.

Proof Proposition 5 shows that the mapping is well-defined, and it is injective by Lemma 1. Hence we are left with proving surjectivity. Let $\rho' \in 2A$ be a ρ-point and A' its alcove. The case where $A' = A$ is clear, so suppose that $A' \neq A$. Consider a minimal gallery from A to A'. Surely, the alcoves of such a gallery all belong to $2A$. Lemma 6 tells us that the next-to-last alcove in the gallery can be written as $s_{\delta-\varphi}A'$ for some positive root $\varphi \in \Phi_+$. Moreover, $s_{\delta-\varphi}\rho' + \varphi = \rho' = s_{\delta-\varphi}(s_{\delta-\varphi}\rho')$ because $s_{\delta-\varphi}A'$ and A' have a common facet lying in the wall $\mathrm{Fix}(s_{\delta-\varphi}) = H_{\delta-\varphi}$. Now

$$\|\rho'\|^2 - \|\rho\|^2 = \|s_{\delta-\varphi}\rho'\|^2 - \|\rho\|^2 + 1$$

by Lemma 3. The conclusion follows from Theorem 2 by induction. □

Since $2A$ is the union of 2^l alcoves, hence contains 2^l ρ-points, Peterson's theorem about the number of abelian ideals follows as a corollary, as already noted by Cellini and Papi [CP1, Theorem 2.9].

Corollary 8 (Peterson) *The number of abelian ideals in $\mathfrak{b}$ is 2^l.*

Our next corollary is a consequence of Theorem 7, Proposition 5, and Theorem 2.

Corollary 9 *If $\widehat{w} \in \widehat{W}$ satisfies $\widehat{w}\rho \in 2A$, then $\|\widehat{w}\rho\|^2 - \|\rho\|^2 = \ell(\widehat{w})$.*

3. Explicit description of the abelian ideals

Our approach to describing the abelian ideals hinges on the observation that for each abelian ideal $\mathfrak{a} \trianglelefteq \mathfrak{b}$ its subspace $\mathfrak{a}^{\not\perp\theta}$ spanned by the root spaces for the roots that are not perpendicular to the highest root θ is again an abelian ideal in $\mathfrak{b}$. This is the content of the next proposition.

Proposition 10 *Let $\mathfrak{a} \trianglelefteq \mathfrak{b}$ be an abelian ideal and $\Phi(\mathfrak{a}) \subseteq \Phi_+$ its set of weights. Then*

$$\Phi^{\not\perp\theta}(\mathfrak{a}) := \left\{\varphi \in \Phi(\mathfrak{a}) \;\middle|\; (\varphi|\theta) > 0\right\}$$

is also the set of weights for an abelian ideal $\mathfrak{a}^{\not\perp\theta} \trianglelefteq \mathfrak{b}$:

$$\Phi^{\not\perp\theta}(\mathfrak{a}) = \Phi(\mathfrak{a}^{\not\perp\theta}) \left(= \Phi^{\not\perp\theta}(\mathfrak{a}^{\not\perp\theta})\right).$$

Proof The abelianess is clear: $\Phi^{\not\perp\theta}(\mathfrak{a}) \dotplus \Phi^{\not\perp\theta}(\mathfrak{a}) \subseteq \Phi(\mathfrak{a}) \dotplus \Phi(\mathfrak{a}) = \varnothing$.

That the ideal property holds is also easy to show. For this we must see that $\Phi^{\not\perp\theta}(\mathfrak{a}) \dotplus \Phi_+ \subseteq \Phi^{\not\perp\theta}(\mathfrak{a})$. Let $\varphi \in \Phi^{\not\perp\theta}(\mathfrak{a})$ and $\varphi' \in \Phi_+$. Of the four a priori possibilities (1) $\varphi + \varphi' \notin \Phi_+$, (2) $\varphi + \varphi' \in \Phi_+ - \Phi(\mathfrak{a})$, (3) $\varphi + \varphi' \in \Phi(\mathfrak{a}) - \Phi^{\not\perp\theta}(\mathfrak{a})$, (4) $\varphi + \varphi' \in \Phi^{\not\perp\theta}(\mathfrak{a})$, we must exclude the cases (2) and (3). That (2) is impossible follows from $\Phi^{\not\perp\theta}(\mathfrak{a}) \subseteq \Phi(\mathfrak{a})$ and the fact that $\mathfrak{a}$ is an ideal. Case (3) cannot occur because $(\varphi'|\theta) \geqslant 0$ since θ lies in the dominant chamber and hence $(\varphi + \varphi'|\theta) > 0$ by the definition of $\Phi^{\not\perp\theta}(\mathfrak{a})$. □

Proposition 11 *Let $\mathfrak{a} \trianglelefteq \mathfrak{b}$ be an abelian ideal. The cardinality of the set $\Phi^{\not\perp\theta}(\mathfrak{a})$ is at most $g - 1$.*

Proof The number of positive roots that are not orthogonal to θ is $2g - 3$ [Su1, Proposition 1]. Consider the involution $-s_\theta$ on the set of positive roots that are not orthogonal to θ. Its only fixed point is θ, and $-s_\theta\varphi = \theta - \varphi$ for $\varphi \neq \theta$. The abelianess implies that

$$\left|\{\varphi, -s_\theta\varphi\} \cap \Phi^{\not\perp\theta}(\mathfrak{a})\right| \leqslant 1,$$

and the conclusion follows.

Alternatively, the result follows from Proposition 5. □

Now the problem of describing the abelian ideals in $\mathfrak{b}$ decomposes into two problems according to the disjoint union decomposition

$$\left\{\text{abelian ideals } \mathfrak{a} \trianglelefteq \mathfrak{b}\right\} = \coprod_{\mathfrak{a}'} \left\{\text{abelian ideals } \mathfrak{a} \trianglelefteq \mathfrak{b} \text{ with } \mathfrak{a}^{\not\perp\theta} = \mathfrak{a}'\right\}.$$

The two tasks are

(1) describe the index set $\left\{\mathfrak{a}' \,\middle|\, \mathfrak{a}' \trianglelefteq \mathfrak{b} \text{ abelian ideal with } \mathfrak{a}' = \mathfrak{a}'^{\not\perp\theta}\right\}$;
(2) for each $\mathfrak{a}' = \mathfrak{a}'^{\not\perp\theta}$ describe the set of abelian ideals $\mathfrak{a}$ with $\mathfrak{a}' = \mathfrak{a}^{\not\perp\theta}$.

We will first deal with task (1) and show that there is a canonical one-to-one correspondence

$$\left\{\mathfrak{a}^{\not\perp\theta} \,\middle|\, 0 \neq \mathfrak{a} \trianglelefteq \mathfrak{b} \text{ abelian ideal}\right\} \overset{\cong}{\longleftrightarrow} \Phi_+^{\text{long}}$$

(see Theorems 15 and 17 below). This will then extend and give an a priori explanation for the observation that the maximal abelian ideals are in canonical one-to-one correspondence with the long simple roots, as it was recorded in [PR].

We need some preparation. A modification of the height function will be important. We define the affine functional $L : \mathfrak{h}_{\mathbb{R}}^* \to \mathbb{R}$ by

$$L(\varphi) := \frac{2\,(\theta - \varphi|\rho)}{(\theta|\theta)}\,. \tag{7}$$

Whereas $\operatorname{ht}(\varphi) > 0$ for $\varphi \in \Phi_+$ and $\operatorname{ht}(\varphi) < 0$ for $\varphi \in \Phi_-$, we have $L(\varphi) \geqslant 0$ for all $\varphi \in \Phi$; more precisely, $L(\theta) = 0$ and $L(\varphi) > 0$ for all $\varphi \in \Phi - \{\theta\}$. A second difference concerns the root lengths. Let us write again $\varphi = \sum_{i=1}^{l} c_i\,\alpha_i$. Then

$$\begin{aligned} L(\varphi) = \frac{2\,(\theta - \varphi|\rho)}{(\theta|\theta)} &= g - 1 - \frac{2}{(\theta|\theta)}\Big(\sum_{i=1}^{l} c_i\,\alpha_i \,\Big|\, \sum_{j=1}^{l} \varpi_j\Big) \\ &= g - 1 - \sum_{i=1}^{l} c_i \frac{(\alpha_i|\alpha_i)}{(\theta|\theta)} \end{aligned}$$

because $(\alpha_i|\varpi_j) = \frac{1}{2}(\alpha_i|\alpha_i)\,\delta_{ij}$. And $L(\varphi) = g - 1 - \langle\rho, \varphi^\vee\rangle \in \mathbb{Z}_{\geqslant 0}$ if φ is a long root. In particular, for $\varphi \in \Phi_+^{\text{long}}$ we have $L(\varphi) = g - 2$ if and only if φ is a long simple root. The affine functional L shows its importance in the following proposition.

Proposition 12 *For each positive long root $\varphi \in \Phi_+^{\text{long}}$ there is a unique Weyl group element $w \in W$ of length $\ell(w) = L(\varphi)$ such that $w\varphi = \theta$ is the highest root. Moreover, $w'\varphi \neq \theta$ for all $w' \in W$ with $\ell(w') < L(\varphi)$.*

Proof We first show the minimality that is expressed in the second sentence. Let s_i be a simple reflection with $s_i\varphi$ positive, too, hence $\varphi \neq \pm\alpha_i$.

We compute

$$
\begin{aligned}
L(\varphi) - L(s_i\varphi) &= \frac{2\,(\theta-\varphi|\rho)}{(\theta|\theta)} - \frac{2\,(\theta-s_i\varphi|\rho)}{(\theta|\theta)} && \text{(by definition (7))}\\
&= -\frac{2\,(\varphi|\rho)}{(\theta|\theta)} + \frac{2\,(\varphi|s_i\rho)}{(\theta|\theta)} && \text{(by orthogonality)}\\
&= -\frac{2\,(\varphi|\rho)}{(\theta|\theta)} + \frac{2\,(\varphi|\rho-\alpha_i)}{(\theta|\theta)} && (s_i \text{ is a simple reflection})\\
&= -\frac{2\,(\varphi|\alpha_i)}{(\theta|\theta)} = -\frac{2\,(\varphi|\alpha_i)}{(\varphi|\varphi)} && (\varphi \text{ is a long root})\\
&= -\langle\alpha_i, \varphi^\vee\rangle \in \{0, \pm 1\}. && (\varphi \neq \pm\alpha_i \text{ is a long root})
\end{aligned}
$$

It follows that $L(\varphi) = L(\varphi) - L(\theta) \leqslant \ell(w)$ if $w\varphi = \theta$.

The same calculation shows that given $\varphi \in \Phi_+^{\text{long}} - \{\theta\}$, there exists a simple reflection s_i such that $L(\varphi) - L(s_i\varphi) = 1$. Otherwise, by the previous computation, we would get $\langle\alpha_i, \varphi^\vee\rangle \geqslant 0$ for all $i = 1, \ldots, l$, so that φ would lie in the dominant chamber. This is absurd because θ is the only long dominant root and $\varphi \neq \theta$ by assumption. Hence there is a sequence $s_{i_1}, \ldots, s_{i_{L(\varphi)}}$ of simple reflections with $L(s_{i_k}s_{i_{k-1}} \ldots s_{i_1}\varphi) = L(\varphi) - k$ (for $k = 0, \ldots, L(\varphi)$). In particular, $s_{i_{L(\varphi)}}s_{i_{L(\varphi)-1}} \ldots s_{i_1}\varphi = \theta$.

Uniqueness follows from the uniqueness of coset representatives of minimal length for standard parabolic subgroups. In fact, let $W_{\perp\theta}$ be the standard parabolic subgroup generated by the simple reflections that fix the highest root θ. Its Coxeter-Dynkin graph is the subgraph of the Coxeter-Dynkin graph of W induced by those nodes that are not adjacent to the affine node (corresponding to α_0). The quotient in question is the set of right cosets $W_{\perp\theta}\backslash W$. □

The following table compiles for each long simple root α_i the Weyl group element w with $\ell(w) = g - 2$ and such that $w\alpha_i = \theta$. The labeling coincides with the labeling in the table on pages 197–201.

X	i	w such that $w\alpha_i = \theta$
A_l	i	$s_1 \overset{\nearrow}{\cdots} s_{i-1}\, s_l \overset{\searrow}{\cdots} s_{i+1}$
C_l	l	$s_1 \overset{\nearrow}{\cdots} s_{l-1}$
B_l	i	$s_2 \overset{\nearrow}{\cdots} s_l\, s_1 \overset{\nearrow}{\cdots} s_{i-1}\, s_{l-1} \overset{\searrow}{\cdots} s_{i+1}$ $\quad(i = 1, \ldots, l-1)$
D_l	i	$s_2 \overset{\nearrow}{\cdots} s_{l-2}\, s_1 \overset{\nearrow}{\cdots} s_{i-1}\, s_l \overset{\searrow}{\cdots} s_{i+1}$ $\quad(i = 1, \ldots, l-2)$
	i	$s_2 \overset{\nearrow}{\cdots} s_{l-2}\, s_1 \overset{\nearrow}{\cdots} s_{l-3}\, s_{2l-i-1}\, s_{l-2}$ $\quad(i = l-1, l)$
E_6	1	$s_1\, s_2\, s_3\, s_4\, s_2\, s_5\, s_3\, s_6\, s_4\, s_2$
	2	$s_1\, s_2\, s_3\, s_4\, s_2\, s_1\, s_5\, s_3\, s_6\, s_4$
	3	$s_1\, s_2\, s_3\, s_4\, s_2\, s_1\, s_5\, s_6\, s_4\, s_2$
	4	$s_1\, s_2\, s_3\, s_4\, s_2\, s_1\, s_5\, s_3\, s_2\, s_6$
	5	$s_1\, s_2\, s_3\, s_4\, s_2\, s_1\, s_6\, s_4\, s_2\, s_3$
	6	$s_1\, s_2\, s_3\, s_4\, s_2\, s_1\, s_5\, s_3\, s_2\, s_4$

