Symmetry and enumerative geometry of Calabi–Yau spaces

Tim-Henrik Bülles

ETH Zürich

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Overview

- Calabi–Yau spaces
- Enumerative geometry
- Symmetry

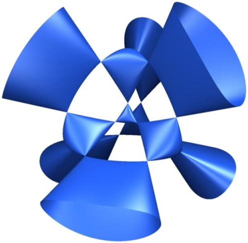
Calabi–Yau spaces: 2-dim

- X - complex smooth projective variety with

$$\omega_X \cong \mathcal{O}_X\,, \quad H^1(S, \mathcal{O}_X) = 0$$

- 2-dim: S - complex smooth projective K3 surface, e.g. Fermat quartic $S \subset \mathbb{P}^3$ zero locus of

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$$



Calabi–Yau spaces: 3-dim

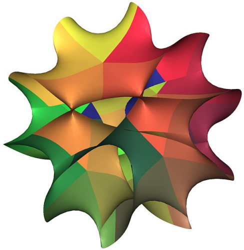
- 3-dim: X - e.g. Fermat quintic $X \subset \mathbb{P}^4$ zero locus of

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0$$

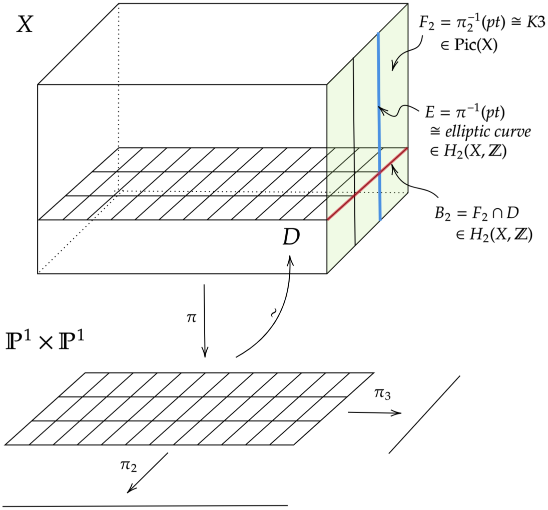
- Often, 2- and 3-dim come together:

$$\pi \colon X \to C$$

with C a curve, fibers $S_t = \pi^{-1}(t)$ are K3 surfaces



STU model - CY3



Enumerative geometry

- $g \in \mathbb{Z}_{\geq 0}$ and curve class $\beta \in H_2(X, \mathbb{Z})$
- **Q:** $\#\{$ genus g curves in X of class β $\} = ??$
- $X \subset \mathbb{P}^4$ quintic, $\beta = [\ell]$ for a line $\ell \subset X$:

$$\begin{array}{llll} n_{0,\beta} = 2875\,, & n_{0,2\beta} = 609250\,, & n_{0,3\beta} = 317206375\,, & \dots \\ \#\{\text{lines}\}\,, & \#\{\text{conics}\}\,, & \#\{\text{twisted cubics}\}\,, & \dots \end{array}$$

(Schubert 19^{th} century, Katz, Ellingsrud–Strømme, Kontsevich, Mirror Symmetry [COGP],...)

Enumerative geometry

Gromov–Witten invariants

$\mathrm{GW}^X_{g,\beta} \in \mathbb{Q}$ counting

$f \colon C \to X$

with C at most nodal,

$|\mathrm{Aut}(f)| < \infty$,

$f_*[C] = \beta$,

$g(C) = g$,

Pandharipande–Thomas invariants

$\mathrm{PT}^X_{n,\beta} \in \mathbb{Z}$ counting

$\mathcal{O}_X \xrightarrow{s} F$

with F pure 1-dimensional,

$\mathrm{coker}(s)$ is 0-dimensional,

$[F] = \beta$,

$\chi(F) = n$.

