Symmetry for stable pair invariants via derived equivalences – the STU model

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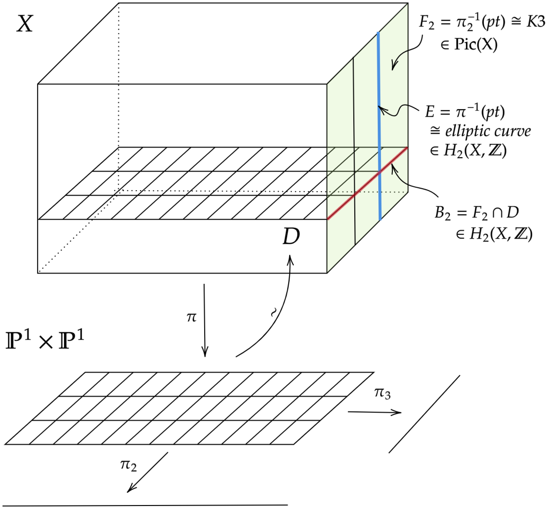
Overview

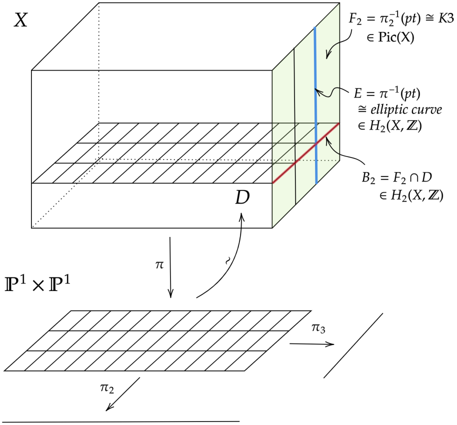
- Geometry of STU model
- Curve counting invariants
- Symmetry for BPS
- Main Question
- Symmetry for PT via derived equivalences
- Seidel–Thomas spherical twist
- Wall-crossing

X - smooth projective Calabi–Yau 3-fold (STU model)



Geometry

- $\pi : X \to \mathbb{P}^1 \times \mathbb{P}^1$ elliptic fibration with section $D \subset X$
- $E = \pi^{-1}(pt) \in H_2(X, \mathbb{Z})$
- $F_i = \pi_i^{-1}(pt) \in \mathrm{Pic}(X),\ i = 2, 3$
- $B_i = F_i \cap D$, i=2, 3 $\in \mathrm{Im}\Big(H_2(D, \mathbb{Z}) \to H_2(X, \mathbb{Z})\Big)$
- $\beta_{h,i,j} = hE + iB_2 + jB_3$ $\in H_2(X, \mathbb{Z})$



Stable pairs

- Stable pair is $I^\bullet = [\mathcal{O}_X \xrightarrow{s} F] \in D^b(X)$ with $\mathrm{ch}(I^\bullet) = (1, 0, -\beta, -n)$ if F pure 1-dim sheaf and $\mathrm{Cok}(s)$ is 0-dim.
- Equivalently:
  (i) $h^i(I^\bullet) = 0$ for $i \neq 0, 1$
  (ii) $h^0(I^\bullet)$ is torsion free and $h^1(I^\bullet)$ is 0-dim
  (iii) $\mathrm{Hom}(Q[-1], I^\bullet) = 0$ for every 0-dim Q.
- $C \subset X$ Cohen–Macaulay with Cartier divisor $\mathcal{O}_C(D)$ then $[\mathcal{O}_X \to \mathcal{O}_C(D)]$ stable pair
- $C = C_1 \cup C_2$ smooth curves with finite intersection Z, then $[\mathcal{O}_X \to \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2}]$ stable pair with $\mathrm{Cok}(s) = Z$

Invariants

- Moduli space $P_n(X,\beta)$ of stable pairs is projective scheme with symmetric perfect obstruction theory and $[P_n(X,\beta)]^{vir}$ of dim 0
- Stable pair invariant

$$P_{n,\beta} = \int_{[P_n(X,\beta)]^{vir}} 1 \in \mathbb{Z}$$

- Generating function

$$\sum_{n\in\mathbb{Z}} P_{n,\beta} q^n$$

is Laurent expansion of rational function invariant under $q \leftrightarrow q^{-1}$ (dualizing $\mathbb{D}$)