E_7	1	$s_1\, s_2\, s_3\, s_4\, s_5\, s_3\, s_2\, s_6\, s_4\, s_3\, s_5\, s_7\, s_6\, s_4\, s_3\, s_2$
	2	$s_1\, s_2\, s_3\, s_4\, s_5\, s_3\, s_2\, s_1\, s_6\, s_4\, s_3\, s_5\, s_7\, s_6\, s_4\, s_3$
	3	$s_1\, s_2\, s_3\, s_4\, s_5\, s_3\, s_2\, s_1\, s_6\, s_4\, s_3\, s_2\, s_5\, s_7\, s_6\, s_4$
	4	$s_1\, s_2\, s_3\, s_4\, s_5\, s_3\, s_2\, s_1\, s_6\, s_4\, s_3\, s_2\, s_5\, s_3\, s_7\, s_6$
	5	$s_1\, s_2\, s_3\, s_4\, s_5\, s_3\, s_2\, s_1\, s_6\, s_4\, s_3\, s_2\, s_7\, s_6\, s_4\, s_3$
	6	$s_1\, s_2\, s_3\, s_4\, s_5\, s_3\, s_2\, s_1\, s_6\, s_4\, s_3\, s_2\, s_5\, s_3\, s_4\, s_7$
	7	$s_1\, s_2\, s_3\, s_4\, s_5\, s_3\, s_2\, s_1\, s_6\, s_4\, s_3\, s_2\, s_5\, s_3\, s_4\, s_6$
E_8	1	$s_1\, s_2\, s_3\, s_4\, s_5\, s_6\, s_7\, s_5\, s_4\, s_3\, s_2\, s_8\, s_6\, s_5\, s_4\, s_3\, s_7\, s_5\, s_4\, s_6\, s_5\, s_7\, s_8\, s_6\, s_5\, s_4\, s_3\, s_2$
	2	$s_1\, s_2\, s_3\, s_4\, s_5\, s_6\, s_7\, s_5\, s_4\, s_3\, s_2\, s_1\, s_8\, s_6\, s_5\, s_4\, s_3\, s_7\, s_5\, s_4\, s_6\, s_5\, s_7\, s_8\, s_6\, s_5\, s_4\, s_3$
	3	$s_1\, s_2\, s_3\, s_4\, s_5\, s_6\, s_7\, s_5\, s_4\, s_3\, s_2\, s_1\, s_8\, s_6\, s_5\, s_4\, s_3\, s_2\, s_7\, s_5\, s_4\, s_6\, s_5\, s_7\, s_8\, s_6\, s_5\, s_4$
	4	$s_1\, s_2\, s_3\, s_4\, s_5\, s_6\, s_7\, s_5\, s_4\, s_3\, s_2\, s_1\, s_8\, s_6\, s_5\, s_4\, s_3\, s_2\, s_7\, s_5\, s_4\, s_3\, s_6\, s_5\, s_7\, s_8\, s_6\, s_5$
	5	$s_1\, s_2\, s_3\, s_4\, s_5\, s_6\, s_7\, s_5\, s_4\, s_3\, s_2\, s_1\, s_8\, s_6\, s_5\, s_4\, s_3\, s_2\, s_7\, s_5\, s_4\, s_3\, s_6\, s_5\, s_4\, s_7\, s_8\, s_6$
	6	$s_1\, s_2\, s_3\, s_4\, s_5\, s_6\, s_7\, s_5\, s_4\, s_3\, s_2\, s_1\, s_8\, s_6\, s_5\, s_4\, s_3\, s_2\, s_7\, s_5\, s_4\, s_3\, s_6\, s_5\, s_4\, s_7\, s_5\, s_8$
	7	$s_1\, s_2\, s_3\, s_4\, s_5\, s_6\, s_7\, s_5\, s_4\, s_3\, s_2\, s_1\, s_8\, s_6\, s_5\, s_4\, s_3\, s_2\, s_7\, s_5\, s_4\, s_3\, s_6\, s_5\, s_4\, s_8\, s_6\, s_5$
	8	$s_1\, s_2\, s_3\, s_4\, s_5\, s_6\, s_7\, s_5\, s_4\, s_3\, s_2\, s_1\, s_8\, s_6\, s_5\, s_4\, s_3\, s_2\, s_7\, s_5\, s_4\, s_3\, s_6\, s_5\, s_4\, s_7\, s_5\, s_6$
F_4	1	$s_1\, s_2\, s_3\, s_2\, s_4\, s_3\, s_2$
	2	$s_1\, s_2\, s_3\, s_2\, s_1\, s_4\, s_3$
G_2	2	$s_2\, s_1$

The notations $\overset{\nearrow}{\ldots}$ and $\overset{\searrow}{\ldots}$ mean that one has to interpolate by increasing and decreasing indices, respectively, with the obvious conventions understood, e. g., $s_1 \overset{\nearrow}{\ldots} s_4 = s_1\, s_2\, s_3\, s_4$ and also $s_1 \overset{\nearrow}{\ldots} s_1 = s_1$ and $s_1 \overset{\nearrow}{\ldots} s_0 = 1$ (empty index set).

Remark Note that if $s_{j_1} \ldots s_{j_{g-2}} \alpha_i = \theta$, then for each $m = 1, \ldots, g-2$,

$$\theta - \sum_{k=1}^{m} \frac{\|\theta\|^2}{\|\alpha_{j_k}\|^2}\, \alpha_{j_k}$$

is a positive root (and for $m = g-2$ equals α_i).

Surely, one can count the number of reduced decompositions. We do not elaborate on this point. But let me mention the two relatively recent papers [St1,St2] by Stembridge about some Weyl group combinatorics like fully commutative elements, minuscule elements, and other interesting topics.

Remark The lengths of the Weyl group elements that occurred in Proposition 12 have the polynomial of degree $g-2$

$$q(t) := \sum_{\substack{\varphi \in \Phi_+^{\text{long}} \\ \left\{ \begin{smallmatrix} w\varphi = \theta \text{ with} \\ \ell(w) \text{ minimal} \end{smallmatrix} \right.}} t^{\ell(w)} = \sum_{\varphi \in \Phi_+^{\text{long}}} t^{L(\varphi)}$$

as a generating function. Since $s_i(\alpha_i) = -\alpha_i$, we have the sum

$$\sum_{\substack{\varphi \in \Phi^{\text{long}} \\ \left\{ \begin{smallmatrix} w\varphi = \theta \text{ with} \\ \ell(w) \text{ minimal} \end{smallmatrix} \right.}} t^{\ell(w)} = q(t) + t^{2g-3} q(t^{-1})$$

for all the (positive and negative) long roots, which is the Poincaré polynomial for the set of minimal coset representatives for $W_{\perp\theta}\backslash W$. For more background about this folklore part in the theory see for instance [Win] and references therein. The usual Poincaré polynomial $W(t)$ of W is defined as $W(t) = \sum_{w\in W} t^{\ell(w)}$. It can be expressed by a product formula. Namely, if $m_1, \ldots, m_l$ are the exponents[2] of W (or of its type X), then using the abbreviation $[n] := (1-t^n)/(1-t)$ one can write $W(t) = \prod_{i=1}^{l}[m_i+1]$.

The Poincaré polynomial of $W_{\perp\theta}\backslash W$ is the quotient $W(t)/W_{\perp\theta}(t)$ of the corresponding Poincaré polynomials. The rightmost column in the next table contains the numbers $\nu(\mathsf{X}) =$ the number of positive long roots in a root system of type X. The Poincaré polynomial evaluated at $t = 1$ equals $2\,\nu(\mathsf{X})$.

X	$\mathsf{X}_{\perp\theta}$	exponents of X	$W(t)/W_{\perp\theta}(t)$	$\nu(\mathsf{X})$
A_l	A_{l-2}	$1, 2, \ldots, l$	$[l][l+1]$	$\dfrac{l(l+1)}{2}$
C_l	C_{l-1}	$1, 3, \ldots, 2l-1$	$[2l]$	l
B_l	$\mathsf{B}_{l-2}+\mathsf{A}_1$	$1, 3, \ldots, 2l-1$	$\dfrac{[2l-2][2l]}{[2]}$	$l(l-1)$
D_l	$\mathsf{D}_{l-2}+\mathsf{A}_1$	$1, 3, \ldots, 2l-3, l-1$	$\dfrac{[l][2l-4][2l-2]}{[2][l-2]}$	$l(l-1)$
E_6	A_5	$1, 4, 5, 7, 8, 11$	$\dfrac{[8][9][12]}{[3][4]}$	36
E_7	D_6	$1, 5, 7, 9, 11, 13, 17$	$\dfrac{[12][14][18]}{[4][6]}$	63
E_8	E_7	$1, 7, 11, 13, 17, 19, 23, 29$	$\dfrac{[20][24][30]}{[6][10]}$	120
F_4	C_3	$1, 5, 7, 11$	$\dfrac{[8][12]}{[4]}$	12
G_2	A_1	$1, 5$	$[6]$	3

The usual conventions are employed for the entries in the column marked $\mathsf{X}_{\perp\theta}$, namely, $\mathsf{A}_{-1} = \mathsf{A}_0 = \mathsf{B}_0 = \varnothing$, $\mathsf{C}_1 = \mathsf{B}_1 = \mathsf{A}_1$, $\mathsf{C}_2 = \mathsf{B}_2$, $\mathsf{D}_2 = \mathsf{A}_1 + \mathsf{A}_1$.

Remark In all cases, $W(t)/W_{\perp\theta}(t)$ is of the form $\dfrac{[a][c][e]}{[b][d]}$ according to the table above (since $1 = [1]$). Of course, one can always take $e = h$.

Here is a little numerological table for the types D_4, E_6, E_7, and E_8.

[2] A word about the labeling: the numbers $m_1, \ldots, m_l$ are not naturally associated to the nodes of the Coxeter-Dynkin graph.

X	$r := \frac{h}{6} = \frac{l+2}{10-l}$	$\frac{[a][c][e]}{[b][d]} = \frac{[4r][5r-1][6r]}{[r+1][2r]}$	$\lvert\text{group}\rvert = (r+1)2r$
D_4	1	$\frac{[4][4][6]}{[2][2]}$	$\lvert\,\mathrm{Dih}_2\,\rvert = 2 \cdot 2$
E_6	2	$\frac{[8][9][12]}{[3][4]}$	$\lvert\,\mathrm{Alt}_4\,\rvert = 3 \cdot 4$
E_7	3	$\frac{[12][14][18]}{[4][6]}$	$\lvert\,\mathrm{Sym}_4\,\rvert = 4 \cdot 6$
E_8	5	$\frac{[20][24][30]}{[6][10]}$	$\lvert\,\mathrm{Alt}_5\,\rvert = 6 \cdot 10$

One could extend the table above to the types A_1 and A_2 but without the entries for the last column. The six types A_1, A_2, D_4, E_6, E_7, and E_8 are precisely the simply laced ones in Deligne's family (see [Del] and follow-up papers by various authors). Further numerology pertaining to the types E_6, E_7, E_8 can be found in the paper of Arnold [Arn] about trinities.

For $w \in W$ one defines $\Phi_w := \Phi_+ \cap w\Phi_-$, the set of positive roots which are of the form $w\varphi$ for a negative root φ. The following fundamental lemma is well-known.

Lemma 13 *For a Weyl group element $w \in W$ with reduced decomposition $w = s_{i_1} \dots s_{i_k}$ ($i_j \in \{1, \dots, l\}$) the set Φ_w consists of the k distinct positive roots*

$$\alpha_{i_1},\ s_{i_1}(\alpha_{i_2}),\ s_{i_1}s_{i_2}(\alpha_{i_3}),\ \dots,\ s_{i_1}\dots s_{i_{k-1}}(\alpha_{i_k}).$$

Proof Clearly $\Phi_1 = \varnothing$. One shows by induction that $\Phi_{s_i w} = s_i\Phi_w \cup \{\alpha_i\}$ if $\ell(s_i w) = \ell(w) + 1$ ($\alpha_i \notin \Phi_w$) using the fact that s_i (for $i = 1, \dots, l$) permutes all positive roots other than α_i and $s_i(-\alpha_i) = \alpha_i$. $\square$

Let $s(w) := \langle\Phi_w\rangle$, the sum of the elements in Φ_w. Now the function $s : W \to \mathfrak{h}_{\mathbb{R}}^*$ satisfies the 1-cocycle condition $s(ww') = ws(w') + s(w)$, and in fact, $s(w) = \rho - w\rho$.

Lemma 14 $\ell(s_i w) = \ell(w) \pm 1 \iff w^{-1}\alpha_i \in \Phi_\pm,$
$\ell(ws_i) = \ell(w) \pm 1 \iff w\alpha_i \in \Phi_\pm.$

Theorem 15 *Let $\varphi \in \Phi_+^{\mathrm{long}}$ be a positive long root. Let $w \in W$ be the Weyl group element such that $w\varphi = \theta$ and with $\ell(w) = L(\varphi)$ as in Proposition 12. Then for all $\psi \in \Phi_w$, $\theta - \psi$ is a positive root and*

$$\mathfrak{a}^{\varphi,\min} := \mathfrak{g}_\theta \oplus \bigoplus_{\psi \in \Phi_w} \mathfrak{g}_{\theta-\psi}$$

is an (obviously nonzero) abelian ideal in $\mathfrak{b}$. The ρ-point of the alcove corresponding to $\mathfrak{a}^{\varphi,\min}$ is $s_0 w\rho$.

Proof For the proof we use Kostant's theorem (Theorem 2). First let us write w as a reduced decomposition $w = s_{i_{L(\varphi)}} \dots s_{i_1}$ as in the proof of Proposition 12. Each root $\psi \in \Phi_w$ is of the form $\psi = s_{i_{L(\varphi)}} \dots s_{i_{k+1}}(\alpha_{i_k})$ and we compute

$$\begin{aligned}
\frac{2\,(\psi|\theta)}{(\theta|\theta)} &= \frac{2}{(\theta|\theta)}\big(s_{i_{L(\varphi)}} \dots s_{i_{k+1}}(\alpha_{i_k}) \bigm| s_{i_{L(\varphi)}} \dots s_{i_1}\varphi\big) \\
&= \frac{2}{(\theta|\theta)}\big(\alpha_{i_k} \bigm| s_{i_k} \dots s_{i_1}\varphi\big) = \frac{2}{(\theta|\theta)}\big(-\alpha_{i_k} \bigm| s_{i_{k-1}} \dots s_{i_1}\varphi\big) \\
&= -\big\langle \alpha_{i_k}, (s_{i_{k-1}} \dots s_{i_1}\varphi)^\vee \big\rangle \\
&= L(s_{i_{k-1}} \dots s_{i_1}\varphi) - L(s_{i_k} \dots s_{i_1}\varphi) = 1.
\end{aligned}$$

Hence we have $s_\theta(\psi) = \psi - \theta$ and $\theta - \psi$ is a positive root. Now we put $\ell := \ell(w) = L(\varphi)$ for abbreviation, so that $|\Phi_w| = \ell$. We check that the $\ell + 1$ element set $\Psi := \{\theta\} \cup \{\theta - \psi | \psi \in \Phi_w\} \subseteq \Phi_+$ satisfies Kostant's criterion (Theorem 2) for an abelian ideal. Using $\sum_{\psi \in \Phi_w} \psi = \rho - w\rho$ we have

$$\begin{aligned}
\rho + \langle \Psi \rangle = \rho + \theta + \sum_{\psi \in \Phi_w} (\theta - \psi) &= w\rho + (\ell + 1)\theta = w\rho + \big(L(\varphi) + 1\big)\theta \\
&= w\rho + \big(g - \langle \rho, \varphi^\vee \rangle\big)\theta = w\rho + \big(g - \langle w\rho, (w\varphi)^\vee \rangle\big)\theta
\end{aligned}$$

and because $w\varphi = \theta$ we get

$$= w\rho - \langle w\rho, \theta^\vee \rangle \theta + g\theta = s_0 w\rho.$$

This proves the assertion about the ρ-point. Now we compute

$$\begin{aligned}
\|s_0 w\rho\|^2 - \|\rho\|^2 &= \big\|(\ell + 1)\theta + w\rho\big\|^2 - \|\rho\|^2 \\
&= (\ell + 1)^2 \|\theta\|^2 + (\ell + 1)\, 2\,(\theta | w\rho)
\end{aligned}$$

and with $\ell\,\|\theta\|^2 = 2\,(\theta - \varphi|\rho)$ and $w^{-1}\theta = \varphi$ the calculation continues

$$\begin{aligned}
&= (\ell + 1)\big(2\,(\theta - \varphi|\rho) + \|\theta\|^2 + 2\,(\varphi|\rho)\big) \\
&= (\ell + 1)\big(\|\rho + \theta\|^2 - \|\rho\|^2\big) = \ell + 1.
\end{aligned}$$

This completes the proof of the theorem. □

Now it is appropriate to digress and review Peterson's description of the abelian ideals in $\mathfrak{b}$, to see how this fits with Theorem 15, and to observe why this gives an equality of the form $\|\rho + \langle \Psi \rangle\|^2 - \|\rho\|^2 = |\Psi|$.

First we have to extend the sets Φ_w to the affine context. Let us briefly recall how one does this. The affine root system $\widehat{\Phi} = \Phi + \mathbb{Z}\delta$ was already mentioned before. (The so-called imaginary roots $\pm n\delta$ for $n \in \mathbb{Z}_{>0}$ are fixed by $\widehat{W}$ and play no role here, so we disregard them.) Let

$$\widehat{\Phi}_+ := \Phi_+ \cup \big\{\varphi + n\delta \bigm| \varphi \in \Phi,\ n \in \mathbb{Z}_{>0}\big\}$$

and $\widehat{\Phi}_- := -\widehat{\Phi}_+$. We further define for $\widehat{w} \in \widehat{W}$ the set $\widehat{\Phi}_{\widehat{w}} := \widehat{\Phi}_+ \cap \widehat{w}\widehat{\Phi}_-$ of cardinality $\ell(\widehat{w})$. The sum of the elements of $\widehat{\Phi}_{\widehat{w}}$ is $\widehat{\rho} - \widehat{w}\,\widehat{\rho}$ where now $\widehat{\rho} = \rho + \frac{1}{2}\Lambda_0$.

