Curves on K3 surfaces - what to do with multiple covers?

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Overview

- K3 surfaces and GW invariants
- Brief history: YZ, MPT, KKV, PT
- Multiple cover formula
- Recent progress

K3 surfaces

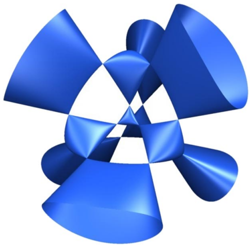
- S - complex smooth projective K3 surface:

$$\omega_S \cong \mathcal{O}_S\,, \quad H^1(S, \mathcal{O}_S) = 0$$

- $\beta \in H_2(S, \mathbb{Z})$ effective curve class
- Example: Fermat quartic $S \subset \mathbb{P}^3$ zero locus of

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$$

and $\beta = c_1\big(\mathcal{O}(1)|_S\big)$



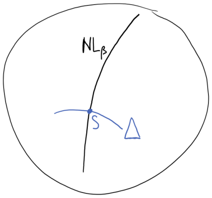
K3 surfaces

- Want to define $\mathrm{GW}^{\mathrm{S}}_{\mathrm{g},\beta}$ (enumerative invariants, string theory, etc.)
- Problem: β deforms outside of NL_β, i.e. no longer $(1,1)$ Hodge class

$$\implies \overline{M}_{g,n}(S,\beta) \text{ has virtual class } [\ \ ]^{vir} = 0$$

- Solution: Remove trivial piece $\mathcal{O}_{\overline{M}}$ from obstruction theory

$$\implies [\overline{M}_{g,n}(S,\beta)]^{red} \in A_{g+n}(\overline{M}_{g,n}(S,\beta))$$



Moduli space of K3 surfaces

K3 surfaces

- Key fact: By deformation (Torelli theorem) $\mathrm{GW}(\mathrm{S},\beta)$ depends only on

$$\beta^2 \in 2\mathbb{Z}\,, \quad \mathrm{div}(\beta)$$

- Often use $S \to \mathbb{P}^1$ elliptic K3 surface with section B and fiber class F:

$$mB + hF \in H_2(S,\mathbb{Z})$$

Definition

For $\psi_i \in A^1(\overline{M}_{g,n}(S,\beta))$, $\gamma_1, \ldots, \gamma_n \in H^*(S,\mathbb{Q})$ define

$$\left\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \right\rangle^S_{g,\beta} = \int_{[\overline{M}_{g,n}(S,\beta)]^{red}} \prod \psi_i^{k_i} ev_i^* \gamma_i \in \mathbb{Q}\,.$$

Primitive curve classes

- $\beta_h \in H_2(S, \mathbb{Z})$ *primitive* with $\beta_h^2 = 2h - 2$ (e.g. $\beta_h = B + hF$)
- Genus $g = 0$: $N_h = \int_{[\overline{M}_0(S,\beta_h)]^{red}} 1 = \#_{vir}(\text{Rational curves})$
- Yau–Zaslow formula: generating series is a modular form

$$\sum_{h \geq 0} N_h q^{h-1} = q^{-1} \prod_{k \geq 1} \frac{1}{(1 - q^k)^{24}}$$
$$= q^{-1} + 24 + 324q + 3200q^2 + \cdots$$

Primitive curve classes

Maulik–Pandharipande–Thomas '10:

- (Quasi-)modularity true for all descendents (β primitive)!
- Algorithm for computation (reducing to YZ formula)
- (Katz–Klemm–Vafa formula): Computation of

$$N_{g,\beta} = \int_{[\overline{M}_g(S,\beta)]^{red}} (-1)^g \lambda_g \in \mathbb{Q}\,,$$

  where $\lambda_g = c_{top}(\mathbb{E}_g)$ and $\mathbb{E}_g$ is the Hodge bundle.

(1) Technique of proof?

(2) Why this particular integral?

Technique

Two types of degenerations of S:

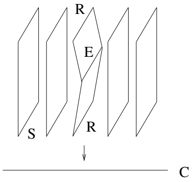
- Degeneration to normal cone of smooth fiber $E \subset S$

$$S \rightsquigarrow S \cup_E \mathbb{P}^1 \times E$$

- Breaking

$$S \rightsquigarrow R \cup_E R \,,$$

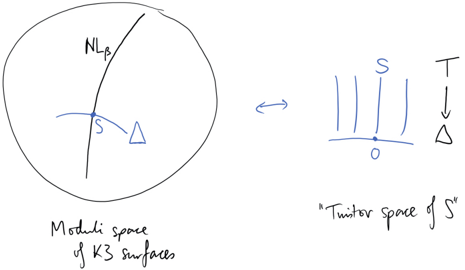
where $R \to \mathbb{P}^1$ rational elliptic surface.