PT/ BPS

- Stable pair invariants is *disconnected* theory
- Define connected invariants $P^{conn}_{n,\beta} \in \mathbb{Q}$ via

$$1 + \sum_{\substack{\beta \neq 0 \\ n \in \mathbb{Z}}} P_{n,\beta} q^n v^\beta = \exp\left(\sum_{\substack{\beta \neq 0 \\ n \in \mathbb{Z}}} P^{conn}_{n,\beta} q^n v^\beta \right)$$

- For all $\beta \neq 0$ there is unique expansion

$$\sum_{n \in \mathbb{Z}} P^{conn}_{n,\beta} q^n = \sum_{\substack{g \geq 0 \\ r | \mathrm{div}(\beta)}} n_{g,\frac{\beta}{r}} \frac{(-1)^{g-1}}{r} \left((-q)^r - 2 + (-q)^{-r} \right)^{g-1}$$

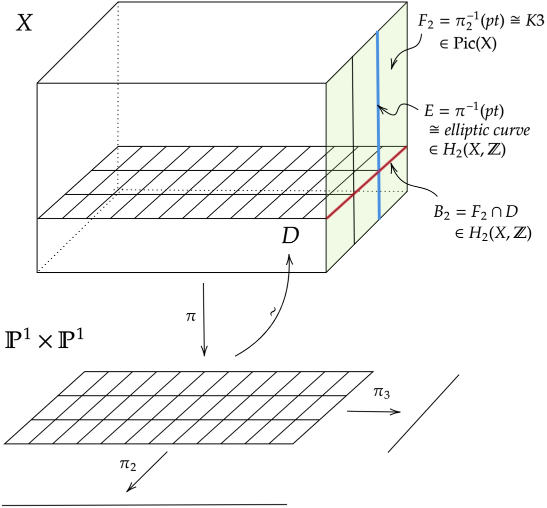
with $n_{g,\beta} \in \mathbb{Z}$ the BPS invariants

Computations

- What do we know about $\{P_{n,\beta}\}$ or $\{n_{g,\beta}\}$ for X - STU model?
- For $\beta_{h,i,j} = hE + iB_2 + jB_3 \in H_2(X,\mathbb{Z})$ write $n_{g,(h,i,j)}$
- Consider $\beta_{(h,i,0)}$, then proof of KKV Conjecture + Noether–Lefschetz computation gives complete solution

Computations

Fiber class $\beta = hE + B_2 \in \mathrm{Im}\big(\mathrm{H}_2(\mathrm{K3}, \mathbb{Z}) \to \mathrm{H}_2(\mathrm{X}, \mathbb{Z})\big)$



Theorem (Pandharipande–Thomas)

(i)

$$\sum_{g\geq 0,\, h>0} n_{g,\beta_{(h,1,0)}} (q+2+q^{-1})^g t^h = \frac{-2E_4(t)E_6(t)}{\prod_{n\geq 1}(1-t^n)^{20}(1+qt^n)^2(1+q^{-1}t^n)^2}$$

(ii) $n_{g,(h,i,0)} = n_{g,(h,0,i)}$,

(iii) $n_{g,(h,i,0)} = n_{g,(h,h-i,0)}$,

(iv) $n_{g,(h,i,0)}$ depends only on $i(h-i)$ and vanishes for $i > h$ except $n_{0,(0,1,0)} = -2$.

- Thus, $n_{g,(h,i,0)}$ and $n_{g,(h,0,j)}$ fully understood
- Genus 0 computable via mirror symmetry
- [Oberdieck–Shen] solution for $n_{g,(h,1,1)}$ (for all g, h via Jacobi forms) application of elliptic transformation law
- Beyond these cases, **widely open**

Jacobi form conjecture

- For $H \in H_2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z})$ (i.e. fixed (i, j)) consider

$$PT_H(q,t) = \sum_{h \geq 0, n \in \mathbb{Z}} P_{n,H+hE} q^n t^h$$

- [Huang–Katz–Klemm] conjecture that

$$\frac{PT_H(q,t)}{PT_0(q,t)}$$

is meromorphic Jacobi form.