One knows from Peterson's work that $\Psi = \Phi(\mathfrak{a})$ for an abelian ideal $\mathfrak{a} \trianglelefteq \mathfrak{b}$ if and only if the set $\{\delta - \varphi | \varphi \in \Psi\}$ is of the form $\widehat{\Phi}_{\widehat{w}}$. (For an explanation of this fact see equation (6) in the proof of Proposition 5.) The abelian ideal that we constructed in Theorem 15 belongs to

$$\begin{aligned}\widehat{\Phi}_{s_{\delta-\theta}w} &= s_{\delta-\theta}\widehat{\Phi}_w \cup \{\delta - \theta\} = s_{\delta-\theta}\Phi_w \cup \{\delta - \theta\}\\ &= \Big\{\delta - (\theta - \psi) \;\Big|\; \psi \in \Phi_w\Big\} \cup \{\delta - \theta\}.\end{aligned}$$

Now if $\mathfrak{a}$ is an abelian ideal in $\mathfrak{b}$ with $\Psi := \Phi(\mathfrak{a}) = \{\varphi_1, \ldots, \varphi_d\}$, $d = \dim \mathfrak{a}$, then we have $\widehat{\Phi}_{\widehat{w}} = \{\delta - \varphi_1, \ldots, \delta - \varphi_d\}$ for some $\widehat{w} \in \widehat{W}$. From

$$d\delta - \langle\Psi\rangle = (\delta - \varphi_1) + \cdots + (\delta - \varphi_d) = \widehat{\rho} - \widehat{w}\,\widehat{\rho}$$

and since $\widehat{w}$ is orthogonal, $\widehat{w} \in \mathrm{O}\big(\mathfrak{h}^*_{\mathbb{R}} \oplus \mathbb{R}\delta \oplus \mathbb{R}\Lambda_0, (\ \mid\ )\big)$, we get

$$\begin{aligned}0 &= \|\widehat{w}\,\widehat{\rho}\|^2 - \|\widehat{\rho}\|^2 = \|\rho + \langle\Psi\rangle - d\delta + \tfrac{1}{2}\Lambda_0\|^2 - \|\rho + \tfrac{1}{2}\Lambda_0\|^2\\ &= \|\rho + \langle\Psi\rangle\|^2 - d - \|\rho\|^2,\end{aligned}$$

so that in fact

$$\|\rho + \langle\Psi\rangle\|^2 - \|\rho\|^2 = |\Psi|.$$

Lemma 16 *If $w \in W$ satisfies $s_0 w\rho \in 2A$, then $\varphi := w^{-1}\theta$ is a positive long root and $s_0 w\rho$ is the ρ-point of the alcove corresponding to $\mathfrak{a}^{\varphi,\min}$.*

Proof First note that $s_0 w 0 = s_0 0 = g\theta$ is a vertex of the alcove with ρ-point $s_0 w\rho$. The special vertex $g\theta$ is a vertex of $|W|$ alcoves. Now we compute which fraction of them lies in $2A$. By elementary geometry this fraction is the reciprocal of the number of alcoves that contain the point $\frac{1}{2}g\theta$. To compute the number of these alcoves, the standard procedure is to delete from the Coxeter-Dynkin graph of the affine Weyl group $\widehat{W}$ the nodes that are adjacent to the node for the reflection s_0. What remains is the Coxeter-Dynkin graph of the group $\mathrm{gp}(s_0) \times W_{\perp\theta}$ of order $2\,|W_{\perp\theta}|$. So the number of alcoves that lie in $2A$ and have $g\theta$ as a vertex is $\frac{|W|}{2\,|W_{\perp\theta}|} = |\Phi_+^{\mathrm{long}}|$.

Thus the alcoves in $2A$ of the form $s_0 wA$ with $w \in W$ are exactly the alcoves that are associated with the abelian ideals that were constructed in Theorem 15. In particular, $\varphi := w^{-1}\theta$ is a positive long root and $s_0 w\rho$ is the ρ-point of the alcove corresponding to $\mathfrak{a}^{\varphi,\min}$. $\square$

Theorem 17 *The mapping*

$$\begin{aligned}\Phi_+^{\mathrm{long}} &\longrightarrow \Big\{\mathfrak{a}^{\not\perp\theta} \;\Big|\; 0 \neq \mathfrak{a} \trianglelefteq \mathfrak{b} \textit{ abelian ideal}\Big\}\\ \varphi &\longmapsto \mathfrak{a}^{\varphi,\min}\end{aligned}$$

is a bijection.

Proof By Theorem 15 the mapping is well-defined and injective. Suppose that $\mathfrak{a} = \mathfrak{a}^{\not\perp\theta}$ is a nonzero abelian ideal that is not of the form $\mathfrak{a}^{\varphi,\min}$. Let $\mathfrak{a}_1 \subseteq \cdots \subseteq \mathfrak{a}_d = \mathfrak{a}$ be a flag of abelian ideals with $\dim \mathfrak{a}_k = k$. Clearly $\mathfrak{a}_k = \mathfrak{a}_k^{\not\perp\theta}$ because $\mathfrak{a} = \mathfrak{a}^{\not\perp\theta}$. By hypothesis we can choose an index k such that $\mathfrak{a}_k = \mathfrak{a}^{\varphi,\min}$ but $\mathfrak{a}_{k+1}$ is not of the form $\mathfrak{a}^{\psi,\min}$. By taking into account Lemma 16 we can write

$$\rho + \langle \Phi(\mathfrak{a}_k)\rangle = s_0 w\rho,$$
$$\rho + \langle \Phi(\mathfrak{a}_{k+1})\rangle = s_0 w s_0\rho,$$

where $w \in W$, and $\varphi = w^{-1}\theta$ is a positive long root. The difference $s_0ws_0\rho - s_0w\rho$ would have to be a positive root such that

$$\big(s_0ws_0\rho - s_0w\rho \;\big|\; \theta\big) > 0.$$

But on the other hand we have $s_0ws_0\rho - s_0w\rho = s_\theta w(s_0\rho - \rho) = s_\theta w\theta$ and hence

$$\big(s_0ws_0\rho - s_0w\rho \;\big|\; \theta\big) = (s_\theta w\theta|\theta) = -(w\theta|\theta) = -(\theta|\varphi) \leqslant 0$$

because φ is a positive root and θ lies in the dominant chamber. This contradiction completes the proof of the theorem. $\square$

Remark As the notation $\mathfrak{a}^{\varphi,\min}$ suggests there will also be abelian ideals $\mathfrak{a}^{\varphi,\max}$. In fact, each nonzero abelian ideal $\mathfrak{a}$ satisfies $\mathfrak{a}^{\varphi,\min} \subseteq \mathfrak{a} \subseteq \mathfrak{a}^{\varphi,\max}$ for some positive long root φ which is characterized by $\mathfrak{a}^{\not\perp\theta} = \mathfrak{a}^{\varphi,\min}$. If φ is not perpendicular to the highest root θ, then $\mathfrak{a}^{\varphi,\max} = \mathfrak{a}^{\varphi,\min}$.

Let us now look closer at the case where $\varphi \perp \theta$. Before giving the general picture, we state a preliminary result.

Proposition 18 *Let $\varphi \in \Phi_+^{\text{long}}$ and $w \in W$ be as in Theorem 15 and suppose in addition that φ is perpendicular to the highest root θ. Then*

$$\mathfrak{a}^{\varphi,\min^+} := \mathfrak{a}^{\varphi,\min} \oplus \mathfrak{g}_{w\theta}$$

is an abelian ideal in $\mathfrak{b}$.

Proof We first show that $w\theta$ is a positive root perpendicular to θ. In fact, $(w\theta|\theta) = (w\theta|w\varphi) = (\theta|\varphi) = 0$. Hence $w\theta$ is a long root spanned by the simple roots α_i that are perpendicular to θ. (To make this assertion clear, let us write $w\theta = \sum_{i=1}^{l} a_i\,\alpha_i$. Here the coefficients a_i are either all nonnegative or all nonpositive. Now we take the inner product with θ and use $(\alpha_i|\theta) \geqslant 0$ because θ lies in the dominant chamber and α_i is a positive root.) For each such root $\alpha_i \perp \theta$ we have $s_iw\varphi = s_i\theta = \theta$. Hence $\ell(s_iw) \geqslant \ell(w)$ by the minimality of $\ell(w)$. Lemma 14 shows that $w^{-1}\alpha_i \in \Phi_+$. Since θ lies in the dominant chamber, we get $0 \leqslant (w^{-1}\alpha_i|\theta) = (\alpha_i|w\theta)$ for all simple roots $\alpha_i \perp \theta$. This means that $w\theta$ lies in the dominant chamber for the root subsystem $\Phi_{\perp\theta}$ (spanned by the simple roots $\alpha_i \perp \theta$). The root $w\theta$ appears to be the highest root of the φ-component of $\Phi_{\perp\theta}$.

Putting again $\ell := \ell(w)$ we define the set Ψ of cardinality $\ell + 2$ as $\Psi := \{\theta\} \cup \{\theta - \psi | \psi \in \Phi_w\} \cup \{w\theta\} \subseteq \Phi_+$. ($w\theta$ is of course different from the elements $\theta - \psi$ because only $w\theta$ is perpendicular to θ.) Now we employ Kostant's criterion (Theorem 2) as in the proof of Theorem 15. Looking back at the proof there we see that we must show the equality

$$\|(\ell+1)\theta + w\rho + w\theta\|^2 - \|(\ell+1)\theta + w\rho\|^2 = 1.$$

It follows from $\theta \perp w\theta$ and the W-invariance of the inner product together with the identity $\|\rho+\theta\|^2 - \|\rho\|^2 = 1$. $\square$

More minimal coset representatives and Poincaré polynomials. Now we define for each positive root $\varphi \in \Phi_+$ the polynomial $P_\varphi(t) \in \mathbb{Z}_{\geqslant 0}[t]$ by setting

$$P_\varphi(t) := \frac{\widehat{W}_{\perp\varphi}(t)}{W_{\perp\varphi}(t)}\,. \tag{8}$$

Here $\widehat{W}_{\perp\varphi}$ is the standard parabolic subgroup of the affine Weyl group $\widehat{W}$ generated by those reflections s_i ($i = 0, \ldots, l$) for which $\alpha_i \perp \varphi$ (here $\alpha_0 \perp \varphi$ means $\theta \perp \varphi$). Note that $\widehat{W}_{\perp\varphi}$ is a finite Coxeter group. Similarly, $W_{\perp\varphi}$ is the standard parabolic subgroup of the finite Weyl group W generated by those simple reflections s_i ($i = 1, \ldots, l$) for which $\alpha_i \perp \varphi$. In particular, $\widehat{W}_{\perp\varphi} = W_{\perp\varphi}$ if $\varphi \not\perp \theta$. The expressions $\widehat{W}_{\perp\varphi}(t)$ and $W_{\perp\varphi}(t)$ stand for the Poincaré polynomials of the Coxeter groups in question, and the quotient $\frac{\widehat{W}_{\perp\varphi}(t)}{W_{\perp\varphi}(t)}$ is the Poincaré polynomial for the set of minimal coset representatives in $W_{\perp\varphi}\backslash\widehat{W}_{\perp\varphi}$.

Let $\widehat{w}^\varphi_\circ$ be the longest element of $\widehat{W}_{\perp\varphi}$ and $w^\varphi_\circ$ the longest element of $W_{\perp\varphi}$. The set of minimal coset representatives in $W_{\perp\varphi}\backslash\widehat{W}_{\perp\varphi}$ is the interval $\left[1, w^\varphi_\circ\widehat{w}^\varphi_\circ\right]$ in the right weak Bruhat order—coming from the covering relation $\widehat{w} \prec s_i\widehat{w} :\Leftrightarrow \ell(s_i\widehat{w}) = \ell(\widehat{w}) + 1$ — (note that $(w^\varphi_\circ)^2 = 1$ and also $(\widehat{w}^\varphi_\circ)^2 = 1$). In particular, the longest element in $\left[1, w^\varphi_\circ\widehat{w}^\varphi_\circ\right]$ has length $\ell(w^\varphi_\circ\widehat{w}^\varphi_\circ) = \ell(\widehat{w}^\varphi_\circ) - \ell(w^\varphi_\circ)$.

In the following long table we show the polynomials $P_\alpha(t)$ for all simple roots $\alpha \in \Pi$. The polynomials $P_\varphi(t)$ can be extracted from this piece of information. This is clear for the simple types different from A_l because then the affine node of the Coxeter-Dynkin graph is a leaf in a tree and hence $P_\varphi(t) = P_\alpha(t)$ for an appropriate simple root $\alpha \in \Pi$. For type A_l we can reduce to the case of a simple root by looking at A_k for appropriate k, namely, $P^{\mathsf{A}_l}_{\alpha_i+\cdots+\alpha_{i+j}}(t) = P^{\mathsf{A}_{l-j}}_{\alpha_i}(t)$.

Definition For a nonnegative integer n let us recall the definition of the polynomial

$$[n] := \frac{1-t^n}{1-t} \in \mathbb{Z}_{\geqslant 0}[t].$$

Moreover, we define the factorials

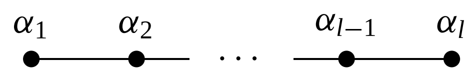
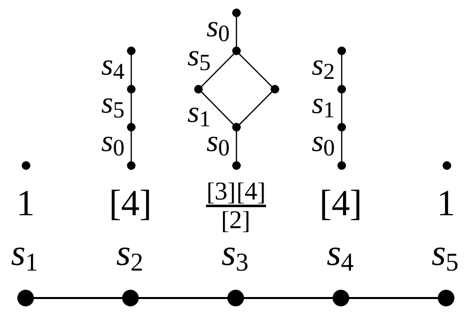
$$[n]! := \prod_{i=1}^{n} [i]$$

and their relatives

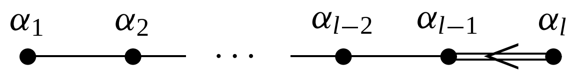
$$[2n]!! := \prod_{i=1}^{n} [2i] \quad \text{and} \quad [2n+1]!! := \prod_{i=0}^{n} [2i+1].$$

Of course, $[0]! = [0]!! = 1$.

The following table shows the polynomials $P_{\alpha_i}(t) = \widehat{W}_{\perp\alpha_i}(t)/W_{\perp\alpha_i}(t)$ and the minimal coset representatives for $W_{\perp\alpha_i}\backslash\widehat{W}_{\perp\alpha_i}$ (the latter for the classical series in the rank 5 case). Above or beneath each node marked by the simple reflection s_i we have depicted along with the polynomial $P_{\alpha_i}(t)$ the Hasse graph of the poset of $W_{\perp\alpha_i}\backslash\widehat{W}_{\perp\alpha_i}$. To read a minimal coset representative we have to start at the bottom node and read upwards along the edges. E. g., the minimal coset representatives for $W_{\perp\alpha_3}\backslash\widehat{W}_{\perp\alpha_3}$ for type A_5 are 1, s_0, s_0s_1, s_0s_5, $s_0s_1s_5 = s_0s_5s_1$, $s_0s_1s_5s_0 = s_0s_5s_1s_0$.