$$\mathrm{GW} \quad \xleftrightarrow{\mathit{MNOP}} \quad \mathrm{PT}$$

Partition function

- **Upshot:** Partition function ($q = -e^{iu}$)

$$Z^X = \exp\left(\sum_{g,\beta} \mathrm{GW}^X_{g,\beta}\, u^{2g-2}\, z^\beta\right) = \sum_{n,\beta} \mathrm{PT}^X_{n,\beta}\, q^n\, z^\beta$$

- **Very hard to study:** Solved for local geometries and $K3 \times E$, otherwise only partial results

Partition function

Key: Understand parts of Z^X as *functions*.

modular forms

GW

Interaction with moduli of curves $\overline{M}_{g,n}$ and stable maps $\overline{M}_{g,n}(S,\beta)$

Tautological relations

rational functions

PT

Interaction with derived category $D^b(X)$ and stability conditions

Hall algebras, wall-crossing

Rationality I (PT)

Theorem (Bridgeland, Toda '16)

For each β the generating series

$$\sum_{n \in \mathbb{Z}} \mathrm{PT}^{X}_{n,\beta}\, q^n$$

is the expansion of a rational function f_β satisfying the symmetry

$$f_\beta(1/q) = f_\beta(q)$$

Typical example (contribution of isolated rational curve):

$$f(q) = \frac{q}{(1-q)^2}$$

Rationality I (PT)

- View stable pairs

$$\mathcal{O}_X \xrightarrow{s} F$$

as 2-term complex in derived category $D^b(X)$

- General principle:

$$\text{Symmetry of the derived category } \phi \in \text{Aut}\left(D^b(X)\right)$$
$$\downarrow$$
$$\text{Constraints on curve counting on } X$$

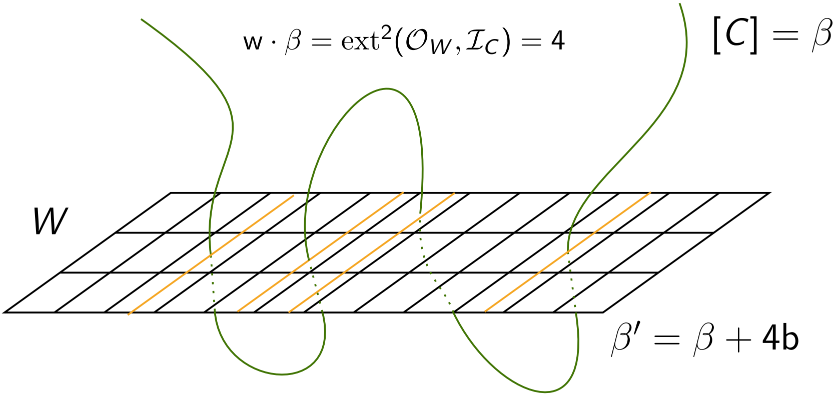
- Rationality/ Symmetry in q:

$$\phi = \mathbb{D}^X = R\mathcal{H}om(-, \mathcal{O}_X)[2]$$
$$\downarrow$$
$$(n\,, \beta) \longmapsto (-n\,, \beta)$$

Rationality II (PT)

- Surface W a $\mathbb{P}^1$-bundle (over some curve)
- X Calabi–Yau 3-fold containing $W \subset X$
- $\mathrm{b} \in H_2(X, \mathbb{Z})$ the class of the $\mathbb{P}^1$-ruling of W

Rationality II (PT)



$\mathbb{Z}_2$-involution: $\beta \mapsto \beta' = \beta + (\mathrm{w} \cdot \beta)\, \mathrm{b}$

Rationality II (PT)

- For all $\beta \in H_2(X, \mathbb{Z})$ define

$$\mathrm{PT}_\beta(q, Q) = \sum_{n,j \in \mathbb{Z}} \mathrm{PT}_{n,\beta + j\mathsf{b}}\, q^n\, Q^j$$

- For $\beta = 0$ compute directly

$$\mathrm{PT}_0(q, Q) = \prod_{j \geq 1} (1 - q^j Q)^{(2g-2)j}$$

Rationality II (PT)