BPS numbers $n_{(h,i,j)}$ in genus 0

(i,j) \ h	0	1	2	3	4	5
(0,1)	-2	480	282888	17058560	477516780	8606976768
(1,1)	-4	1440	-226080	51516800	107913873744	17263743561792
(2,1)	-6	2400	-452160	103374720	-16013531460	19768877695872
	$\uparrow$ $-\frac{2}{(1-t)^2}$	$\uparrow$ $\frac{480(1+t)}{(1-t)^2}$	$\uparrow$ $\frac{p_{2,1}}{(1-t)^2}$	$\uparrow$ $\frac{p_{3,1}}{(1-t)^2}$	$\dots$	$\dots$

- Rows = quasimodular forms (Taylor expansion of [HKK] Jacobi form)
- Columns $= -\frac{2}{(1-t)^2},\ 480\frac{1+t}{(1-t)^2},\ \frac{282888(1+t^2)-791856t}{(1-t)^2},$ $\frac{17058560(1+t^3)+17399680(t+t^2)}{(1-t)^2}$
- **Question I:** $\sum_{i\geq 0} n_{(h,i,j)} t^i$ = Laurent expansion of rational function with functional equation?

BPS numbers $n_{(h,i,j)}$ in genus 0

j=1

(i,j) \ h	0	1	2	3	4	5
(0,1)	-2	480	282888	17058560	477516780	8606976768
(1,1)	-4	1440	-226080	51516800	107913873744	17263743561792
(2,1)	-6	2400	-452160	103374720	-16013531460	19768877695872
	$\uparrow$ $\frac{-2}{(1-t)^2}$	$\uparrow$ $\frac{480(1+t)}{(1-t)^2}$	$\uparrow$ $\frac{p_{2,1}}{(1-t)^2}$	$\uparrow$ $\frac{p_{3,1}}{(1-t)^2}$	...	...

j=2

(i,j) \ h	0	1	2	3	4	5
(0,2)	0	0	-2	282888	477516780	115311621680
(1,2)	-6	2400	-452160	103374720	-16013531460	19768877695872
(2,2)	-32	16800	-4093920	789875200	-115113738240	63247732459200
	$\uparrow$ $\frac{-2t(3+4t+3t^2)}{(1-t)^6(1+t)^2}$	$\uparrow$ $2400\,\frac{t(1+t)}{(1-t)^6}$	$\uparrow$ $\frac{p_{2,2}}{(1-t)^6(1+t)^2}$	$\uparrow$ $\frac{p_{3,2}}{(1-t)^6}$	...	...

BPS numbers $n_{(h,i,j)}$ in genus 1

(i,j) \ h	0	1	2	3	4	5
(0,1)	0	4	-948	-568640	-35818260	-1059654720
(1,1)	0	12	-4752	-469072	96219816	-120434126760
(2,1)	0	20	-9504	298704	-1107564076	-185860380240

↑	↑	↑	↑		
0	$\frac{4(1+t)}{(1-t)^2}$	$\frac{p_{2,1}^{(g=1)}}{(1-t)^2}$	$\frac{p_{3,1}^{(g=1)}}{(1-t)^2}$	$\cdots$	$\cdots$

$$p_{2,1}^{(g=1)} = -948(1+t^2) - 2856t\,,$$
$$p_{3,1}^{(g=1)} = -568640(1+t^3) + 668208(t+t^2)$$

Heterotic string

- In physics literature appears symmetry from heterotic string + mirror symmetry
- For all $g \geq 0$ expect some relation

$$n_{g,(h,i,j)} \sim n_{g,(h,h-i-2j,j)}$$

- For classes in K3 fiber the symmetry reduces to

$$n_{g,(h,i,0)} = n_{g,(h,h-i,0)}$$

- Origin of this symmetry is monodromy for elliptic K3 surfaces exchanging $F \leftrightarrow B + F$

Heterotic string

- However, in general should **not** hold as equality at level of BPS counts, e.g. in genus 0

$$n_{(3,0,1)} = 17\,058\,560 \neq 51\,516\,800 = n_{(3,1,1)}$$

- What relation

$$n_{g,(h,i,j)} \sim n_{g,(h,h-i-2j,j)}$$

  should hold?
- Related to columns $\sim$ rational function (with functional equation)?