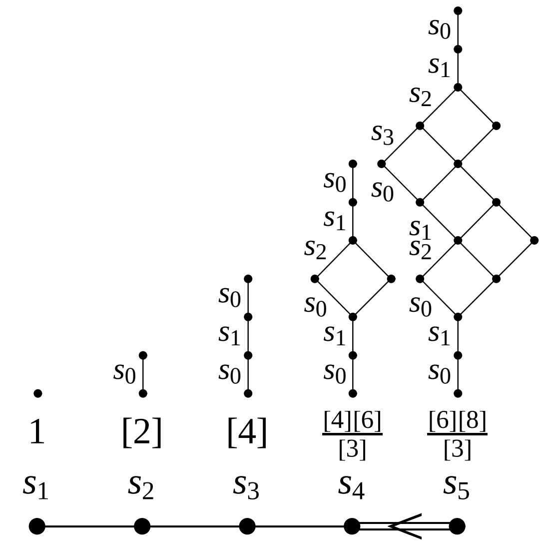
A_l	 $P_{\alpha_i}(t) = \dfrac{[l-1]!}{[i-1]!\,[l-i]!} \quad (i = 1,\dots,l)$
A_5	

C_l

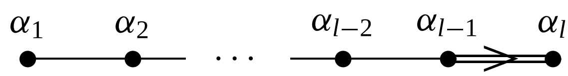


$$P_{\alpha_i}(t) = \frac{[2i-2]!!}{[i-1]!} \quad (i = 1, \ldots, l)$$

C_5



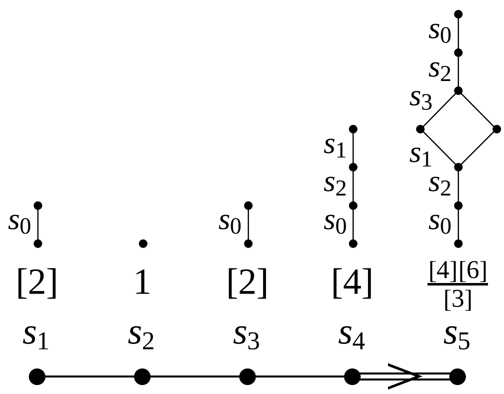
B_l

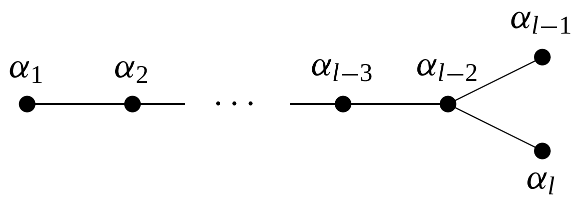
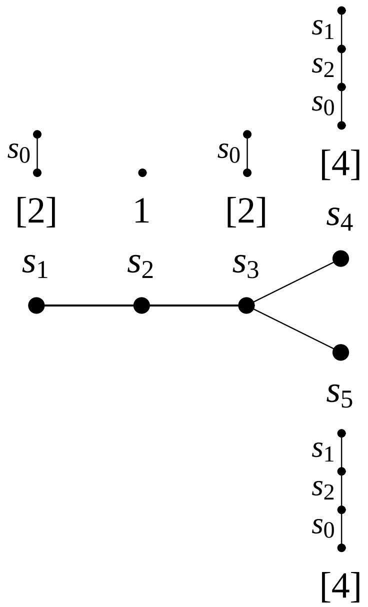
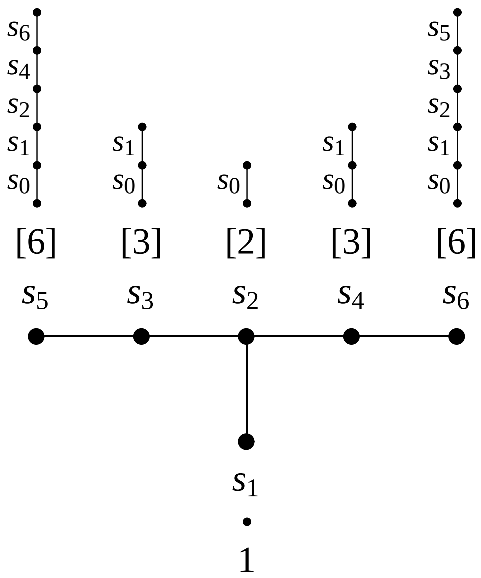


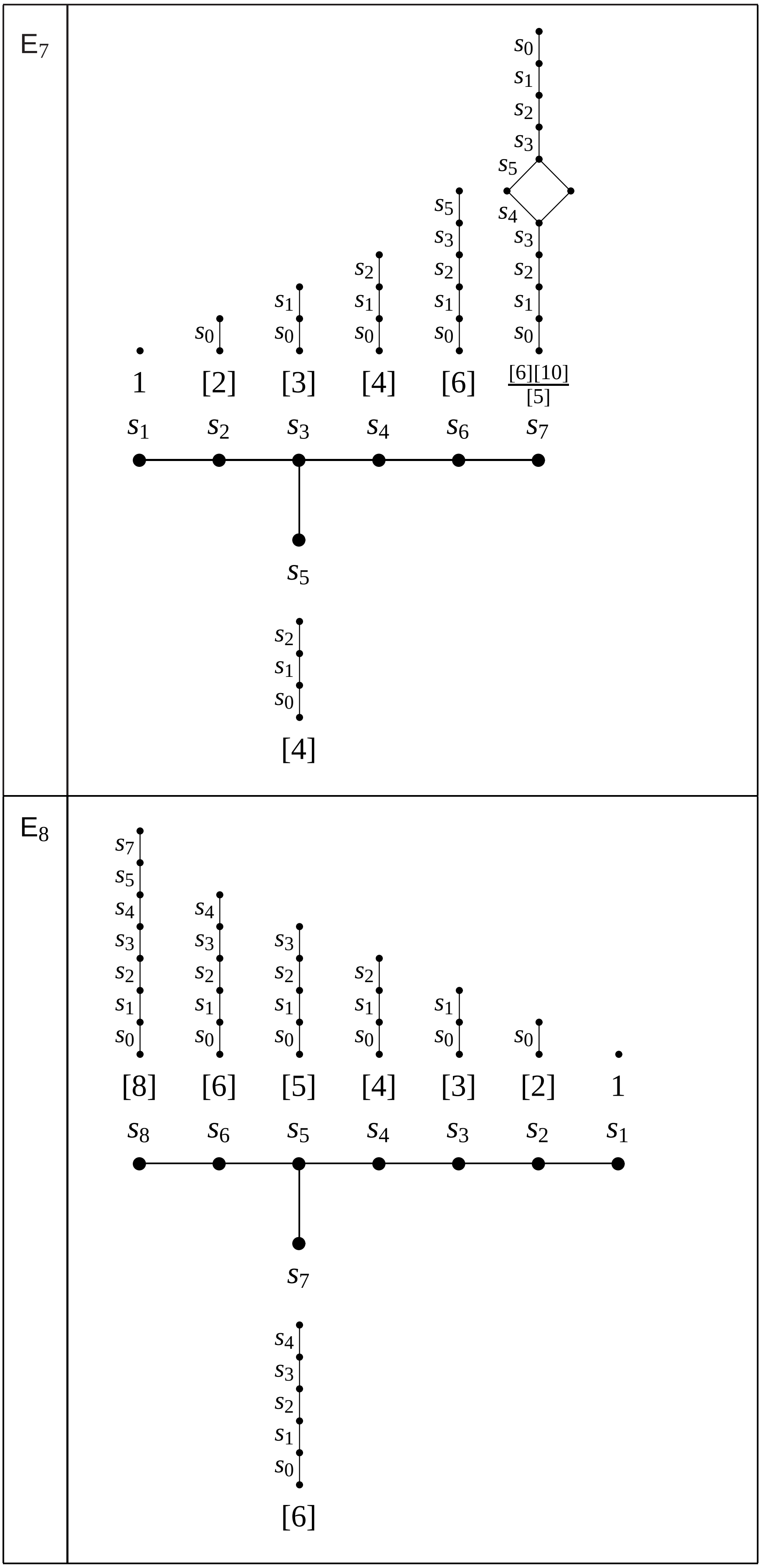
$$P_{\alpha_1}(t) = [2]$$

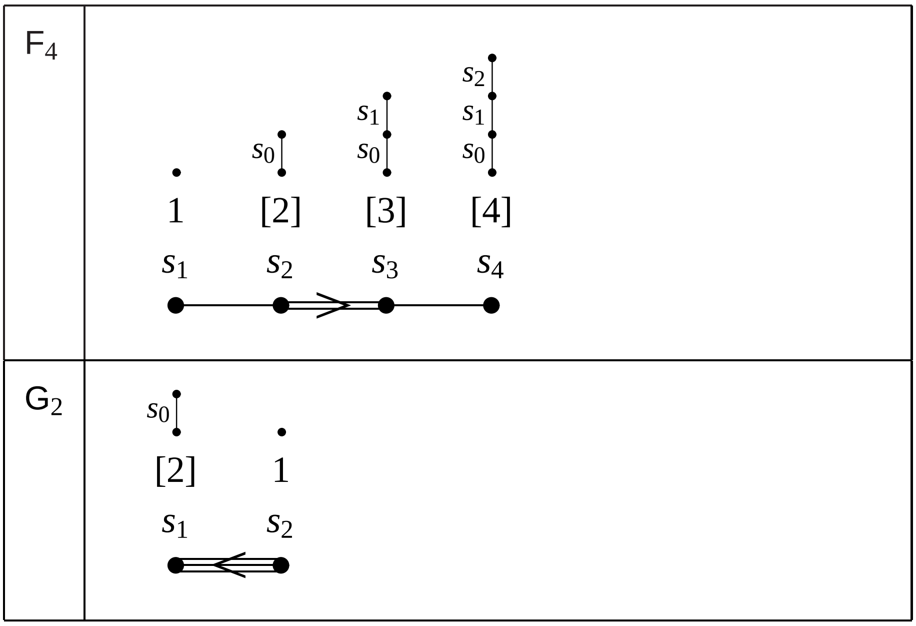
$$P_{\alpha_i}(t) = \frac{[2i-4]!!}{[i-2]!} \quad (i = 2, \ldots, l)$$

B_5



D_l	 $P_{\alpha_1}(t) = [2]$ $P_{\alpha_i}(t) = \dfrac{[2i-4]!!}{[i-2]!} \quad (i = 2, \ldots, l-1)$ $P_{\alpha_l}(t) = P_{\alpha_{l-1}}(t)$
D_5	
E_6	


E7
s0
s1
s2
s3
s5
s4
s3
s2
s1
s0
s5 s3 s2 s1 s0
s2 s1 s0
s1 s0
s0
1 [2] [3] [4] [6] [6][10]/[5]
s1 s2 s3 s4 s6 s7
s5
s2 s1 s0
[4]
E8
s7 s5 s4 s3 s2 s1 s0
s4 s3 s2 s1 s0
s3 s2 s1 s0
s2 s1 s0
s1 s0
s0
[8] [6] [5] [4] [3] [2] 1
s8 s6 s5 s4 s3 s2 s1
s7
s4 s3 s2 s1 s0
[6]



Lemma 19 *Let $\widehat{w} \in \widehat{W}_{\perp\varphi}$ be a minimal coset representative for a coset in $W_{\perp\varphi}\backslash\widehat{W}_{\perp\varphi}$. Then $\widehat{w}\rho$ lies in the dominant chamber.*

Proof We have $\ell(s_i\widehat{w}) > \ell(\widehat{w})$ for all $i = 1, \dots, l$, namely, for those i for which $\alpha_i \perp \varphi$ by the minimality of $\widehat{w}$, and for the remaining i because $s_i \notin \widehat{W}_{\perp\varphi}$. Hence $\widehat{w}$ is a minimal coset representative for a coset in $W\backslash\widehat{W}$. The assertion $\widehat{w}\rho \in C$ is now clear. (Since C is a fundamental domain for W, there is a unique $w \in W$ and a (minimal) gallery from the fundamental alcove A to $w\widehat{w}A$ which stays inside the dominant chamber C and so neither of the walls $H_1, \dots, H_l$ is crossed. By the minimality of $\widehat{w}$ we obtain $w = 1$, i. e., $\widehat{w}A \subseteq C$, or equivalently, $\widehat{w}\rho \in C$.) □

The next lemma generalizes in part (i) the orthogonality $w\theta \perp \theta$ in the proof of Proposition 18, which corresponds to $\widehat{w} = s_0$ and requires $\varphi \perp \theta$ in Lemma 20.

Lemma 20 *Let $\varphi \in \Phi_+^{\mathrm{long}}$. Let $w \in W$ be such that $w\varphi = \theta$ and let $\widehat{w} \in \widehat{W}_{\perp\varphi}$.*

(i) $s_0 w\widehat{w}\rho - s_0 w\rho$ is perpendicular to the highest root θ.
(ii) If $s_0 w\widehat{w}s_i\rho - s_0 w\rho \perp \theta$, then $s_i \in \widehat{W}_{\perp\varphi}$.

Proof We first note that $\widehat{w}(\lambda + \varphi^\perp) \subseteq \lambda + \varphi^\perp$ for each $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$ by the definition of $\widehat{W}_{\perp\varphi}$. To prove (i) we compute (recall that s_θ is the linear part of s_0)

$$\begin{aligned}\big(s_0 w\widehat{w}\rho - s_0 w\rho \mid \theta\big) &= \big(s_\theta(w\widehat{w}\rho - w\rho) \mid \theta\big)\\ &= \big(w(\widehat{w}\rho - \rho) \mid -\theta\big) = \big(\underbrace{\widehat{w}\rho - \rho}_{\perp\,\varphi} \mid -\varphi\big) = 0.\end{aligned}$$

To prove (ii) we write

$$\left(s_0 w\widehat{w} s_i\rho - s_0 w\rho \;\middle|\; \theta\right) = \Big(\underbrace{\widehat{w} s_i\rho - \rho}_{\substack{\in s_i\rho + \varphi^\perp - \rho \\ = -\alpha_i + \varphi^\perp}} \;\Big|\; -\varphi\Big) = (\alpha_i|\varphi).$$

Hence $s_0 w\widehat{w} s_i\rho - s_0 w\rho \perp \theta$ is actually equivalent to $s_i \in \widehat{W}_{\perp\varphi}$. □

Proposition 21 *Let $\varphi \in \Phi_+^{\mathrm{long}}$ be a positive long root and $\widehat{w} \in \widehat{W}_{\perp\varphi}$ be the minimal coset representative for a coset in $W_{\perp\varphi}\backslash\widehat{W}_{\perp\varphi}$. Then $\widehat{w}\rho \in 2A$.*

Proof By the definition of $\widehat{W}_{\perp\varphi}$ at least one of the simple reflections $s_1, \ldots$ $\ldots, s_l$ is not contained in the standard parabolic subgroup $\widehat{W}_{\perp\varphi}$ of $\widehat{W}$. Say $s_i \notin \widehat{W}_{\perp\varphi}$. All other Coxeter generators of $\widehat{W}$ fix the vertex $\frac{\overset{\vee}{\varpi}_i}{n_i}$ of the fundamental alcove, and hence so does the group $\widehat{W}_{\perp\varphi}$. In particular, the points ρ, $\frac{\overset{\vee}{\varpi}_i}{n_i}$, and $\widehat{w}\rho$ all belong to the half space $\left\{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \;\middle|\; (\lambda|\theta) < 1\right\}$.

Moreover, by Lemma 19, $\widehat{w}\rho$ lies in the dominant chamber. This concludes the proof. □

Let us recall that for $\widehat{w} \in \widehat{W}$ the implication

$$\widehat{w}\rho \in 2A \implies \|\widehat{w}\rho\|^2 - \|\rho\|^2 = \ell(\widehat{w})$$

holds by Corollary 9. The converse implication is not true. However, the next lemma states a partial converse.