Key in both cases: $\beta \cdot E = 1$

3-fold invariants

- $N_{g,\beta} = \int_{[\overline{M}_g(S,\beta)]^{red}} (-1)^g \lambda_g$ naturally arises from 3-fold geometry:



- As moduli spaces there is an equality

$$\overline{M}_g(T, \iota_*\beta) = \overline{M}_g(S, \beta)$$

with virtual class $[\ ]^{vir} = [\ ]^{red} \cap (-1)^g \lambda_g$

3-fold invariants

- Let β primitive and $m \geq 1$

  $\rightsquigarrow$ formula for $m\beta$ (from 3-fold point of view)?
- Example $X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$ non-compact Calabi–Yau 3-fold:

$$N^X_{g=0,d[\mathbb{P}^1]} = \frac{1}{d^3} \qquad \text{(compare with YZ formula)}$$

$$N^X_{g,d[\mathbb{P}^1]} = \frac{|B_{2g}|}{2g(2g-2)!)} d^{2g-3}$$

KKV/ PT

Theorem (Pandharipande–Thomas '14)

$$N_{g,m\beta} = \sum_{d|m} d^{2g-3} N_{g,\beta_{h_d}} \,,$$

where $2h_d - 2 = (\frac{m}{d}\beta)^2$.

Using Calabi–Yau 3-fold + MNOP move problem to *sheaf theory* (stable pairs) on K3 surfaces:

Certain integrals on Hilbert schemes depend on β only through β^2, *independent of divisibility*

Multiple cover formula

- At the heart: Let $d \mid m$ and

$$C' \xrightarrow{d:1} C\,, \quad C \xrightarrow{f} S$$

with $f_*[C] = \frac{m}{d}\beta$, then $f'_*[C'] = m\beta$ for the composition

- $\overline{M}_{g,n}(S, m\beta)$ has (besides contracted components) all sorts of complicated components from multiple covers

- Formula for $m\beta$ via divisors $d|m$?

Multiple cover formula

Conjecture (Oberdieck–Pandharipande '14)

$$\left\langle \alpha; \gamma_1 \ldots \gamma_n \right\rangle_{g,m\beta} = \sum_{d|m} d^{2g-3+\sum \deg(\gamma_i)} \left\langle \alpha; \tilde{\gamma}_1 \ldots \tilde{\gamma}_n \right\rangle_{g,\beta_{h_d}}$$

Supporting evidence:

- $g \leq 1$ all curve classes
- $(-1)^g \lambda_g$ in all genus, all curve classes
- Example (Oberdieck): Genus 2 invariant $\left\langle \tau_0(p)^2 \right\rangle_{2,2\beta_2}$

Recent progress

Theorem (Bae-B. '20)

For divisibility two curve classes 2β:

(i) *(Quasi-)modularity in all genus, all descendents*

(ii) *Holomorphic anomaly equation (recursive structure for certain derivate of generating series)*

(iii) *Extend algorithm [MPT] to compute all* $\mathrm{GW}^{\mathrm{S}}_{\mathrm{g},2\beta}$

What's new?

DR relations

- $\mathfrak{M}_{g,n}$ - stack of prestable curves, $\mathfrak{Pic}_{g,n} \to \mathfrak{M}_{g,n}$ Picard stack for the universal curve
- $\mathfrak{Pic}_{g,n}$ classifies pairs (C, L) of prestable curve with line bundle
- S - K3 surface, $\mathcal{L} \in \mathrm{Pic}(S)$ get a morphism

$$\varphi_{\mathcal{L}} \colon \overline{M}_{g,n}(S, \beta) \to \mathfrak{Pic}_{g,n}, \ [f \colon C \to S] \mapsto (C, f^*\mathcal{L})$$