Heterotic string

- [Klemm–Kreuzer–Rieger–Scheidegger] propose expansion of GW prepotential in genus 0 as

$$\mathcal{F}_0 = \sum_{h,j\geq 0} f_{h,j}(t_2) t_1^h t_3^j \,,$$

and the relation imposes

$$f_{h,j}(1/t) = t^{2j-h} f_{h,j}(t)$$

- Note that equality $n_{g,(h,i,j)} = n_{g,(h,h-i-2j,j)}$ would be equivalent to $f_{h,j}$ symmetric polynomial of degree $h-2j$

Heterotic string

- More specifically [KKRS] consider

$$f_{h,j}(t) = \frac{p_{h,j}(t)}{(1-t)^{4j-2}}$$

  with $p_{h,j}(t)$ symmetric polynomial of degree $\leq 2j - 2 + h$
- Verified for $h = 0$ and small (i, j), i.e. $\mathcal{O}(-2,-2) \to \mathbb{P}^1 \times \mathbb{P}^1$ non-compact CY3
- Exactly matches the rationality of columns in the tables
- **Upshot:** Symmetry expected not on level of coefficients but for generating functions

Heterotic string

- **Question II:** For all $(h, j) \geq 0$ and $n \in \mathbb{Z}$ is

$$f(t) = \sum_{i \geq 0} P_{n, \beta_{(h,i,j)}} t^i$$

Laurent expansion of rational function with functional equation

$$f(1/t) = t^{2j-h} f(t) \quad ?$$

- True for $j = 0$ (K3 fiber classes)
- Not so clear: relation to modularity/ [HKK] Jacobi form

Symmetries for PT generating functions I

- Fix $\beta \in H_2(X, \mathbb{Z})$, then

$$\sum_{n \in \mathbb{Z}} P_{n,\beta} q^n$$

is Laurent expansion of rational function invariant under $q \leftrightarrow q^{-1}$

- Does not hold at level of coefficients, e.g.

$$\frac{q}{(1+q)^2} = q - q^2 + 2q^3 - \dots$$

- Proved using derived dual $\mathbb{D} \in \mathrm{Aut}(D^b(X))$ and wall-crossing [Bridgeland, Toda]

Symmetries for PT generating functions I

- Why wall-crossing?
- $I^\bullet$ stable pair, $\phi \in \mathrm{Aut}(D^b(X))$ then $\phi(I^\bullet)$ usually not stable pair
- But if $\phi(I^\bullet)$ stable for some (weak) stability condition σ

$$\sigma\text{-stability} \overset{\text{wall-crossing}}{\dashrightarrow} \text{stable pair stability}$$

Symmetries for PT generating functions II

- Fix $H \in H_2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z})$, then

$$Z_H(q,t) = \frac{PT_H(q,t)}{PT_0(q,t)}$$

  should satisfy elliptic transformation law for all $\lambda \in \mathbb{Z}$

$$Z_H(qt^\lambda, t) = t^{-(\ell-1)\lambda^2} q^{-2(\ell-1)\lambda} Z_H(q,t)$$

  as function in $\mathbb{Q}(q)[[t]]$, not on level of coefficients
- [Oberdieck–Shen] for H irreducible using Poincaré sheaf Fourier–Mukai transform $\phi_{\mathcal{P}} \in \mathrm{Aut}(D^b(X))$ and wall-crossing

Symmetries for PT generating functions III

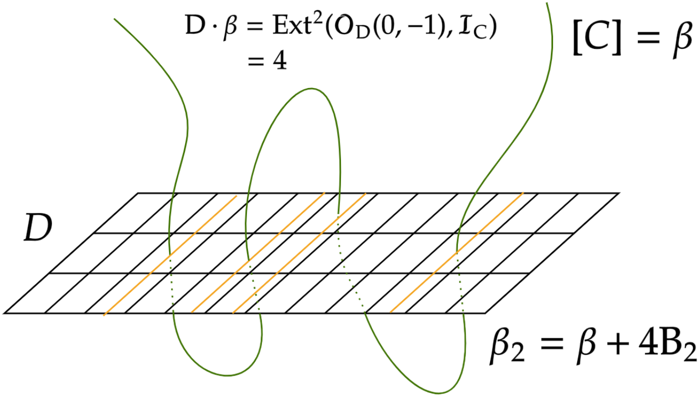
- **Goal:** connect $n_{g,(h,i,j)} \sim n_{g,(h,h-i-2j,j)}$ to derived equivalence
- Note that $D \cdot \beta_{(h,i,j)} = h - 2i - 2j$ since $\mathcal{N}_{D/X} = \omega_D$, thus equivalently

$$n_{g,\beta} \sim n_{g,\beta+(D\cdot\beta)B_2}$$

- Consider $\mathcal{O}_D(0,-1) \in D^b(X)$ and curve $C \subset X$ of class β, then

$$\sum(-1)^i \mathrm{Ext}^i(\mathcal{O}_D(0,-1), \mathcal{I}_C) = D \cdot \beta$$