Lemma 22 *Let $\widetilde{w}\rho$ be the ρ-point of an alcove that lies outside the dominant chamber C but is adjacent to an alcove of $2A$. Then*

$$\|\widetilde{w}\rho\|^2 - \|\rho\|^2 = \ell(\widetilde{w}) - 1.$$

Proof Say the alcove with ρ-point $\widetilde{w}\rho$ lies in the chamber s_iC. This means that $\widetilde{w} = s_i\widehat{w}$ with $\widehat{w}\rho \in 2A$ and $\ell(\widetilde{w}) = \ell(\widehat{w}) + 1$. It follows that

$$\|\widetilde{w}\rho\|^2 - \|\rho\|^2 = \|\widehat{w}\rho\|^2 - \|\rho\|^2 = \ell(\widehat{w}) = \ell(\widetilde{w}) - 1.$$

□

Here is the main theorem of this paper.

Theorem 23 *Let $\varphi \in \Phi_+^{\mathrm{long}}$ be a positive long root and $\widehat{w} \in \widehat{W}_{\perp\varphi}$ be the minimal coset representative for a coset in $W_{\perp\varphi}\backslash\widehat{W}_{\perp\varphi}$. To the pair $(\varphi, \widehat{w})$ we associate the ρ-point $s_0 w\widehat{w}\rho$ where $w \in W$ is the Weyl group element such that $w\varphi = \theta$ and with $\ell(w) = L(\varphi)$ as in Proposition 12. Then $s_0 w\widehat{w}\rho \in 2A - A$ and hence $s_0 w\widehat{w}\rho$ is the ρ-point of an alcove that corresponds to a nonzero abelian ideal $\mathfrak{a}^{\varphi,\widehat{w}} \trianglelefteq \mathfrak{b}$. Moreover, each nonzero abelian ideal occurs in this way.*

Hence we have the parametrization we looked for

$$\coprod_{\varphi\in\Phi_+^{\mathrm{long}}} W_{\perp\varphi}\backslash\widehat{W}_{\perp\varphi} \stackrel{1:1}{\longleftrightarrow} \left\{\mathfrak{a} \;\middle|\; 0\neq\mathfrak{a}\trianglelefteq\mathfrak{b},\ \mathfrak{a}\ \textit{abelian}\right\} \tag{9}$$

where the right coset $W_{\perp\varphi}\widehat{w}$ in the component for φ on the left hand side with $\widehat{w}\in\widehat{W}_{\perp\varphi}$ its minimal coset representative corresponds to the abelian ideal $\mathfrak{a}^{\varphi,\widehat{w}}$ of dimension $L(\varphi)+1+\ell(\widehat{w})$.

Remark The abelian ideals $\mathfrak{a}^{\varphi,\min}$, $\mathfrak{a}^{\varphi,\min^+}$ if $\varphi\perp\theta$, and $\mathfrak{a}^{\varphi,\max}$ that were mentioned earlier have the following descriptions.

$$\begin{aligned}\mathfrak{a}^{\varphi,\min} &= \mathfrak{a}^{\varphi,1}\\ \mathfrak{a}^{\varphi,\min^+} &= \mathfrak{a}^{\varphi,s_0}\text{ if }\varphi\perp\theta\\ \mathfrak{a}^{\varphi,\max} &= \mathfrak{a}^{\varphi,w_\circ^\varphi\widehat{w}_\circ^\varphi}\,.\end{aligned}$$

Proof By Lemma 20 we know that $s_0w\widehat{w}\rho\in s_0w\rho+\theta^\perp$. And Theorem 15 says that $s_0w\rho$ is the ρ-point of a nonzero abelian ideal. Hence we have $s_0w\rho\in 2A-A$. It follows that $s_0w\widehat{w}\rho$ lies in the strip between the two walls $H_0=\left\{\lambda\in\mathfrak{h}_{\mathbb{R}}^* \;\middle|\; (\lambda|\theta)=\frac{1}{2}\right\}$ and $2H_0=\left\{\lambda\in\mathfrak{h}_{\mathbb{R}}^* \;\middle|\; (\lambda|\theta)=1\right\}$.

Let $\widehat{w}=s_{i_1}\dots s_{i_m}$ be a reduced decomposition of $\widehat{w}$. Each initial subword $\widehat{w}_k:=s_{i_1}\dots s_{i_k}$ (for $0\leqslant k\leqslant m$) is again a minimal coset representative for a coset in $W_{\perp\varphi}\backslash\widehat{W}_{\perp\varphi}$. In fact $\ell(s_is_{i_1}\dots s_{i_m})=m+1$ for $i=1,\dots,l$ implies $\ell(s_is_{i_1}\dots s_{i_k})=k+1$ for $i=1,\dots,l$ for each $k=0,\dots,m$. Hence we compute for each $k=0,\dots,m$

$$\begin{aligned}&\|s_0w\widehat{w}_k\rho\|^2-\|\rho\|^2\\ &=\left\|s_0w\widehat{w}_k\rho-s_0w\rho\right\|^2+2\left(\underbrace{s_0w\widehat{w}_k\rho-s_0w\rho}_{\perp\theta\text{ (by Lemma 20)}}\;\middle|\;s_0w\rho\right)+\underbrace{\|s_0w\rho\|^2-\|\rho\|^2}_{=\ell(s_0w)}\\ &=\left\|s_\theta w(\widehat{w}_k\rho-\rho)\right\|^2+2\left(\underbrace{s_\theta w(\widehat{w}_k\rho-\rho)}_{\perp\theta}\;\middle|\;s_\theta w\rho+g\theta\right)+\ell(s_0w)\\ &=\left\|\widehat{w}_k\rho-\rho\right\|^2+2\left(\widehat{w}_k\rho-\rho\;\middle|\;\rho\right)+\ell(s_0w)=\underbrace{\|\widehat{w}_k\rho\|^2-\|\rho\|^2}_{=\ell(\widehat{w}_k)}+\ell(s_0w)\\ &\geqslant\ell(s_0w\widehat{w}_k).\end{aligned}\tag{10}$$

In the list $s_0w\rho=s_0w\widehat{w}_0\rho, s_0w\widehat{w}_1\rho,\dots,s_0w\widehat{w}_m\rho=s_0w\widehat{w}\rho$ the first ρ-point belongs to $2A-A$. We show that actually all of them belong to $2A-A$. Suppose not, and let $s_0w\widehat{w}_k\rho$ be the first ρ-point outside $2A-A$. We know that $s_0w\widehat{w}_k\rho$ lies in the strip between the walls H_0 and $2H_0$.

Hence $s_0 w\widehat{w}_k\rho$ would lie outside the dominant chamber. By Lemma 22 we would have $\|s_0 w\widehat{w}_k\rho\|^2 - \|\rho\|^2 = \ell(s_0 w\widehat{w}_k) - 1$, which according to (10) is absurd.

So far we have proved that the mapping $(\varphi, \widehat{w}) \mapsto \mathfrak{a}^{\varphi,\widehat{w}}$ in (9) is well-defined. It is injective because $(\mathfrak{a}^{\varphi,\widehat{w}})^{\not\perp\theta} = \mathfrak{a}^{\varphi,\min}$.

Finally, we must show that our construction is exhaustive. So let $\mathfrak{a}$ be a nonzero abelian ideal in $\mathfrak{b}$. Its subideal $\mathfrak{a}^{\not\perp\theta}$ is of the form $\mathfrak{a}^{\not\perp\theta} = \mathfrak{a}^{\varphi,\min}$ by Theorem 17, and its alcove has $s_0 w\rho$ as its ρ-point where $w \in W$ with $\ell(w) = L(\varphi)$ for $\varphi = w^{-1}\theta$ by Theorem 15. By Lemma 4 we can choose a flag

$$\mathfrak{a}^{\not\perp\theta} = \mathfrak{a}_0 \subseteq \cdots \subseteq \mathfrak{a}_m = \mathfrak{a}$$

of abelian ideals with $\dim \mathfrak{a}_k = \dim \mathfrak{a}^{\not\perp\theta} + k$ for $k = 0, \ldots, m$. The ρ-point of the alcove of $\mathfrak{a}_k$ can be written as

$$\rho + \langle \Phi(\mathfrak{a}_k)\rangle = s_0 w s_{i_1} \ldots s_{i_k}\rho$$

where $s_{i_1}, \ldots, s_{i_m}$ are simple reflections in $\widehat{W}_{\perp\varphi}$ by Lemma 20 and the definition of the ideal $\mathfrak{a}^{\not\perp\theta}$. From Proposition 5 we get

$$\ell(s_0 w s_{i_1} \ldots s_{i_k}) = \ell(s_0 w) + k$$

so that $\ell(s_{i_1} \ldots s_{i_k}) = k$. Now we compute as above

$$\|s_0 w s_{i_1} \ldots s_{i_k}\rho\|^2 - \|\rho\|^2 = \|s_{i_1} \ldots s_{i_k}\rho\|^2 - \|\rho\|^2 + \ell(s_0 w).$$

The left hand side is $\ell(s_0 w s_{i_1} \ldots s_{i_k})$ by Proposition 5 and Corollary 9 so that

$$\|s_{i_1} \ldots s_{i_k}\rho\|^2 - \|\rho\|^2 = k.$$

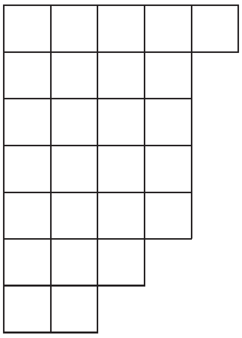
Again using Lemma 22 we conclude that

$$s_{i_1} \ldots s_{i_k}\rho \in 2A \subseteq C.$$

In particular, $s_{i_1} \ldots s_{i_m}\rho \in C$ so that $s_{i_1} \ldots s_{i_m}$ is the minimal coset representative for the coset $W s_{i_1} \ldots s_{i_m} \in W\backslash\widehat{W}$, and since $s_{i_1}, \ldots, s_{i_m}$ are simple reflections in $W_{\perp\varphi}$, we conclude that $s_{i_1} \ldots s_{i_m}$ is the minimal coset representative for the coset $W_{\perp\varphi} s_{i_1} \ldots s_{i_m} \in W_{\perp\varphi}\backslash\widehat{W}_{\perp\varphi}$. Hence we have shown that $\mathfrak{a} = \mathfrak{a}^{\varphi, s_{i_1}\ldots s_{i_m}}$. □

Another parametrization. In the previous theorem we have parametrized the nonzero abelian ideals in $\mathfrak{b}$ as $\mathfrak{a}^{\varphi,\widehat{w}}$. The ρ-point $\widehat{w}\rho$ belongs to $2A$ by Proposition 21. So either $\widehat{w} = 1$ or $\widehat{w}\rho$ is again the ρ-point of a nonzero abelian ideal $\mathfrak{a}^{\varphi',\widehat{w}'}$. By iteration this procedure yields a list of positive long roots $\varphi_1, \ldots, \varphi_r$ corresponding to the abelian ideal $\mathfrak{a}^{\varphi,\widehat{w}} = \mathfrak{a}^{(\varphi_1,\ldots,\varphi_r)}$. Now we can put the zero ideal back into the picture and write $0 = \mathfrak{a}^{(\,)}$, the abelian ideal for the empty sequence.

In the classical A_l case an abelian ideal corresponds to a Young diagram. The associated positive roots $\varphi_1, \ldots, \varphi_r$ correspond to the decomposition of the Young diagram into hooks. Let us make that more concrete by looking at an example. Take A_{11} and examine the abelian ideal corresponding to the following Young diagram.



s_0	s_{11}	s_{10}	s_9	s_8
s_1	s_0	s_{11}	s_{10}	
s_2	s_1	s_0	s_{11}	
s_3	s_2	s_1	s_0	
s_4	s_3	s_2	s_1	
s_5	s_4	s_3		
s_6	s_5			

The filled diagram on the right yields the following affine Weyl group element when read hook-wise.

$$\widehat{w} = s_0 \underbrace{s_1\, s_2\, s_3\, s_4\, s_5\, s_6\, s_{11}\, s_{10}\, s_9\, s_8}_{=:\, w_1} s_0 \underbrace{s_1\, s_2\, s_3\, s_4\, s_5\, s_{11}\, s_{10}}_{=:\, w_2} s_0 \underbrace{s_1\, s_2\, s_3\, s_{11}}_{=:\, w_3} s_0 \underbrace{s_1}_{=:\, w_4}$$

The abelian ideal with ρ-point $\widehat{w}\rho$ is $\mathfrak{a}^{(\varphi_1,\ldots,\varphi_4)}$, where

$$\begin{aligned}
\varphi_1 &= w_1^{-1}\theta = \alpha_7, \\
\varphi_2 &= w_2^{-1}\theta = \alpha_6 + \cdots + \alpha_9, \\
\varphi_3 &= w_3^{-1}\theta = \alpha_4 + \cdots + \alpha_{10}, \\
\varphi_4 &= w_4^{-1}\theta = \alpha_2 + \cdots + \alpha_{11}.
\end{aligned}$$

Our next theorem is a corollary of Theorem 23 and Corollary 8. The two sets on the left and the right hand sides in (9) have the same cardinality.

Theorem 24 (First Sum Formula) *The following formula holds.*

$$\sum_{\varphi \in \Phi_+^{\mathrm{long}}} \frac{|\widehat{W}_{\perp\varphi}|}{|W_{\perp\varphi}|} = 2^l - 1$$

Proof As mentioned, the First Sum Formula is already proved. Nevertheless we shall give an alternative (case by case) proof. This is then an alternative for the exhaustiveness part in Theorem 23 and also for Theorem 17.

Let us use the abbreviation $P_\varphi(1) = \frac{|\widehat{W}_{\perp\varphi}|}{|W_{\perp\varphi}|}$ in accordance with (8). Let $S := \sum_{\varphi \in \Phi_+^{\mathrm{long}}} P_\varphi(1)$.

$\boxed{\mathsf{A}_l}$ Using $P_{\alpha_i+\cdots+\alpha_{i+j}}(1) = \frac{(l-j-1)!}{(i-1)!\,(l-j-i)!}$ $(i = 1, \ldots, l,\ j = 0, \ldots, l-i)$ we get

$$S = \sum_{i=1}^{l} \sum_{j=0}^{l-i} P_{\alpha_i+\cdots+\alpha_{i+j}}(1) = \sum_{i=1}^{l} \sum_{j=0}^{l-i} \frac{(l-j-1)!}{(i-1)!\,(l-j-i)!} = 2^l - 1.$$

For the other simple types the affine Coxeter-Dynkin graph has a tree as its underlying simple graph. Each root φ is of the form $\varphi = \sum_{i=1}^{l} a_i\,\alpha_i$ and has support $\operatorname{supp}\varphi := \{\alpha_i \mid a_i \neq 0\}$. Let $\operatorname{pr}(\varphi) \in \operatorname{supp}\varphi$ be the simple root which is nearest to α_0 when considered as nodes in the affine Coxeter-Dynkin tree. It is clear that $P_\varphi(t) = P_{\operatorname{pr}(\varphi)}(t)$. For $i = 1, \ldots, l$ let r_i be the number of positive long roots φ for which $\operatorname{pr}(\varphi) = \alpha_i$. The sum S can now be rewritten as $S = \sum_{i=1}^{l} r_i\,P_{\alpha_i}(1)$. The numbers r_i can be expressed via the numbers $\nu(\mathsf{X}) =$ the number of positive long roots of a root system of type X (for $\nu(\mathsf{X})$ see the table on page 191).