Theorem (B.-Moreira '21)

$$\frac{\mathrm{PT}_\beta(q,Q)}{\mathrm{PT}_0(q,Q)} \in \mathbb{Q}(q,Q)$$

is the expansion of a rational function $f_\beta(q,Q)$ *which satisfies the functional equations*

$$f_\beta(q^{-1},Q) = f_\beta(q,Q)\,,$$
$$f_\beta(q,Q^{-1}) = Q^{-\mathrm{w}\cdot\beta} f_\beta(q,Q)$$

Predicted by physics, at least in the local case K_W (Klemm–Kreuzer–Riegler–Scheidegger '05)

Rationality II (PT)

- Geometry $W \subset X$ leads to a spherical twist

$$t_\Phi \in \mathrm{Aut}\left(D^b(X)\right)$$

- Rationality/ Symmetry in Q:

$$\begin{array}{c} \phi = t_\Phi \circ \mathbb{D}^X \\ \downarrow \\ (n\,,\beta) \longmapsto (-n\,,\beta') \end{array}$$

Rationality II (PT)

For $X = K_{\mathbb{P}^1 \times \mathbb{P}^1}$ and β class of a line:

$$
\begin{aligned}
f_\beta(-q, Q) &= \frac{2q}{(1-q)^2(1-Q)^2} \\
f_{2\beta}(-q, Q) &= \frac{2q^4}{(1-q)^2(1-q^2)^2(1-qQ)^2(1-Q)^2} \\
&\quad + \frac{2q^4}{(1-q)^2(1-q^2)^2(q-Q)^2(1-Q)^2} \\
&\quad + \frac{2q^4}{(1-q)^4(1-qQ)^2(q-Q)^2}
\end{aligned}
$$

Partition function

- **Upshot:** Partition function ($q = -e^{iu}$)

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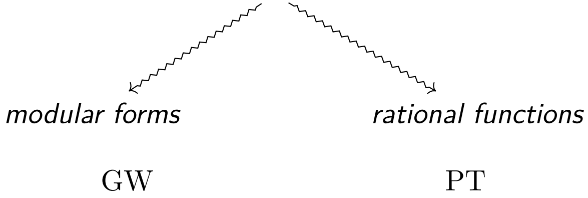

Modularity I (GW)

- S a K3 surface, can define descendent theory $\mathrm{GW}^S_{g,\beta}(\gamma) \in \mathbb{Q}$
- For K3-fibration $\pi\colon X \to C$, Noether–Lefschetz theory connects

$$\mathrm{GW}^X \xleftrightarrow{\mathrm{NL}^\pi} \mathrm{GW}^S(\lambda)$$

- For fixed genus g and divisibility $m \in \mathbb{Z}_{>0}$ define generating series

$$\mathrm{GW}^S_{g,m}(\gamma) = \sum_{h \geq 0} \mathrm{GW}^S_{g,\beta_{m,h}}(\gamma)\, q^{h-m} \in q^{-m}\,\mathbb{Q}[[q]]$$

Modularity I (GW)

Theorem (Maulik–Pandharipande–Thomas '10)

For $m = 1$, i.e. primitive curve classes,

$$\mathrm{GW}_1^S(\gamma) \in \frac{1}{\Delta(q)}\,\mathrm{QMod}$$

is the q-expansion of weakly holomorphic quasimodular form.

- Algorithm to compute all GW^S for $m = 1$ [MPT, Johannes Schmitt et al.]
- Conjectured to hold for all $m \in \mathbb{Z}_{>0}$

Modularity II (GW)

Theorem (Bae-B. '20)

For divisibility two curve classes ($m = 2$):

(i) *Quasimodularity of level 2 in all genus, all descendents,*

$$\mathrm{GW}_2^S(\gamma) \in \frac{1}{\Delta(q)^2}\,\mathsf{QMod}(2)$$

(ii) *Holomorphic anomaly equation (recursive structure for certain derivate of generating series)*

(iii) *Extend algorithm [MPT] to compute all $\mathrm{GW}^S_{g,2,h}(\gamma)$*

Thank you!