- Pull-back tautological relations in A^* (or H^*) via this map

DR relations

Let $d = \int_\beta \mathcal{L}$, $A = (a_1, \dots, a_n) \in \mathbb{Z}^n$ be a vector of integers satisfying

$$\sum_i a_i = d\,.$$

Denote by $P^{\,c}_{g,A,d}$ the codimension c component in $A^c_{\mathrm{op}}(\mathfrak{Pic}_{g,n})$ (followed by taking constant part in r) of

$$\sum_{\substack{\Gamma\in \mathsf{G}_{g,n,d}\\ w\in \mathsf{W}_{\Gamma,r}}} \frac{r^{-h^1(\Gamma_\delta)}}{|\operatorname{Aut}(\Gamma_\delta)|}\, j_{\Gamma*}\Bigg[\prod_{i=1}^n \exp\left(\frac{1}{2}a_i^2\psi_i + a_i\xi_i\right) \prod_{v\in V(\Gamma_\delta)} \exp\left(-\frac{1}{2}\eta(v)\right)$$

$$\prod_{e=(h,h')\in E(\Gamma)} \frac{1-\exp\left(-\frac{w(h)w(h')}{2}(\psi_h+\psi_{h'})\right)}{\psi_h+\psi_{h'}}\Bigg].$$

DR relations

Theorem (BHPSS '20, Pixton)

(i) $P^{\,c}_{g,A,d} = 0$ *for all* $c > g$ *in* $A^c_{\mathrm{op}}(\mathfrak{Pic}_{g,n})$

(ii) *Dependence on* $\{a_i\}$ *is polynomial*

Application: Choose $\mathcal{L} \in \mathrm{Pic}(S)$ and pullback equation to obtain

$$\varphi^*_{\mathcal{L}}\left(P^{\,c}_{g,A,d} = 0\right) \cap [\ \ ]^{red} \in A_{g+n-c}(\overline{M}_{g,n}(S,\beta))$$

then pick monomial in $\{a_i\}$

Upshot: relation among descendents

Calculation

- Let $S \to \mathbb{P}^1$ elliptic with section B, fiber class F:

$$\beta = mB + hF \in H_2(S, \mathbb{Z})$$

- For $\mathrm{div}(\beta) \geq 2$ consider genus 2 invariant $\langle \tau_1(F)\tau_0(\mathsf{p}) \rangle_{2,\beta}$, where $p \in H^4(S)$ class of a point
- (Refined) algorithm [MPT] breaks down in this case
- Use relation $P^3_{2,A,m} = 0$ and reduce to descendents where algorithm applies

Calculation

- Line bundle $\mathcal{L} = \mathcal{O}_S(F)$, then $\int_\beta \mathcal{L} = m$
- For $a_1 + a_2 = m$ the $[a_1^4]$-coefficient is (up to lower genus data)

$$-\frac{1}{2}\psi_1 \, \mathrm{ev}_1^*(F) \, \mathrm{ev}_2^*(F) - \frac{1}{2}\psi_2 \, \mathrm{ev}_1^*(F) \, \mathrm{ev}_2^*(F)$$

- Integrating

$$\mathrm{ev}_2^*(B) P^3_{2,A,m}|_{a_2 = m - a_1}$$

against the reduced class $[\overline{M}_{2,2}(S,\beta)]^{red}$, we find

$$-\frac{1}{2}\langle \tau_1(F)\tau_0(\mathrm{p})\rangle_{2,\beta} \underbrace{-\frac{m}{2}\langle \tau_1(\mathrm{p})\rangle_{2,\beta} + \text{(lower genus data)}}_{\text{algorithm applies}} .$$

Calculation

- Relation on $\overline{M}_{2,2}(S, \beta)$:

$$-\frac{1}{2}\psi_1 \, \mathrm{ev}_1^*(F) \, \mathrm{ev}_2^*(F) - \frac{1}{2}\psi_2 \, \mathrm{ev}_1^*(F) \, \mathrm{ev}_2^*(F) + (\text{lower genus data}) = 0$$

- Traded $\psi_1 \longleftrightarrow \psi_2$ after $(\dots) \cup \mathrm{ev}_1^*(F) \, \mathrm{ev}_2^*(F)$
- Not available on $\overline{M}_{2,2}$

Outlook

- For $\mathrm{div} = 2$ sufficient to calculate some initial data in this way
- For $\mathrm{div} > 2$ not clear how much initial data is sufficient
- **Pro:** Relations for all genus, all curve classes
- **Con:** Complicated to study systematically
- [MPT] algorithm used Ionel-Getzler vanishing: any $\alpha \in R^{\geq g}(\overline{M}_{g,n})$ is pushforward from boundary $\partial\overline{M}_{g,n} \to \overline{M}_{g,n}$
- Attempt to use DR relations for such a study on $\overline{M}_{g,n}(S, \beta)$