Symmetries for PT generating functions III



Seidel–Thomas spherical twist

- More generally, for $\mathcal{E} \in D^b(X)$ define $\widetilde{\mathcal{E}}$ by exact triangle

$$\bigoplus \mathrm{Ext}^i(\mathcal{O}_D(0,-1), \mathcal{E}) \otimes \mathcal{O}_D(0,-1)[-i] \to \mathcal{E} \to \widetilde{\mathcal{E}}$$

- This defines equivalence "Seidel–Thomas spherical twist"

$$\mathrm{ST}_{\mathcal{O}_D(0,-1)} = \widetilde{(-)} \in \mathrm{Aut}(D^b(X))$$

- Let $\beta \in H_2(X,\mathbb{Z})$, $\beta_2 = \beta + (D \cdot \beta) B_2$ and $I^\bullet = [\mathcal{O}_X \to F]$ stable pair with $\mathrm{ch}(I^\bullet) = (1, 0, -\beta_2, -n)$
- Let $\mathrm{Coh}_{\leq 1}(X)$ be sheaves on X with support of dimension ≤ 1 then

$$I^\bullet \in \left\langle \mathcal{O}_X, \mathrm{Coh}_{\leq 1}(X)[-1] \right\rangle_{tr}$$

with exact triangle

$$F[-1] \to I^\bullet \to \mathcal{O}_X$$

- Note: $H^*(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2,-1)) = 0$ implies $\widetilde{\mathcal{O}_X} = \mathcal{O}_X$

- Thus, there is exact triangle

$$\widetilde{F[-1]} \to \widetilde{I^\bullet} \to \mathcal{O}_X$$

- Let $\mathrm{Coh}^D_{\leq 1}(X)$ be sheaves on X with support of dimension ≤ 1 outside D and

$$\mathcal{D}_X = \left\langle \mathcal{O}_X, D^b(\mathrm{Coh}^D_{\leq 1}(X)) \right\rangle_{tr}$$

then $\widetilde{I^\bullet} \in \mathcal{D}_X$ with $\mathrm{ch}(\widetilde{I^\bullet}) = \left(1, (D \cdot \beta)D, -\beta, -n + \frac{D \cdot \beta}{3}\right)$

Wall-crossing

- Attempt to relate images $\widetilde{I^\bullet}$ to stable pairs via wall-crossing
- $\beta \in H_2(X, \mathbb{Z})$, $m = D \cdot \beta$ and $\beta_2 = \beta + mB_2$
- Let $\gamma = (1, mD, -\beta, -n + \frac{m}{3}) = \mathrm{ch}(\widetilde{I^\bullet_{n,\beta_2}})$, then

$$\gamma = \mathrm{ch}\Big(I^\bullet_{n',\beta'} \otimes \mathcal{O}_X(mD)\Big)$$

where $I^\bullet_{n',\beta'}$ stable pair with

$$\beta' = \beta - m^2(B_2 + B_3)\,, \quad n' = n - m^2 - \frac{m}{3}(8m^2 + 1)$$

Much related to work of [Toda]:

- Let $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$ space of weak stability conditions on $\mathcal{D}_X$
- (in progress) there is a heart $\mathcal{A}_X$ of bounded t-structure on $\mathcal{D}_X$ and path $\{\sigma_t^{(2)} = (Z_t, \mathcal{A}_X)\}_{t\in\mathbb{R}_{\geq 0}} \subset \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$ such that

$$\left(\mathcal{F} \text{ is } \sigma^{(2)}_{0<t\ll 1} - \text{stable with } \mathrm{ch}(\mathcal{F}) = \gamma\right) \iff \mathcal{F} \simeq \mathrm{ST}_{\mathcal{O}_D(0,-1)}(I^\bullet_{n,\beta_2})$$

$$\Big\downarrow \begin{smallmatrix} t \\ \downarrow \\ \infty \end{smallmatrix}$$

$$\left(\mathcal{F} \text{ is } \sigma^{(2)}_{t\gg 1} - \text{stable with } \mathrm{ch}(\mathcal{F}) = \gamma\right) \iff \mathcal{F} \simeq I^\bullet_{n',\beta'} \otimes \mathcal{O}_X(mD)$$

Wall-crossing