$\boxed{\mathsf{C}_l}$ $r_i = 1$ $(i = 1, \ldots, l)$

$$S = \sum_{i=1}^{l} 2^{i-1} = 2^l - 1$$

$\boxed{\mathsf{B}_l}$ $r_1 = \nu(\mathsf{A}_1) = 1$

$r_2 = \nu(\mathsf{B}_l) - \nu(\mathsf{B}_{l-2}) - \nu(\mathsf{A}_1) = 4l - 7$

$r_i = \nu(\mathsf{B}_{l-i+1}) - \nu(\mathsf{B}_{l-i}) = 2l - 2i$ $(i = 3, \ldots, l-2)$

$r_{l-1} = \nu(\mathsf{B}_2) = 2$ (for $l \geqslant 4$)

$r_l = 0$

$$S = 1 \cdot 2 + (4l-7) \cdot 1 + \sum_{i=3}^{l-1} (2l-2i) \cdot 2^{i-2} + 0 \cdot 2^{l-2} = 2^l - 1$$

$\boxed{\mathsf{D}_l}$ $r_1 = \nu(\mathsf{A}_1) = 1$

$r_2 = \nu(\mathsf{D}_l) - \nu(\mathsf{D}_{l-2}) - \nu(\mathsf{A}_1) = 4l - 7$

$r_i = \nu(\mathsf{D}_{l-i+1}) - \nu(\mathsf{D}_{l-i}) = 2l - 2i$ $(i = 3, \ldots, l-3)$

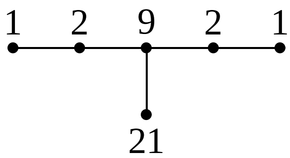
$r_{l-2} = \nu(\mathsf{A}_3) - 2\,\nu(\mathsf{A}_1) = 4$ (for $l \geqslant 5$)

$r_{l-1} = r_l = \nu(\mathsf{A}_1) = 1$

$$S = 1 \cdot 2 + (4l-7) \cdot 1 + \sum_{i=3}^{l-2} (2l-2i) \cdot 2^{i-2} + 1 \cdot 2^{l-3} + 1 \cdot 2^{l-3}$$
$$= 2^l - 1$$

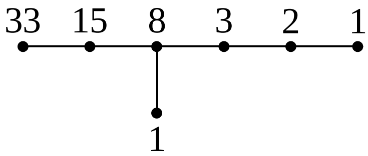
For the exceptional types we write the numbers r_i directly near the corresponding node in the Coxeter-Dynkin graph. It is clear how to compute them, e.g., for E_6, $r_1 = \nu(\mathsf{E}_6) - \nu(\mathsf{A}_5)$, $r_2 = \nu(\mathsf{A}_5) - 2\,\nu(\mathsf{A}_2)$, and so on.

$\boxed{\mathsf{E}_6}$



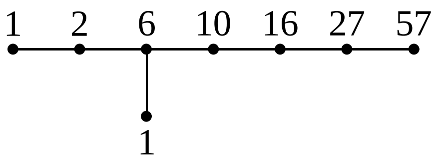
$$S = 21\cdot 1 + 9\cdot 2 + 2\cdot 3 + 2\cdot 3 + 1\cdot 6 + 1\cdot 6 = 2^6 - 1$$

$\boxed{\mathsf{E}_7}$



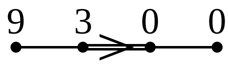
$$S = 33\cdot 1 + 15\cdot 2 + 8\cdot 3 + 3\cdot 4 + 1\cdot 4 + 2\cdot 6 + 1\cdot 12 = 2^7 - 1$$

$\boxed{\mathsf{E}_8}$



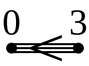
$$S = 57\cdot 1 + 27\cdot 2 + 16\cdot 3 + 10\cdot 4 + 6\cdot 5 + 2\cdot 6 + 1\cdot 6 + 1\cdot 8 \\ = 2^8 - 1$$

$\boxed{\mathsf{F}_4}$



$$S = 9\cdot 1 + 3\cdot 2 + 0\cdot 3 + 0\cdot 4 = 2^4 - 1$$

$\boxed{\mathsf{G}_2}$



$$S = 3\cdot 1 + 0\cdot 2 = 2^2 - 1$$

□

4. Maximal abelian ideals

Among the maximal abelian ideals are those whose dimension is maximal. We can express these dimensions in a uniform way as $g - 1 + N' - N''$ where g is the dual Coxeter number of $\mathfrak{g}$ and N', N'' are the numbers of positive roots of certain root subsystems of the root system of $\mathfrak{g}$. In fact, for a long simple root α we have

$$\begin{aligned}\dim \mathfrak{a}^{\alpha,\max} &= \dim \mathfrak{a}^{\alpha, w_\circ^\alpha \widehat{w}_\circ^\alpha} \\ &= L(\alpha) + 1 + \ell\big(w_\circ^\alpha \widehat{w}_\circ^\alpha\big) = g - 1 + \ell\big(\widehat{w}_\circ^\alpha\big) - \ell\big(w_\circ^\alpha\big).\end{aligned}$$

So we have our next corollary which gives a clear explanation of these maximal dimensions that hitherto appeared somewhat mysterious.

Corollary 25 *The maximal dimension of an abelian ideal in* $\mathfrak{b}$ *can be expressed as* $g - 1 + \widehat{N}_{\perp\alpha} - N_{\perp\alpha}$ *where* $\alpha \in \Pi^{\mathrm{long}}$ *is a long simple root such that the difference* $\widehat{N}_{\perp\alpha} - N_{\perp\alpha}$ *is maximal. Here* $\widehat{N}_{\perp\alpha} = \ell(\widehat{w}_\circ^\alpha)$ *is the length of the longest element* $\widehat{w}_\circ^\alpha \in \widehat{W}_{\perp\alpha}$*, or, equivalently, the number of positive roots of a root system of type* $\widehat{\mathsf{X}}_{\perp\alpha}$*. Analogously,* $N_{\perp\alpha} = \ell(w_\circ^\alpha)$ *for* $w_\circ^\alpha \in W_{\perp\alpha}$ *its longest element.*

The maximal dimension of an abelian ideal in $\mathfrak{b}$ coincides with the maximal dimension of a commutative subalgebra of $\mathfrak{g}$ (as first calculated by A. Malcev [Mal] case by case). Let us briefly state the algebraic reason why these dimensions are the same. For a k-dimensional ideal $\mathfrak{a} \trianglelefteq \mathfrak{b}$ consider the 1-dimensional subspace $\bigwedge^k \mathfrak{a} \subseteq \bigwedge^k \mathfrak{g}$. Since $\mathfrak{a}$ is an ideal, the line $\bigwedge^k \mathfrak{a}$ is the highest weight space of a simple $\mathfrak{g}$-module $L_\mathfrak{a} \subseteq \bigwedge^k \mathfrak{g}$. One can show that

$$\mathcal{A}_k := \bigoplus_{\mathfrak{a}} L_\mathfrak{a} = \sum_{\mathfrak{c}} \bigwedge^k \mathfrak{c}$$

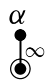
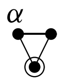
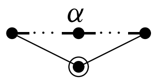
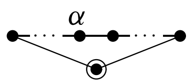
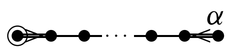
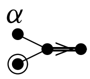
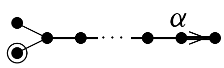
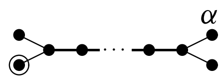
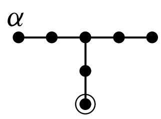
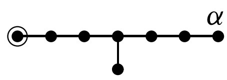
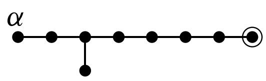
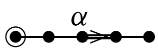
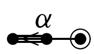
where $\mathfrak{a}$ runs through the k-dimensional abelian ideals in $\mathfrak{b}$ and $\mathfrak{c}$ is taken over the k-dimensional commutative subalgebras of $\mathfrak{g}$. The maximal dimension is $\max\left\{k \;\middle|\; \mathcal{A}_k \neq 0\right\}$. For the details see [Ko1].

Remark In the table on the next page, in some cases there are several possibilities for the long simple root α that yields an abelian ideal of maximal dimension. By inspection we see that the number of abelian ideals in $\mathfrak{b}$ of maximal dimension is

3 for type D_4,
2 for types A_l (l even), D_l ($l > 4$), and E_6, and
1 for the other types.

Remark Instead of taking $\widehat{\mathsf{X}}_{\perp\alpha}$ and $\mathsf{X}_{\perp\alpha}$ one could already delete the common components (nonvoid for the types B_3, D_l, and F_4).

Abelian ideals of maximal dimension. The fourth column in the table shows the Coxeter-Dynkin graph CD_X of type X and the affine Coxeter-Dynkin graph $\mathrm{CD}_{\widehat{\mathsf{X}}}$ with the node corresponding to α_0 encircled.

X	$g_{\mathsf{X}}-1$	N_{X}	CD_{X} $\mathrm{CD}_{\widehat{\mathsf{X}}}$ α	$\widehat{\mathsf{X}}_{\perp\alpha}$	$\mathsf{X}_{\perp\alpha}$	max dim
A_1	1	1		$\varnothing$	$\varnothing$	1
A_2	2	3		$\varnothing$	$\varnothing$	2
A_l $l\geqslant 3$ l odd	l	$\dfrac{l(l+1)}{2}$		A_{l-2}	$\mathsf{A}_{\frac{l-3}{2}}+\mathsf{A}_{\frac{l-3}{2}}$	$\dfrac{(l+1)^2}{4}$
A_l $l\geqslant 4$ l even	l	$\dfrac{l(l+1)}{2}$		A_{l-2}	$\mathsf{A}_{\frac{l-2}{2}}+\mathsf{A}_{\frac{l-4}{2}}$	$\dfrac{l^2+2l}{4}$
C_l $l\geqslant 2$	l	l^2		C_{l-1}	A_{l-2}	$\dfrac{l^2+l}{2}$
B_3	4	9		$\mathsf{A}_1+\mathsf{A}_1$	A_1	5
B_l $l\geqslant 4$	$2l-2$	l^2		D_{l-2}	A_{l-3}	$\dfrac{l^2-l+2}{2}$
D_l $l\geqslant 4$	$2l-3$	$l(l-1)$		$\mathsf{D}_{l-2}+\mathsf{A}_1$	$\mathsf{A}_{l-3}+\mathsf{A}_1$	$\dfrac{l^2-l}{2}$
E_6	11	36		A_5	A_4	16
E_7	17	63		D_6	D_5	27
E_8	29	120		A_7	A_6	36
F_4	8	24		$\mathsf{A}_1+\mathsf{A}_1$	A_1	9
G_2	3	6		$\varnothing$	$\varnothing$	3

The usual conventions apply, namely, $\mathsf{A}_0=\varnothing$, $\mathsf{C}_1=\mathsf{A}_1$, $\mathsf{B}_2=\mathsf{C}_2$, $\mathsf{D}_2=\mathsf{A}_1+\mathsf{A}_1$.

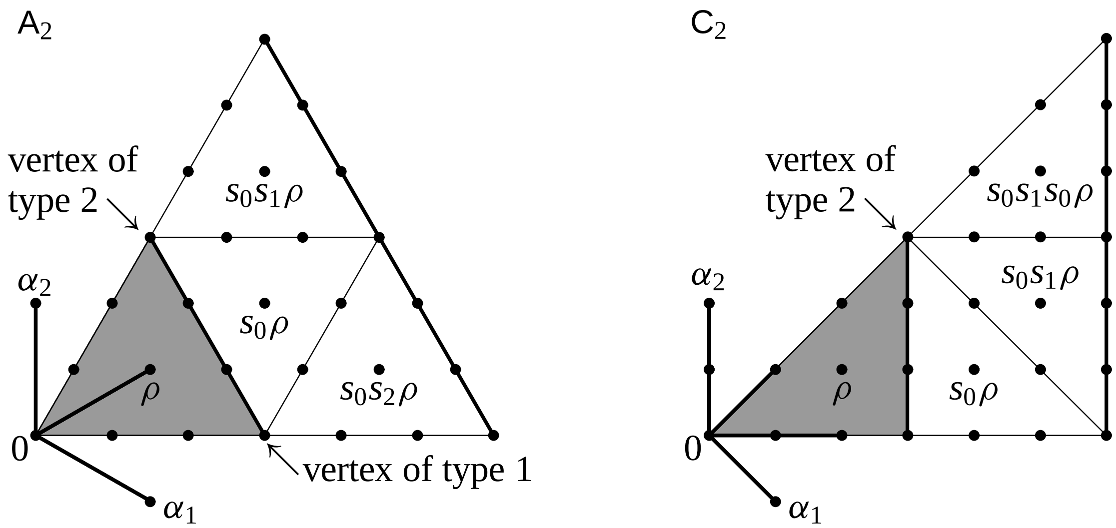
Remark As already mentioned the numbers in the rightmost column of the table above were first computed case by case by A. Malcev [Mal].

In the paper [Boe] B. Boe computed, again case by case, the maximal length $\ell(\widehat{w})$ of an affine Weyl group element $\widehat{w}$ such that $\widehat{w}A \subseteq (k+1)A$; see [Boe, Table 1] but with the types C_l and B_l interchanged because there the highest *short* root is used to define the tessellation by alcoves. Neither Boe's paper nor its review paper [Sri] mentions the connexion with Malcev's result.

We are now interested in the maximal abelian ideals in $\mathfrak{b}$. It has been observed in [PR] that the number of maximal abelian ideals in a fixed Borel subalgebra of $\mathfrak{g}$ equals the number of long simple roots. A canonical one-to-one correspondence was exhibited between the two sets. However, the proof was based on a case by case consideration and was therefore rather unsatisfactory. Here we will give a geometric approach which makes the whole picture very transparent.

We know by Proposition 5 that each abelian ideal $\mathfrak{a} \trianglelefteq \mathfrak{b}$ corresponds to an alcove $\widehat{w}A \subseteq 2A$. If no facet of $\widehat{w}A$ lies in the wall $2H_0$, then by convexity a minimal gallery between A and $\widehat{w}A$ can be extended beyond $\widehat{w}A$ but still inside $2A$. Hence each maximal abelian ideal has an alcove with one facet lying in the wall $2H_0$. It is convenient to have some terminology which describes this geometric situation.

Definition An **upper alcove** $\widehat{w}A$ is an alcove in $2A$ such that one facet of $\widehat{w}A$ lies in the wall $2H_0$. For an upper alcove $\widehat{w}A$ the **lower vertex** is the vertex that sticks out, i. e., does not lie in the wall $2H_0$.



Let us look at some examples. For type A_2 there are two upper alcoves, namely, those with ρ-points $s_0s_2\rho$ and $s_0s_1\rho$. Both belong to maximal abelian ideals, namely, $\mathfrak{a}^{\alpha_1,1} = \mathfrak{a}^{(\alpha_1)}$ and $\mathfrak{a}^{\alpha_2,1} = \mathfrak{a}^{(\alpha_2)}$. For the former alcove, the lower vertex has type 1 and for the latter type 2. For type C_2 there are again two upper alcoves, with ρ-points $s_0s_1\rho$ (ideal $\mathfrak{a}^{\alpha_2,1} = \mathfrak{a}^{(\alpha_2)}$) and $s_0s_1s_0\rho$ (ideal $\mathfrak{a}^{\alpha_2,s_0} = \mathfrak{a}^{(\alpha_2,\theta)}$), both with the same lower vertex of type 2. Only the latter belongs to the maximal abelian ideal. For type G_2 (see the picture on page 184) there is only one upper alcove, with ρ-point $s_0 s_2 s_1 \rho$ and lower vertex of type 2.

From the previous results we already know that the lower vertices are in one-to-one correspondence with the long simple roots.

Another way for proving that each lower vertex necessarily has the type of a long simple root can be deduced from the following proposition which we also use for our Second Sum Formula (Theorem 27).

Proposition 26 $\mathrm{vol}_{l-1}(F_0) : \cdots : \mathrm{vol}_{l-1}(F_l) = \|\alpha_0\|\, n_0 : \cdots : \|\alpha_l\|\, n_l$.

Proof Recall that the vertices of the fundamental alcove A with facets $F_0, \ldots, F_l$ are $0, \frac{\overset{\vee}{\varpi}_1}{n_1}, \ldots, \frac{\overset{\vee}{\varpi}_l}{n_l}$. We compute the volume of an alcove in two different ways.

The volume of the pyramid A over F_0 with apex 0 is $\frac{1}{l}$ times $\mathrm{vol}_{l-1}(F_0)$ times the distance of the apex 0 from the wall H_0 supporting the facet F_0. This distance is $\frac{1}{2\|\theta\|}$ because $\frac{\theta}{2\|\theta\|^2} = \frac{1}{2} g\theta \in H_0$ is the orthogonal projection of the apex 0 to H_0. On the other hand, the volume of A is $\frac{1}{l!}$ times the volume $D = \left| \frac{\overset{\vee}{\varpi}_1}{n_1} \wedge \cdots \wedge \frac{\overset{\vee}{\varpi}_l}{n_l} \right|$ of the parallelepiped spanned by the vectors $\frac{\overset{\vee}{\varpi}_i}{n_i}$ $(i = 1, \ldots, l)$. Hence

$$\mathrm{vol}_{l-1}(F_0) = 2\,\|\theta\|\, \frac{D}{(l-1)!} = 2\,\|\alpha_0\|\, n_0\, \frac{D}{(l-1)!}\,. \tag{11}$$

Now we compute the $(l-1)$-dimensional volume of an $(l-1)$-simplex F_i $(i = 1, \ldots, l)$ as the l-dimensional volume of the prism $F_i \times I$ where I is a unit interval perpendicular to F_i. Hence

$$\begin{aligned}\mathrm{vol}_{l-1}(F_i) &= \frac{1}{(l-1)!} \left| \frac{\overset{\vee}{\varpi}_1}{n_1} \wedge \cdots \wedge \frac{\overset{\vee}{\varpi}_{i-1}}{n_{i-1}} \wedge \frac{\alpha_i}{\|\alpha_i\|} \wedge \frac{\overset{\vee}{\varpi}_{i+1}}{n_{i+1}} \wedge \cdots \wedge \frac{\overset{\vee}{\varpi}_l}{n_l} \right| \\ &= 2\,\|\alpha_i\|\, n_i\, \frac{D}{(l-1)!} \end{aligned} \tag{12}$$

because $\alpha_i = 2 \sum_{k=1}^{l} \left(\alpha_i \,\middle|\, \overset{\vee}{\varpi}_k\right) \alpha_k = 2 \sum_{k=1}^{l} \left(\alpha_i \,\middle|\, \alpha_k\right) \overset{\vee}{\varpi}_k$.

The proof follows from (11) and (12). □

Remark The two formulae

$$\mathrm{vol}_l(A) = \frac{1}{l} \cdot \mathrm{vol}_{l-1}(F_0) \cdot \frac{1}{2\,\|\theta\|}$$

and

$$\begin{aligned}
\mathrm{vol}_l(A) &= \sum_{i=0}^{l} \mathrm{vol}_l(\text{pyramid with base } F_i \text{ and apex } \rho) \\
&= \frac{1}{l} \cdot \sum_{i=0}^{l} \mathrm{dist}(\rho, F_i) \cdot \mathrm{vol}_{l-1}(F_i) \\
&= \frac{1}{l} \cdot \sum_{i=0}^{l} \frac{1}{2} \, \|\alpha_i\| \cdot \frac{n_i \, \|\alpha_i\|}{n_0 \, \|\alpha_0\|} \cdot \mathrm{vol}_{l-1}(F_0) \\
&= \frac{1}{l} \cdot \frac{1}{2 \, \|\theta\|} \cdot \mathrm{vol}_{l-1}(F_0) \cdot \sum_{i=0}^{l} n_i \, \|\alpha_i\|^2
\end{aligned}$$

show that $\sum_{i=0}^{l} n_i \, \|\alpha_i\|^2 = 1$.

The previous proposition makes clear that the lower vertex of an upper alcove cannot have the type of a short simple root for commensurability reasons. (Here the convention is that a root is long and not short if the root system is simply laced.) We next observe that no lower vertex can have type 0. For volume reasons such a vertex would have to lie in F_0 which is absurd.

Theorem 27 (Second Sum Formula) *The following sum formula holds.*

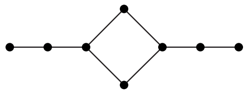
$$\sum_{\alpha_i \in \Pi^{\mathrm{long}}} n_i \, \frac{|\widehat{W}_{\perp\alpha_i}|}{|W_{\perp\alpha_i}|} = \sum_{\alpha_i \in \Pi^{\mathrm{long}}} n_i \, P_{\alpha_i}(1) = 2^{l-1}.$$

Proof We look at $\mathrm{vol}_{l-1}(2F_0)$ and compute the volume in two ways. First, of course, $\mathrm{vol}_{l-1}(2F_0) = 2^{l-1} \, \mathrm{vol}_{l-1}(F_0)$. Second, consider the tessellation of $2F_0$ induced by the tessellation of $\mathfrak{h}_{\mathbb{R}}^*$ by the alcoves. Namely, $\mathrm{vol}_{l-1}(F_i) = n_i \cdot \mathrm{vol}_{l-1}(F_0)$ and for each $\alpha_i \in \Pi^{\mathrm{long}}$ there are $P_{\alpha_i}(1)$ simplices of type i in the tessellation of $2F_0$. □

5. Symmetries of the Hasse graphs

In this section we look at the Hasse graph of the poset of abelian ideals in $\mathfrak{b}$ and determine its group of symmetries. A natural geometric realization of this Hasse graph lives in $\mathfrak{h}_{\mathbb{R}}^*$. The nodes are the ρ-points of the alcoves contained in $2A$. Two ρ-points are connected if and only if their alcoves are adjacent. Surely, the geometric symmetry group of this 1-dimensional complex is a subgroup of the abstract symmetry group of the Hasse graph. In fact, it turns out that the two symmetry groups coincide unless $\mathfrak{g}$ has type

C_3 or G_2. In the former case the abstract Hasse graph has the following shape with symmetry group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.



In the natural geometric realization the cycle of length four is actually not a square but a rectangle with side ratio $\sqrt{2} : 1$. Thus the geometric symmetry group collapses to $\mathbb{Z}/2\mathbb{Z}$. In the case of G_2 the two groups are $\mathbb{Z}/2\mathbb{Z}$ (see page 184) and 1.

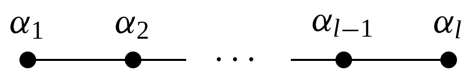
Loosely speaking, the geometric symmetry group is the symmetry group of $2A$, hence isomorphic to the symmetry group of the affine Coxeter-Dynkin graph. Going through the classification one sees that the abstract symmetry group is the same as the geometric one, except for the two cases mentioned above.

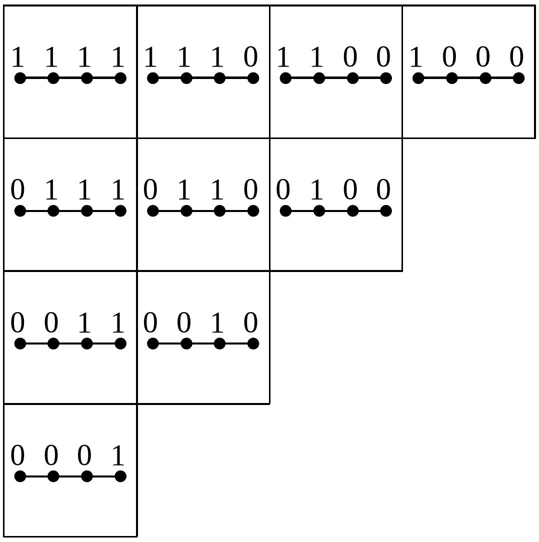
In the next few pages we show the Hasse graphs of the posets of abelian ideals in $\mathfrak{b}$ for the five simple types of rank 4. Each node of the Hasse graph consists of a diagram of a shape of which an enlarged version is drawn before the Hasse graph. The boxes of the enlarged version are filled with the nonforbidden[3] positive roots. Each node in the Hasse graph corresponds to the abelian ideal $\bigoplus_{\varphi} \mathfrak{g}_\varphi$ where φ runs over the positive roots marked by a dot.

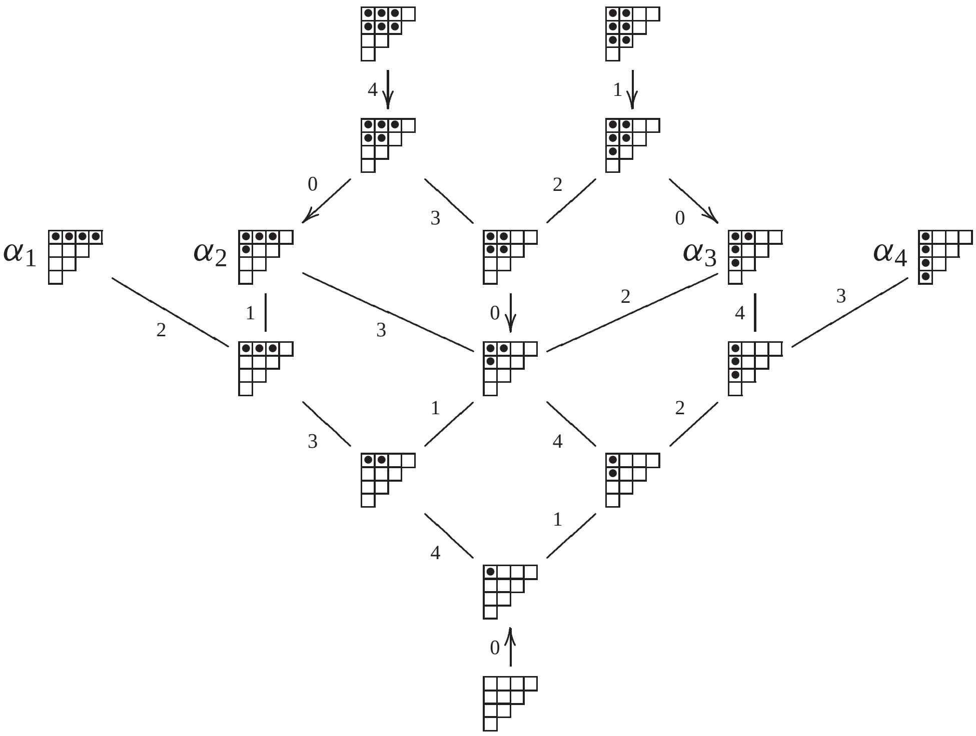
The arrows in the Hasse graphs have the following meaning. Each node which is not the source of an arrow corresponds to an ideal of the form $\mathfrak{a}^{\varphi,\min}$ for some $\varphi \in \Phi_+^{\text{long}}$. For φ a long simple root, we have labeled the node belonging to $\mathfrak{a}^{\varphi,\min}$. The passage from $0 \neq \mathfrak{a}$ to $\mathfrak{a}^{\not\perp\theta}$ corresponds to following the arrows till one arrives at a sink. Finally, an arrow points from the empty diagram ($\mathfrak{a} = 0$) to the diagram filled with one dot ($\mathfrak{a} = \mathfrak{g}_\theta$). The numbers along the edges show the types of the facets between adjacent alcoves. Disregard the arrows for the automorphism groups.

[3] A forbidden positive root φ is such that $\theta - 2\varphi$ is a sum of positive roots. Then the root space $\mathfrak{g}_\varphi$ cannot belong to an abelian ideal in $\mathfrak{b}$.

A_l



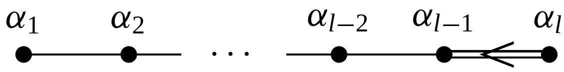


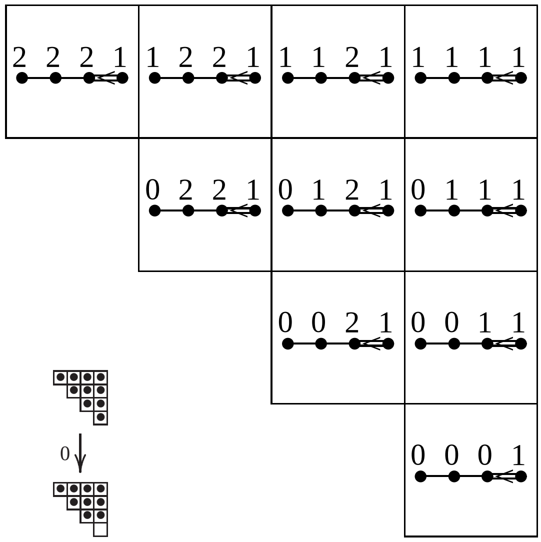


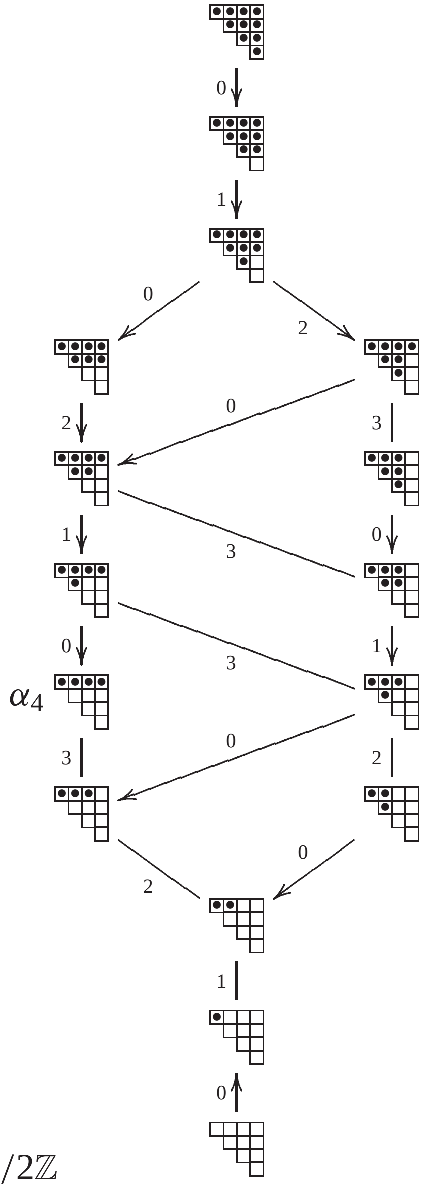
$\mathrm{Aut}\big(\mathrm{Hasse}(\mathsf{A}_1)\big) \cong \mathbb{Z}/2\mathbb{Z}$

$\mathrm{Aut}\big(\mathrm{Hasse}(\mathsf{A}_l)\big) \cong \mathrm{Dih}_{l+1} \quad (l \geqslant 2)$

C_l





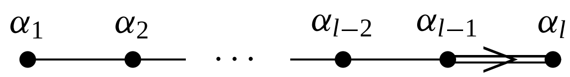


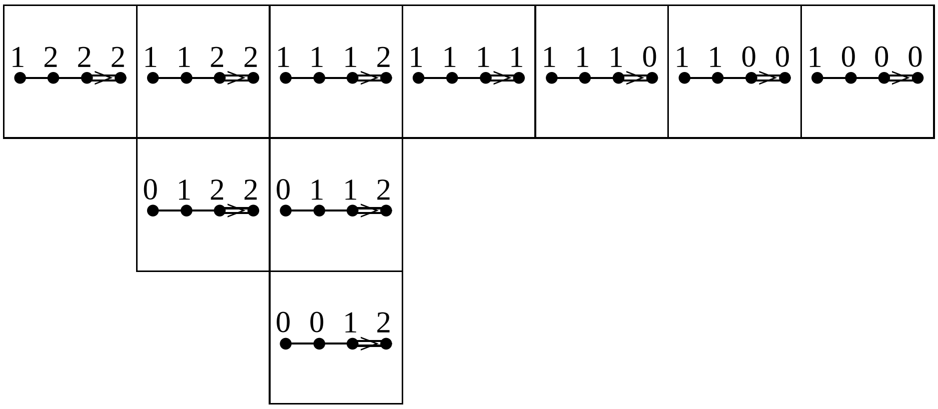
$\mathrm{Aut}\big(\mathrm{Hasse}(\mathsf{C}_2)\big) \cong \mathbb{Z}/2\mathbb{Z}$

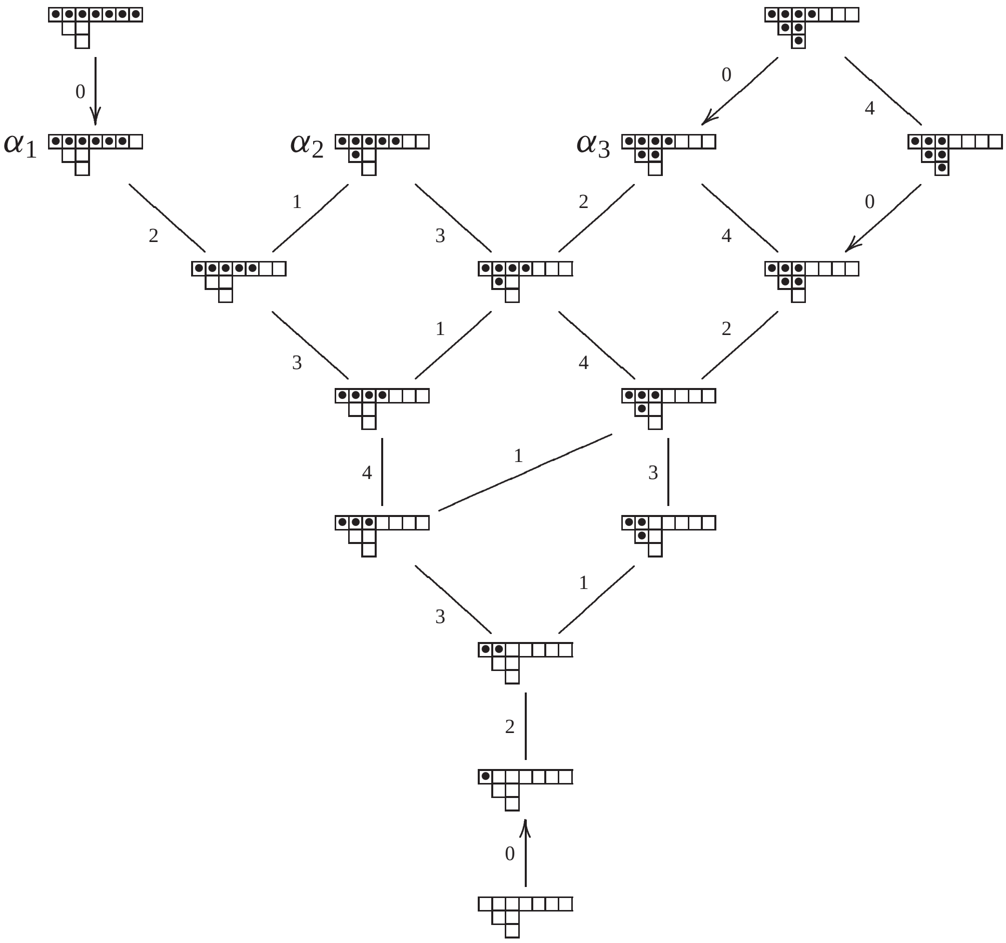
$\mathrm{Aut}\big(\mathrm{Hasse}(\mathsf{C}_3)\big) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

$\mathrm{Aut}\big(\mathrm{Hasse}(\mathsf{C}_l)\big) \cong \mathbb{Z}/2\mathbb{Z} \quad (l \geqslant 4)$

B_l

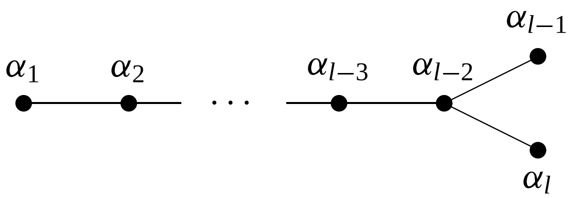


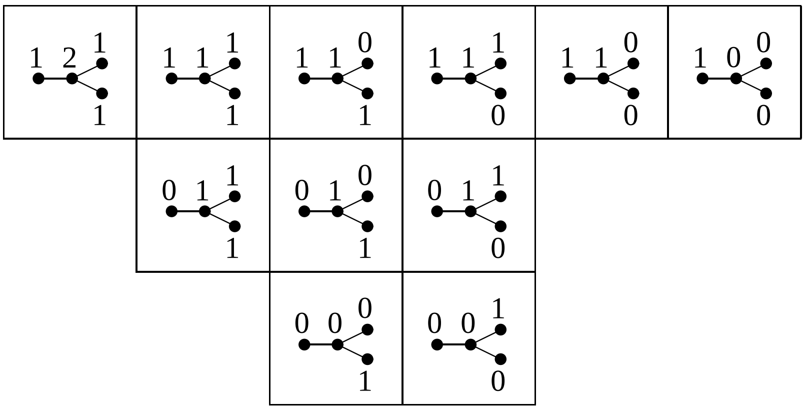


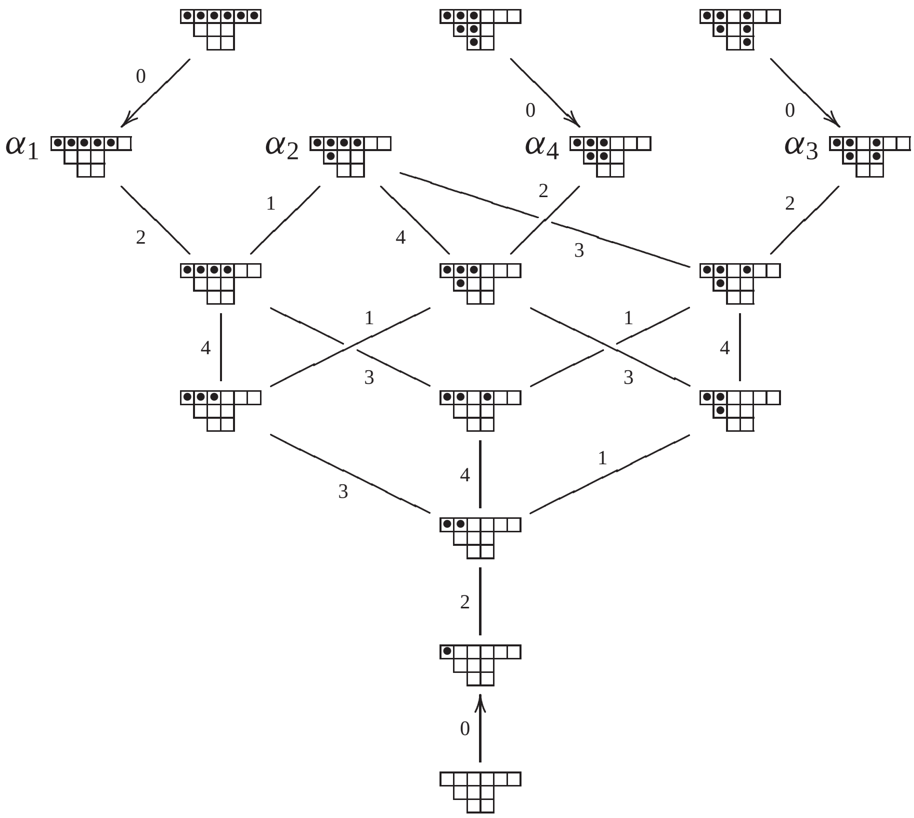


$$\mathrm{Aut}\big(\mathrm{Hasse}(\mathsf{B}_l)\big) \cong \mathbb{Z}/2\mathbb{Z} \quad (l \geqslant 2)$$

D_l



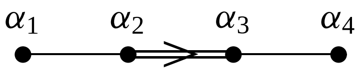


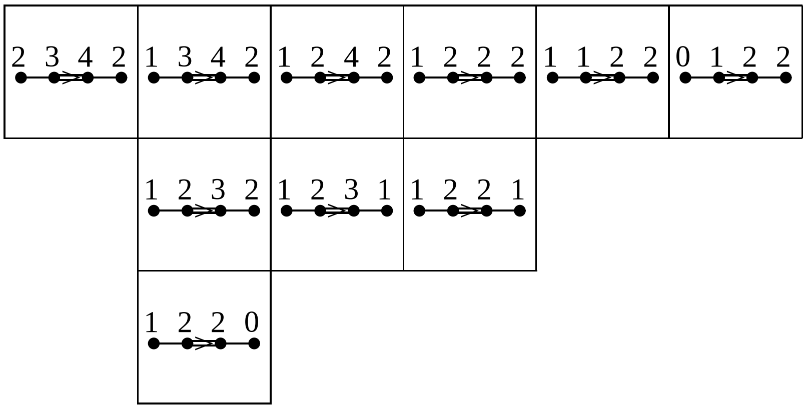


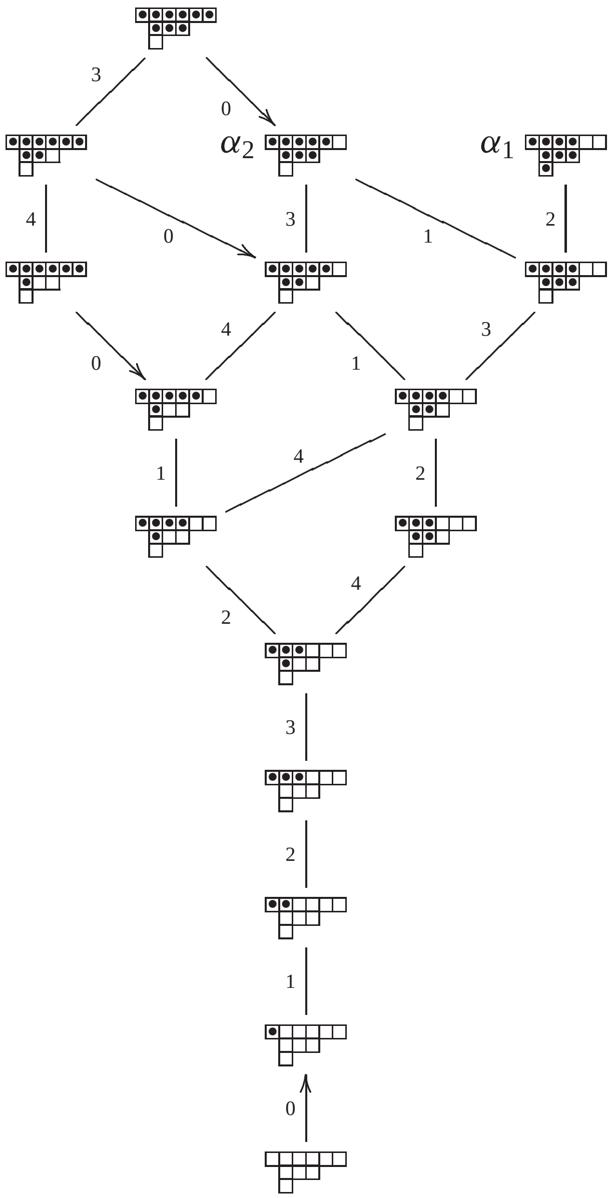
$\mathrm{Aut}\big(\mathrm{Hasse}(\mathsf{D}_4)\big) \cong \mathrm{Sym}_4$

$\mathrm{Aut}\big(\mathrm{Hasse}(\mathsf{D}_l)\big) \cong \mathrm{Dih}_4 \quad (l \geqslant 5)$

F_4

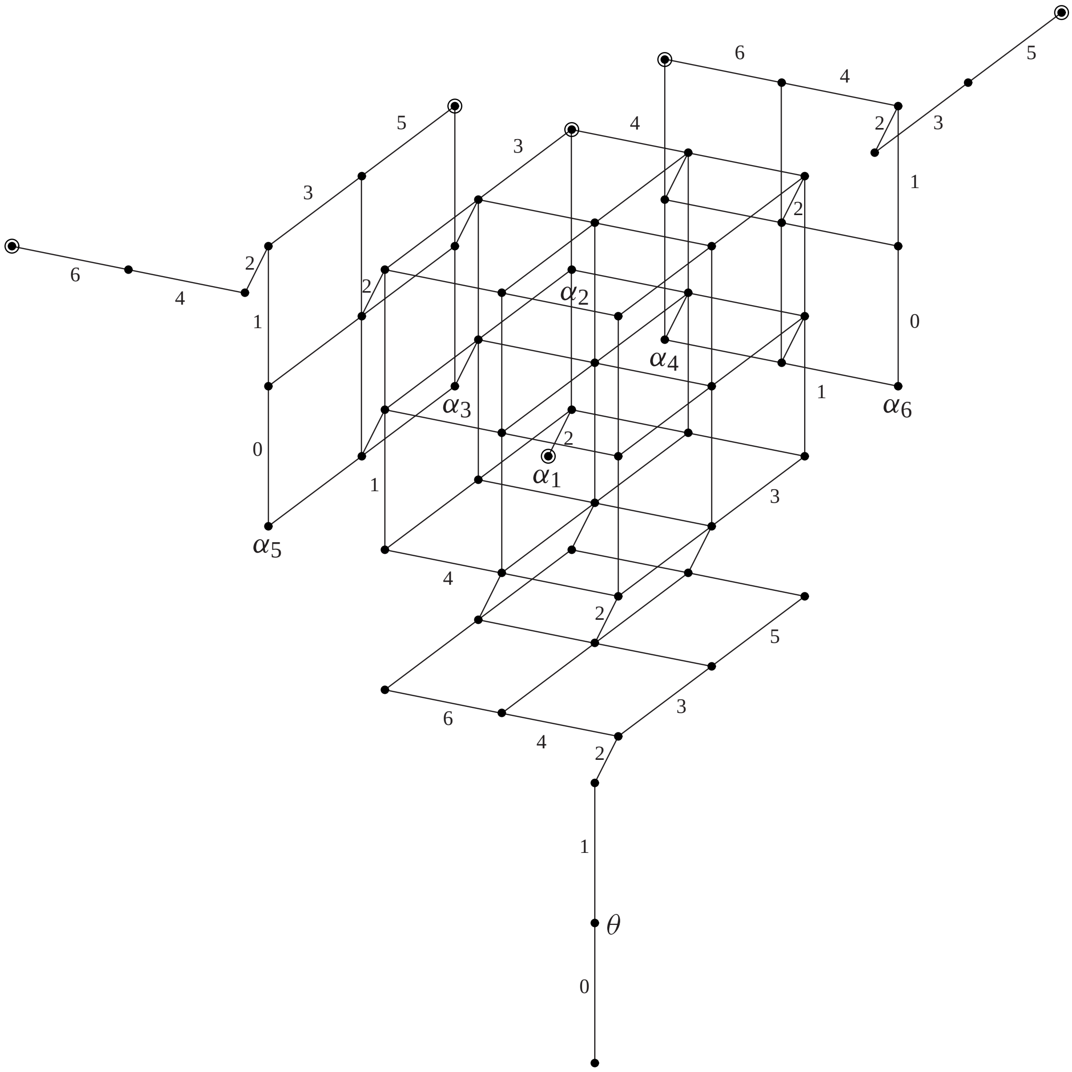




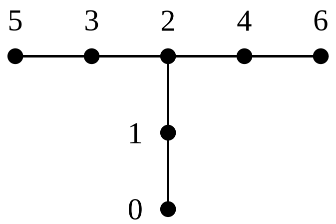


$$\mathrm{Aut}\big(\mathrm{Hasse}(\mathsf{F}_4)\big) = 1$$

Finally, let us display the Hasse graph of the poset of abelian ideals in $\mathfrak{b}$ for type E_6. I chose to draw it in a way in which the symmetry becomes manifest. The nodes marked by $\varphi = \theta, \alpha_1, \ldots, \alpha_6$ carry the abelian ideals $\mathfrak{a}^{\varphi,\min}$. The encircled nodes mark the maximal abelian ideals $\mathfrak{a}^{\alpha_i,\max}$ $(i = 1, \ldots, 6)$.



$$\mathrm{Aut}\big(\mathrm{Hasse}(\mathsf{E}_6)\big) \cong \mathrm{Sym}_3$$



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